Irada A. IBADOBA

MATHEMATICAL MODELS OF MOVEMENT OF TWO PARTICLES ON THE RING

Abstract

In this paper we consider the movement model of two particles over n isolated and equidstant points of circumference without overtaking. It is assumed that movemet occurs stepwise in time $0, h, 2h, ...$ $(h > 0)$. The necessary and sufficient condition when separatelyconsidered particle may perforv random binomial walk is found.

1. Introduction. Mathematical models of moving particles are widely used in different applications. The transport problems which are described by behaviour of moving particles both on straightline and on the closed contours are one of applications of such models. On such models it is succedded to find out unexpected effects arising in transport systems [1-3] as jam phenomenon, road capacity and so on.

In [1,2] for models of moving particles without overtaking an unexpected effect - random binomial walk of separately considered particle is detected. This effect allows to compute road capacity and to discover unwanted "jam" phenomena in transport systems [2].

In this paper the models of moving particles on the ring which are extension of investigations led in [1-3] are considered. In the considered models movement occurs step-wise in discrete time. Investigation of continuous models entails definite difficulties, althuogh they can be considered as extension of investigations of discrete models. Some simplified models with continuous time of movement are considered in [4].

The goal of our paper is finding the necessary and sufficient condition for random binomial walk of particles for models considered in [3].

2. Mathematical model of movement of particels without overtaking. Following [3] we consider the movement of two particles on the ring without overtaking. The particles move counterclockwise. We number n equistand points along the ring clockwise. Let $\xi_{i,t}$ be coordinates of i-th particles in time t $(i = 1, 2)$. In discrete time $t \in T = \{0, h, 2h, ...\}$, $h > 0$ every particle can be in jump per unit length or stand stil. Denote by $\rho_{i,t}$- distance between particles in time t, in direction of movement

$$\rho_{i,t} = \begin{cases} \xi_{i+1,t} - \xi_{i,t} & \text{if} \quad \xi_{i+1,t} > \xi_{i,t} \\ \\ n + (\xi_{i+1,t} - \xi_{i,t}) \text{ if} & \xi_{i+1,t} < \xi_{i,t}. \end{cases}$$

We defne $\varepsilon_{i,t} = \left|\xi_{i,t+1} - \xi_{i,t}\right|$. From description of model it follows that $\varepsilon_{i,t} = 1$,if particle is in jump in time t; $\varepsilon_{i,t} = 0$,if the particle stands stil.

Asymmetric model. Consider asymmetric model considered in [3].

Let the parameters of movement of one particles $r,l\quad |r+l=1|\quad (0<r<1)$,and the parameters of movement of the other particles depend on distance to the previous particle, i.e.,

$$P\{\varepsilon_{1,t}=1|\rho_{1,t}=k\}=r_k;$$

$$P\{\varepsilon_{1,t}=0|\rho_{1,t}=k\}=l_k,\ \ (k=\overline{2,n-1}),\ r_k+l_k=1;$$

$$P\{\varepsilon_{1,t}=1|\rho_{1,t}=1,\varepsilon_{2,t}=1\}=r_1;$$

$$P\{\varepsilon_{1,t}=0|\rho_{1,t}=1,\varepsilon_{2,t}=1\}=l_1,\ r_1+l_1=1;$$

$$P\{\varepsilon_{1,t}=1|\rho_{1,t}=1,\varepsilon_{2,t}=0\}=0;$$

$$0<r_i<1,\quad (i=\overline{1,n-1});$$

$$P\{\varepsilon_{1,t}=0|\rho_{1,t}=1,\varepsilon_{2,t}=0\}=1;$$

$$P\{\varepsilon_{2,t}=1|\rho_{2,t}=k\}=r,\ (k=\overline{2,n-1});$$

$$P\{\varepsilon_{2,t}=1|\rho_{2,t}=1,\varepsilon_{1,t}=1\}=r;$$

$$P\{\varepsilon_{2,t}=1|\rho_{2,t}=1,\varepsilon_{1,t}=0\}=0.$$

From description of model it follows that $\rho_{i,t}$ form ergodic Markov chain with the finite number of states and with positive transient probability. Consequently, there exists steady-state condition [5], i.e., the distribution $\rho_{i,t}$ independent of t

$$P\{\rho_{i,t}=k\}=a_k,\ \ \sum_{k=1}^{n-1}a_k=1.$$

For steady-state probability of distribution of distance between particles we rewrite the reccurence equation

$$P\{\rho_{1,t+h}=1\}=P\{\rho_{1,t}=1\}[P\{\varepsilon_{1,t}=1|\rho_{1,t}=1\}\times$$
$$\times P\{\varepsilon_{2,t}=1|\rho_{1,t}=1\}+P\{\varepsilon_{2,t}=0|\rho_{1,t}=1\}]+$$
$$+P\{\rho_{1,t}=2\}P\{\varepsilon_{2,t}=0|\rho_{1,t}=2\}\times P\{\varepsilon_{1,t}=1|\rho_{1,t}=2\};$$
$$P\{\rho_{1,t+h}=k\}=P\{\rho_{1,t}=k-1\}P\{\varepsilon_{2,t}=1|\rho_{1,t}=k-1\}\times$$
$$\times P\{\varepsilon_{1,t}=0|\rho_{1,t}=k-1\}+P\{\rho_{1,t}=k\}[P\{\varepsilon_{2,t}=1|\rho_{1,t}=k\}\times$$
$$\times P\{\varepsilon_{1,t}=1|\rho_{1,t}=k\}+P\{\varepsilon_{2,t}=0|\rho_{1,t}=k\}\times \tag{1}$$
$$P\{\varepsilon_{1,t}=0|\rho_{1,t}=k\}]+P\{\rho_{1,t}=k+1\}\times$$
$$\times P\{\varepsilon_{2,t}=0|\rho_{1,t}=k+1\}P\{\varepsilon_{1,t}=1|\rho_{1,t}=k+1\},\ k=\overline{2,n-2};$$
$$P\{\rho_{1,t+h}=n-1\}=P\{\rho_{1,t}=n-2\}P\{\varepsilon_{2,t}=1|\rho_{1,t}=n-2\}\times$$

$$\times P\{\varepsilon_{1,t}=0|\rho_{1,t}=n-2\}+P\{\rho_{1,t}=n-1\}[P\{\varepsilon_{1,t}=1|\rho_{1,t}=n-1\}\times$$

$$\times P\{\varepsilon_{2,t}=1|\rho_{1,t}=n-1\}+P\{\varepsilon_{1,t}=0|\rho_{1,t}=n-1\}];$$

Passing to the limit as $t\to\infty$ and using that there exists steady-state condition, from (1) we obtain

$$a_1=a_1(rr_1+l)+a_2lr_2;$$

$$a_k=a_{k-1}rl_{k-1}+a_k(rr_k+ll_k)+a_{k+1}lr_{k+1},\quad (k=\overline{2,n-2}); \tag{2}$$

$$a_{n-1}=a_{n-2}rl_{n-2}+a_{n-1}(rr_{n-1}+l_{n-1}).$$

Let

$$A_k=\left(\tfrac{r}{l}\right)^{k-1}\frac{l_1...l_{k-1}}{r_2...r_k},\qquad A_1=1,\qquad A=\sum_{j=1}^{n-1}A_j. \tag{3}$$

By immediate substitution we are convinced that $a_k=\dfrac{A_k}{A}$ is a solution of (2). From (3) we obtain

$$a_krl_k=a_{k+1}lr_{k+1}. \tag{4}$$

The sufficient condition $r_{n-1}=1$ when separately considered particle may perform random binomial walk with the parameters r,l, is obtained in [3]. There arises a question: is this condition also necessary, i.e., may the results on random walk for the movement on line stated in [1,2] be transfered fro ring or is an additional condition required for this?

For probability of jumps of particles we have

$$P\{\varepsilon_{1,t}=1\}=\sum_{k=2}^{n-1}P\{\varepsilon_{1,t}=1,\rho_{1,t}=k\}+P\{\varepsilon_{1,t}=1,\rho_{1,t}=1,\varepsilon_{2,t}=1\}=$$

$$=\sum_{k=2}^{n-1}P\{\varepsilon_{1,t}=1|\rho_{1,t}=k\}P\{\rho_{1,t}=k\}+P\{\varepsilon_{1,t}=1|\rho_{1,t}=1,\varepsilon_{2,t}=1\}\times$$

$$\times P\{\varepsilon_{2,t}=1|\rho_{1,t}=1\}P\{\rho_{1,t}=1\}=\sum_{k=2}^{n-1}r_ka_k+r_1ra_1, \tag{5}$$

$$P\{\varepsilon_{2,t}=1\}=\sum_{k=1}^{n-2}P\{\varepsilon_{2,t}=1,\rho_{1,t}=k\}+$$

$$+P\{\varepsilon_{2,t}=1,\rho_{1,t}=n-1,\varepsilon_{1,t}=1\}=\sum_{k=1}^{n-2}ra_k+rr_{n-1}a_{n-1}. \tag{6}$$

Lemma 1. *For asymmetric model*

$$P\{\varepsilon_{1,t}=1\}=P\{\varepsilon_{2,t}=1\}.$$

is satisfied.

Proof.

$$r = r\sum_{k=1}^{n-1}(r_k + l_k)a_k = \sum_{k=1}^{n-1} rr_k a_k + \sum_{k=1}^{n-1} rl_k a_k =$$

$$= \sum_{k=1}^{n-1} rr_k a_k + rl_{n-1}a_{n-1} + \sum_{k=1}^{n-2} rl_k a_k = rl_{n-1}a_{n-1} +$$

$$+ \sum_{k=1}^{n-1} rr_k a_k + \sum_{k=1}^{n-2} lr_{k+1}a_{k+1} = rl_{n-1}a_{n-1} + rr_1 a_1 +$$

$$+ \sum_{k=2}^{n-1} rr_k a_k + \sum_{k=2}^{n-1} lr_k a_k = rl_{n-1}a_{n-1} + P\{\varepsilon_{1,t} = 1\},$$

$$r = P\{\varepsilon_{1,t} = 1\} + rl_{n-1}a_{n-1}. \tag{7}$$

$$P\{\varepsilon_{2,t} = 1\} = \sum_{k=1}^{n-2} ra_k + rr_{n-1}a_{n-1} =$$

$$= r(1 - a_{n-1}) + rr_{n-1}a_{n-1} = r - rl_{n-1}a_{n-1},$$

$$r = P\{\varepsilon_{2,t} = 1\} + rl_{n-1}a_{n-1}. \tag{8}$$

From equalities (7) and (8) we obtain the statement of lemma 1.

Lemma 2. *In order that* $P\{\varepsilon_{i,t} = 1\} = r,\ (i = 1,2)$ *it is necessary and sufficient that*

$$r_{n-1} = 1.$$

Proof. The sufficiency follows from (7) and (8), since if $r_{n-1} = 1$,

$$r = P\{\varepsilon_{1,t} = 1\} = P\{\varepsilon_{2,t} = 1\}.$$

Analogously, from (7 and (8) it follows that if

$$r = P\{\varepsilon_{1,t} = 1\} = P\{\varepsilon_{2,t} = 1\},$$

then $ra_{n-1}l_{n-1} = 0$.Since $r \neq 0$ and $a_{n-1} \neq 0 \Rightarrow l_{n-1} = 0$. Lemma 2 is proved.

Following [3] we introduce the following notation

$$b(\varepsilon_1, ..., \varepsilon_m) = P\{\varepsilon_{1,t+h} = \varepsilon_1, ..., \varepsilon_{1,t+mh} = \varepsilon_m\} =$$

$$= \sum_{k=1}^{n-1} P\{\varepsilon_{1,t+h} = \varepsilon_1, ..., \varepsilon_{1,t+mh} = \varepsilon_m, \rho_{1,t+mh} = k\},$$

$$\varepsilon_m^+ = \sum_{j=1}^{m} \varepsilon_j, \quad \varepsilon_m^- = m - \varepsilon_m^+,\ \varepsilon_j = 0 \quad \text{or } 1.$$

Theorem 1. *In order that* $b(\varepsilon_1, ..., \varepsilon_m) = r^{\varepsilon_m^+} l^{\varepsilon_m^-}$, $(r + l = 1)$ *it is necessary and sufficient that* $r_{n-1} = 1$.

Proof. Sufficiency. Using the method stated in [1], we prove the statement of theorem by mathematical induction method with respect to m. For $m = 1$ the statement of theorem follows from lemma 2. Let the statement be valid for m steps. We prove it for $m+1$ using (4) and lemma 2. For definiteness we assume that $\varepsilon_{m+1}=1$,

$$b(\varepsilon_1,...,\varepsilon_{m+1})=\sum_{k=1}^{n-1}P\{\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m,\varepsilon_{1,t+(m+1)h}=1,\rho_{1,t+(m+1)h}=k\}=$$

$$=\sum_{k=1}^{n-1}P\{\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m,\varepsilon_{1,t+(m+1)h}=1,\rho_{1,t+mh}=k,\varepsilon_{2,t+(m+1)h}=1\}+$$

$$+\sum_{k=1}^{n-2}P\{\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m,\varepsilon_{1,t+(m+1)h}=1,\rho_{1,t+mh}=(k+1),\varepsilon_{2,t+(m+1)h}=0\}=$$

$$=\sum_{k=1}^{n-1}P\{\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m\}P\{\rho_{1,t+mh}=k|\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m\}\times$$

$$\times P\{\varepsilon_{2,t+(m+1)h}=1|\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m,\rho_{1,t+mh}=k\}\times$$

$$\times P\{\varepsilon_{1,t+(m+1)h}=1|\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m,\rho_{1,t+mh}=k,\varepsilon_{2,t+(m+1)h}=1\}+$$

$$+\sum_{k=1}^{n-2}P\{\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m\}P\{\rho_{1,t+mh}=(k+1)|\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m\}\times$$

$$\times P\{\varepsilon_{2,t+(m+1)h}=0|\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m,\rho_{1,t+mh}=(k+1)\}\times$$

$$\times P\{\varepsilon_{1,t+(m+1)h}=1|\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m,\rho_{1,t+mh}=(k+1),\varepsilon_{2,t+(m+1)h}=0\}=$$

$$=\sum_{k=1}^{n-1}r_kra_kb(\varepsilon_1,...,\varepsilon_m)+\sum_{k=1}^{n-2}r_{k+1}la_{k+1}b(\varepsilon_1,...,\varepsilon_m)=b(\varepsilon_1,...,\varepsilon_m)\times$$

$$\times\left(\sum_{k=1}^{n-1}r_kra_k+\sum_{k=1}^{n-2}r_{k+1}a_{k+1}l\right)=b(\varepsilon_1,...,\varepsilon_m)\left(rr_1a_1+\sum_{k=2}^{n-1}r_kra_k+\right.$$

$$\left.+\sum_{k=2}^{n-1}r_ka_kl\right)=b(\varepsilon_1,...,\varepsilon_m)\left(rr_1a_1+\sum_{k=2}^{n-1}r_ka_k\right)=b(\varepsilon_1,...,\varepsilon_m)\cdot r.$$

The case $\varepsilon_{m+1}=0$ is analogously proved.

Necessity. Let $b(\varepsilon_1,...,\varepsilon_m)=P\{\varepsilon_{1,t+h}=\varepsilon_1,...,\varepsilon_{1,t+mh}=\varepsilon_m\}=r^{\varepsilon_m^+}l^{\varepsilon_m^-}$. For simplicity we assume $\varepsilon_{m+1}=1$.Then we have

$$b(\varepsilon_1,...,\varepsilon_{m+1})=b(\varepsilon_1,...,\varepsilon_m)P\{\varepsilon_{1,t}=1\}.$$

On the other hand $b(\varepsilon_1,...,\varepsilon_{m+1})=r^{\varepsilon_{m+1}^+}l^{\varepsilon_{m+1}^-}=r^{\varepsilon_m^+}l^{\varepsilon_m^-}\cdot r=b(\varepsilon_1,...,\varepsilon_m)\cdot r$.Hence it follows $P\{\varepsilon_{1,t}=1\}=r$. From lemma 2 it follows $r_{n-1}=1$. Theorem 1 is proved.

Symmetric model. Let the movement occurs by the following law:

$$P\{\varepsilon_{1,t}=1(0)|\rho_{1,t}=k\}=r_k(l_k),\quad(k=\overline{2,n-1}),\quad r_k+l_k=1;$$

$$P\{\varepsilon_{1,t}=1(0)|\rho_{1,t}=1,\varepsilon_{2,t}=1\}=r_1(l_1),\quad r_1+l_1=1\ ;$$

$$P\{\varepsilon_{1,t}=1(0)|\rho_{1,t}=1,\varepsilon_{2,t}=0\}=0(1);$$

$$P\{\varepsilon_{1,t}=0,\varepsilon_{2,t}=0|\rho_{1,t}=1\}=l_{n-1};$$

$$P\{\varepsilon_{1,t}=1,\varepsilon_{2,t}=1|\rho_{1,t}=1\}=r_1r_{n-1};$$

$$P\{\varepsilon_{1,t}=0,\varepsilon_{2,t}=1|\rho_{1,t}=1\}=l_1r_{n-1}; \tag{9}$$

$$P\{\varepsilon_{1,t}=1,\varepsilon_{2,t}=0|\rho_{1,t}=1\}=0;$$

$$P\{\varepsilon_{2,t}=1(0)|\rho_{1,t}=n-k\}=r_k(l_k),\quad (k=\overline{2,n-1}),\ r_k+l_k=1;$$

$$P\{\varepsilon_{2,t}=1(0)|\rho_{1,t}=n-1,\varepsilon_{1,t}=1\}=r_1(l_1),\ r_1+l_1=1\ ;$$

$$P\{\varepsilon_{2,t}=1(0)|\rho_{1,t}=n-1,\varepsilon_{1,t}=0\}=0(1);$$

$$\rho_{1,t}+\rho_{2,t}=n,\quad 0<r_k<1,\ k=\overline{1,n-1;}$$

Thus, this model is also ergodic Markov chain with the finite 2^n number of states. When the distance $\rho_{i,t}$ does not depend on t and invariant with respect to i, we call it steady-state condition. Since we have ergodic Markov chain with the finite number of states, then there exists steady-state condition

$$P\{\rho_{i,t}=k\}=a_k,\ \sum_{k=1}^{n-1}a_k=1.$$

Such distribution must satisfy the system of equations

$$a_1=a_1(r_1r_{n-1}+l_{n-1})+a_2l_{n-2}r_2;$$

$$a_k=a_{k+1}l_{n-k-1}r_{k+1}+a_k(r_kr_{n-k}+l_kl_{n-k})+$$
$$+a_{k-1}r_{n-k+1}l_{k-1},(k=\overline{2,n-2}); \tag{10}$$

$$a_{n-1}=a_{n-2}r_2l_{n-2}+a_{n-1}(r_1r_{n-1}+l_{n-1}).$$

We introduce

$$A_k=\left(\frac{r}{l}\right)^{k-1}\frac{r_{n-k+1}...r_{n-1}}{l_{n-k}...l_{n-2}}\frac{l_1...l_{k-1}}{r_2...r_k},\qquad A_1=1,\qquad A=\sum_{j=1}^{n-1}A_j. \tag{11}$$

Then $a_k=\frac{A_k}{A}$ is a solution of (10). Note that $A_k=A_{n-k}$. From expression (1) we obtain the reccurence fromula for a_k:

$$a_{k+1}l_{n-k-1}r_{k+1}=a_kr_{n-k}l_k.$$

For probability of jumps of particles we have

$$P\{\varepsilon_{1,t}=1\}=\sum_{k=2}^{n-1}P\{\varepsilon_{1,t}=1,\rho_{1,t}=k\}+$$

$$+P\{\varepsilon_{1,t}=1,\rho_{1,t}=1,\varepsilon_{2,t}=1\}=\sum_{k=2}^{n-1}r_k a_k+r_1 r_{n-1}a_1, \tag{12}$$

$$P\{\varepsilon_{2,t}=1\}=\sum_{k=1}^{n-2}P\{\varepsilon_{2,t}=1,\rho_{1,t}=k\}+P\{\varepsilon_{2,t}=1,\rho_{1,t}=n-1,\varepsilon_{1,t}=1\}=$$

$$=\sum_{k=1}^{n-2}r_{n-k}a_k+r_1 r_{n-1}a_1=\sum_{k=2}^{n-1}r_k a_k+r_1 r_{n-1}a_1. \tag{13}$$

Such model where random binomial walk of separately considered particle is proved, is investigated in [3].

The result on random walk of separately considered particle in case of symmetric model holds without requirement of fulfilment of the condition $r_{n-1}=1$.

Remark. For asymmetric model, for binomial random walk of separately considered particle, $r_{n-1}=1$ is necessary and sufficient condition, whereas for symmetric model this condition is not required. This follows from the proof of theroem 5 stated in the paper [3]. We can explain this by the fact that according to construction of model in symmetric case the probabilities of jumps of particles coincide for a step, and in case of asymmetric model according to lemma 2 these probabilities coincide if and only if $r_{n-1}=1$.

The author express their gratitude to corr.mem. NAS of Azerb. A.G.Gadjiev for the statement of the problem and associate professor F.G.Rahimov for useful discussion.

References

[1]. Belyaev Yu.K. *On simplified model of movement without overtaking.* Izv. ANSSR, Tekhnicheskaya Kibernetika, 1969, No3. (Russian)

[2]. Tsele U. *Generalization of model of movement without overtaking.* Izv. ANSSR, Tekhnicheskaya Kibernetika, 1972, No5. (Russian)

[3]. Hajiev A.G. *Model of moving of particles over the closed contour without overtaking.* Izv. ANSSR, Tekhnicheskaya Kibernetika, 1976, No5. (Russian)

[4]. Hajiev A.G., Gafarov I.A. *Mathematical models of moving particles without overtaking.* Proc. of IMM NAS of Azerb., 2002, v.16(24), No5, pp.50-56. (Russian)

[5]. Feller B. *Introduction to theory of probability and its application.* "Mir", 1967, v.2. (Russian)

Irada A. Ibadova
Institute of Matehmatics and Mechanics of NAS Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 92 74 (off.)

Received November 26, 2003; Revised March 10, 2004.
Translated by Mirzoyeva K.S.