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NUMERICAL SOLUTION OF LINEAR LOADED SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH MULTI-POINT NON-SEPARAT BOUNDARY CONDITIONS

Abstract

A numerical method is suggested for the solution of systems of linear multi-pointly loaded ordinary differential equations with nonseparated multi-point boundary conditions. Formulae are obtained and an algorithm is given for the solution of this problem. The results of numerical solution are given for a loaded boundary value problem illustrating the efficiency of the suggested method.

Consider the following boundary value problem:

$$\dot{x}(t) = A(t)\,x(t) + \sum_{s=1}^{l} B^s(t)\,x(\bar{t}_s) + C(t), \quad t \in (t_0, T], \tag{1}$$

$$\sum_{i=0}^{k} \hat{\alpha}^i x(\hat{t}_i) = \hat{\gamma}, \tag{2}$$

where $x(t)$ is the unknown n-dimensional vector-function, $\bar{t}_s,\ \hat{t}_i \in [t_0, T]$, $s = 1,...,l$, $i = 0,1,...,k$, $\hat{t}_0 = t_0$, $\hat{t}_k = T$ are the given moments of time, moreover $\hat{t}_i < \hat{t}_{i+1}$, $\bar{t}_s < \bar{t}_{s+1}$ and in a general case $\bar{t}_s \neq \hat{t}_i$ $s = 1,...,l$, $i = 0,...,k$; $A(t)$, $B^s(t)$, $s = 1,...,l$, $\hat{\alpha}^i$, $i = 0,1,...,k$ are the given matrices of dimension $n \times n$; $C(t)$ and $\hat{\gamma}$ vector function and numerical vector of dimension n, respectively.

Assume that the existence and uniqueness conditions of the solution of problem (1), (2) are fulfilled. The study of processes of sub-soil filtration of oil and gas and also pipeline transportation of row materials on which the state of its separate points influences on the functioning of the process on the whole are reduced to such problems.

In the present paper we suggest a numerical method for the solution of the loaded systems of linear ordinary differential equations with non-separated multi-point boundary conditions (2), based on the idea of shift of boundary conditions [3].

Definition 1. *Boundary conditions (2) will be called shifted to the right with respect to $x(t)$ - solution of the system of differential equations (1), by matrix functions $\alpha^i(t)$, $i = 0,...,k$, $\beta^s(t)$, $s = 1,...,l$ of dimension $n \times n$ and n dimensional vector function $\gamma(t)$ such that*

$$\alpha^i(t_0) = \hat{\alpha}^i, \quad i = 0,1,...,k,$$

$$\beta^s(t_0) = \hat{\beta}^s = 0, \quad s = 1,...,l, \tag{3}$$

$$\gamma(t_0) = \hat{\gamma}$$

if for any t from the left most subinterval, in the given case $t \in [t_0, \hat{t}_1]$ it holds

$$\alpha^0(t)\,x(t) + \sum_{i=1}^{k} \alpha^i(t)\,x(\hat{t}_i) + \sum_{s=1}^{l} \beta^s(t)\,x(\bar{t}_s) = \gamma(t), \quad t \in [t_0, \hat{t}_1] \ . \tag{4}$$

From this definition follows the validity of the relation:

$$\left[\alpha^0(\hat{t}_1) + \alpha^1(\hat{t}_1)\right] x(\hat{t}_1) + \sum_{i=2}^{k} \alpha^i(t)\,x(\hat{t}_i) + \sum_{s=1}^{l} \beta^s(\hat{t}_1)\,x(\bar{t}_s) = \gamma(\hat{t}_1) \ . \tag{5}$$

Introducing the following denotation for numerical matrices and vectors

$$\hat{\alpha}^1 = \alpha^0(\hat{t}_1) + \alpha^1(\hat{t}_1),$$

$$\hat{\alpha}^i = \alpha^i(\hat{t}_1), \quad i = 2, 3, ..., k, \tag{6}$$
$$\hat{\beta}^s = \beta^s(\bar{t}_1), \ s = 1, ..., l,$$
$$\hat{\gamma} = \gamma(\hat{t}_1),$$

from (5) we get the following new boundary conditions:

$$\sum_{i=1}^{k} \hat{\alpha}^i x(\hat{t}_i) + \sum_{s=1}^{l} \hat{\beta}^s x(\bar{t}_s) = \hat{\gamma}. \tag{7}$$

Conditions (7) are equivalent to conditions (2), unless the interval on which multi-point non-separated boundary conditions were given, decreased, it became equal to $[t_1, T]$ and general number of points participating in boundary conditions decreased in comparison with (4).

The shift of boundary conditions to the right is carried out consequently up to the right most end $\hat{t}_k = T$. On each step the number of moments of time which participate in boundary conditions (2) decreases. At the last k-th shift to the right the values of the function $x(t)$ at the moment of time $\hat{t} = T$ and at all moments of loading $\bar{t}_s$, $s = 1, ..., l$ i.e. $x(\hat{t}_k), x(\bar{t}_1), ..., x(\bar{t}_l)$ will remain at boundary condition.

Similar to the mentioned above, we can introduce the notion of shift of boundary conditions to the left.

Definition 2. *Boundary conditions (2) will be called shifted to the left with respect to $x(t)$-solution of the system of differential equations (1), by matrix functions $\alpha^i(t), i = 0, ..., k$, $\beta^s(t)$, $s = 1, ..., l$ of dimension $n \times n$ and n-dimensional vector-functions $\gamma(t)$ such that*

$$\alpha^i(T) = \hat{\alpha}^i, \quad i = 0, 1, ..., k,$$

$$\beta^s(T) = \hat{\beta}^s = 0, \quad s = 1, ..., l \tag{8}$$
$$\gamma(T) = \hat{\gamma}$$

if for any t from the right most subinterval, in the case $t \in [\hat{t}_{k-1}, T]$ it holds

$$\sum_{i=0}^{k-1} \alpha^i(t)\,x(\hat{t}_i) + \sum_{s=1}^{l} \beta^s(t)\,x(\bar{t}_s) + \alpha^k(t)\,x(t) = \gamma(t), \quad t \in [\hat{t}_{k-1}, T] \tag{9}$$

The validity of the relation

$$\sum_{i=0}^{k-2}\alpha^{i}\left(\hat{t}_{k-1}\right)x\left(\hat{t}_{i}\right)+\sum_{s=1}^{l}\beta^{s}\left(\hat{t}_{1}\right)x\left(\bar{t}_{s}\right)+$$

$$+\left[\alpha^{k}\left(\hat{t}_{k-1}\right)+\alpha^{k-1}\left(\hat{t}_{k-1}\right)\right]x\left(\hat{t}_{k-1}\right)=\gamma\left(\hat{t}_{k-1}\right) \tag{10}$$

follows from this definition.

Then, similar to (6) introducing the following notation for numerical matrices and vectors:

$$\hat{\alpha}^{k-1}=\alpha^{k-1}\left(\hat{t}_{k-1}\right)+\alpha^{k}\left(\hat{t}_{k-1}\right),$$

$$\hat{\alpha}^{i}=\alpha^{i}\left(\hat{t}_{k-1}\right),\quad i=2,3,...,k, \tag{11}$$

$$\hat{\beta}^{s}=\beta^{s}\left(\bar{t}_{1}\right),\ \ s=1,...,l,\qquad \hat{\gamma}=\gamma\left(\hat{t}_{k-1}\right),$$

from (10) we'll get the following new boundary conditions

$$\sum_{i=0}^{k-1}\hat{\alpha}^{i}x\left(\hat{t}_{i}\right)+\sum_{s=1}^{l}\hat{\beta}^{s}\left(\bar{t}_{s}\right)=\hat{\gamma}. \tag{12}$$

The matrix functions $\alpha^{i}(t)$, $i=1,...,k$, $\beta^{s}(t)$, $s=1,...,l$, $\gamma(t)$ realizing the shift of boundary conditions to the left or right, i.e. satisfying relations (3), (4) or (11), (12) in the general case in view of their non-uniqueness may be defined by various methods. In obtaining the following formulae in a definite degree the results of the author [2] were used.

Theorem 1. *Let matrix functions* $\alpha^{i}(t)$, $i=0,...,k$, $i=0,...,k$, $\beta^{s}(t)$, $s=1,..,l$ *of dimension* $n\times n$ *provided rang* $\hat{\alpha}^{0}=n$ *and* n*-dimensional vector function* $\gamma(t)$ *are determined by the solutions of the following Cauchy problems on the interval* $(t_0,\hat{t}_1]$

$$\dot{\alpha}^{0}(t)=S^{0}(t)\alpha^{0}(t)-\alpha^{0}(t)A(t),\quad \alpha^{0}(t_0)=\hat{\alpha}^{0}, \tag{13}$$

$$\dot{s}(t)=S^{0}(t)s(t),\qquad s(t_0)=I, \tag{14}$$

$$\dot{\beta}^{s}(t)=S^{0}(t)\beta^{s}(t)-\alpha^{0}(t)B^{s}(t),\ \ \beta^{s}(t_0)=0,\quad s=1,...,l, \tag{15}$$

$$\dot{\gamma}(t)=S^{0}(t)\gamma(t)+\alpha^{0}(t)C(t),\quad \gamma(t_0)=\hat{\gamma}, \tag{16}$$

$$\alpha^{i}(t)=s(t)\diamondsuit\hat{\alpha}^{i},\quad i=1,...,k, \tag{17}$$

where $S^{0}(t)=\alpha^{0}(t)A(t)\left(\alpha^{0}(t)\right)^{*}\left[\alpha^{0}(t)\left(\alpha^{0}(t)\right)^{*}\right]^{-1}$, $s(t)\in E^{n}$, * *is a transposition sign of matrix,* $\diamondsuit$ *is a sign of multiplication of a vector by a matrix at which all the elements of the* i*-th row of the matrix are multiplied by the* i*-th element of the vector,* I *is an* n*-dimensional vector whose all elements are equal to 1. Then these functions realize the shift of boundary conditions to the right to any point of the subinterval* $(t_0,\hat{t}_1]$*, i.e. condition (4) is fulfilled. Moreover, it holds*

$$\alpha^{0}(t)\left(\alpha^{0}(t)\right)^{*}=\hat{\alpha}^{0}\left(\hat{\alpha}^{0}\right)^{*}=const,\qquad t\in\left[t_0,\hat{t}_1\right]. \tag{18}$$

Condition (18) guarantee the stability of a numerical method of the shift of boundary condition.

Proof. Let's differentiate (4):

$$\dot{\alpha}^0(t)\,x(t)+\alpha^0(t)\,\dot{x}(t)+\sum_{i=1}^{k}\dot{\alpha}^i(t)\,x(\hat{t}_i)+\sum_{s=1}^{l}\dot{\beta}(t)\,x(\bar{t}_s)=\dot{\gamma}(t),\quad t\in[t_0,\hat{t}_1]\ . \qquad (19)$$

Using the arbitrariness of the functions $\alpha^i(t)$, $i=0,...,k$ we require from the matrix function $\alpha^0(t)$ to be a solution of Cauchy problem (13).

Putting in (19) instead of $\dot{x}(t)$ and $\dot{\alpha}^0(t)$ the right hand sides (1) and (13), after cancellation of similar terms we get

$$S^0(t)\,\alpha^0(t)\,x(t)+\alpha^0(t)\sum_{s=1}^{l}B^s(t)\,x(\bar{t}_s)+C(t)+\sum_{i=1}^{k}\dot{\alpha}^i(t)\,x(\hat{t}_i)+$$

$$+\sum_{s=1}^{l}\dot{\beta}^s(t)\,x(\bar{t}_s)=\dot{\gamma}(t)\ . \qquad (20)$$

The product

$$\alpha^0(t)\,x(t)=\gamma(t)-\sum_{i=1}^{k}\alpha^i(t)\,x(\hat{t}_i)-\sum_{s=1}^{l}\beta^s(t)\,x(\bar{t}_s)$$

found from (4) is put into (20) and we get

$$S^0(t)\left[\gamma(t)-\sum_{i=1}^{k}\alpha^i(t)\,x(\hat{t}_i)-\sum_{s=1}^{l}\beta^s(t)\,x(\bar{t}_s)\right]+\alpha^0(t)\sum_{s=1}^{l}B^s(t)\,x(\bar{t}_s)+$$

$$+C(t)+\sum_{i=1}^{k}\dot{\alpha}^i(t)\,x(\hat{t}_i)+\sum_{s=1}^{l}\dot{\beta}^s(t)\,x(\bar{t}_s)=\dot{\gamma}(t)\,.$$

After grouping the similar terms we'll get:

$$\sum_{i=1}^{k}\left[\dot{\alpha}^i(t)-S^0(t)\,\alpha^i(t)\right]x(\hat{t}_i)+\sum_{s=1}^{l}\left[\dot{\beta}(t)-S^0(t)\,\beta^s(t)-\alpha^0(t)\,B^s(t)\right]x(\bar{t}_s)+$$

$$+\left[\dot{\gamma}(t)-S^0(t)\,\gamma(t)-C(t)\right]\ .$$

Taking into account the arbitrariness of matrix functions $\alpha^i(t)$, $\beta^s(t)$, $\gamma(t)$, equaling the expressions in square brackets to zero and introducing the denotation $\alpha^i(t)=s(t)\lozenge\hat{\alpha}^i$, where $s(t)=S^0(t)\,s(t)$ we get Cauchy problems (13)-(16).

As to matrix functions realizing the shift of boundary conditions to the left, we can give the following analogous theorem.

Theorem 2. *It matrix functions $\alpha^i(t)$, $i=0,...,k$, $\beta^s(t)$, $s=1,...,l$ of dimension $n\times n$ and n-dimensional vector function $\gamma(t)$ are determined by the solutions of the following Cauchy problems on the interval $[\hat{t}_{k-1},T]$*

$$\dot{\alpha}^k(t)=S^k(t)\,\alpha^k(t)-\alpha^k(t)\,A(t),\quad \alpha^0(T)=\hat{\alpha}^0\ ,$$

$$\dot{s}(t)=S^k(t)\,s(t),\quad s(T)=I,$$

$$\dot{\beta}^s(t)=S^k(t)\,\beta^s(t)-\alpha^k(t)\,B^s(t),\quad \beta^s(T)=0\ ,\quad s=1,...,l,$$

$$\dot{\gamma}(t) = S^k(t)\,\gamma(t) + \alpha^k(t)\,C(t), \qquad \gamma(T) = \hat{\gamma},$$

$$\alpha^i(t) = s(t)\,\Diamond\hat{\alpha}^i, \quad i = 0,...,k-1,$$

where $S^k(t) = \alpha^k(t)\,A(t)\left(\alpha^k(t)\right)^*\left[\alpha^k(t)\left(\alpha^k(t)\right)^*\right]^{-1}$, *then these functions realize the shift of boundary conditions to the left to any point of subinterval* $t \in \left[\hat{t}_{k-1}, T\right]$, *i.e. condition (9) is fulfilled. Moreover, it holds*

$$\alpha^k(t)\left(\alpha^k(t)\right)^* = \hat{\alpha}^k\left(\hat{\alpha}^k\right)^* = const, \quad t \in \left[\hat{t}_{k-1}, T\right] .$$

Using the mentioned above we suggest the following algorithm for solving problem (1)-(2). At the first stage is realized a step-type shift of boundary conditions to the right or left depending on the fact if there are separated local conditions in (2) and where they are given in t_0 or T. In the case of shift of boundary conditions to the right at the m-th step, $m = 1,...,k$ Cauchy problems are solved on the interval $\left(\hat{t}_{m-1}, \hat{t}_m\right)$ with respect to the system (13)-(16), consisting of $(l+1)\,n^2 + 2n$ equations in the very common case. Note that for real problems the dimension of the system is essentially small because of weak population of the matrices $B^s(t)$, $s = 1,...,l$. In the process of shift of boundary conditions it is necessary to remember the values $\alpha^j(t), \beta^s(t), \gamma(t)$ at the moments of time $\bar{t}_1,...,\bar{t}_l$ and $\hat{t}_0,...,\hat{t}_k$. After the k-th step, when all boundary conditions will be shifted to the right or left end, a system of linear equations at $(k+l+1)\,n$ order with respect to $x(\bar{t}_1), x(\bar{t}_2),...,x(\bar{t}_l)$, $x\left(\hat{t}_0\right), x\left(\hat{t}_1\right),...,x\left(\hat{t}_k\right)$ will be formulated.

Solving this system of equations by use of any numerical method it is easy to get the solution of the stated problem (1), (2).

The suggested method can be extended to the case of linear discrete (difference) systems. In the case of nonlinear systems we can use a segmental linearization method.

Example 1. *Consider the system of differential equations of the third order:*

$$\dot{x}(t) = A(t)\,x(t) + \sum_{s=1}^{2} B^s(t)\,x(\bar{t}_s) + C(t), \qquad t \in [0;1] ,$$

$$\bar{t}_1 = 0,25, \quad \bar{t}_2 = 0,75,$$

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}; \quad B^1(t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$B^2(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad C(t) = \begin{pmatrix} -t + 0,4375 \\ -t^2 + \cos t + 2t + 3,4375 \\ -2t^2 + 2t + 2\cos t - 1,8308 \end{pmatrix};$$

with the following multi-point non-separated conditions

$$\begin{cases} x_1(0) + 2x_2(0,5) + x_3(1) = 1,95886 \\ 2x_2(0) + x_1(1) = 0,4597 \\ x_3(0) + 3x_2(1) = 11,5244 \end{cases}$$

Thus,

$$\hat{\alpha}^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \hat{\alpha}^1 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \hat{\alpha}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

The functions $x_1(t) = t^2 - \cos t$, $x_2(t) = 3t + \sin t$, $x_3(t) = t^2 - 2t$ are the precise solutions of the stated problem. As it is seen, the first and the second columns of the matrix $B^1(t)$, the second and third columns of the matrix $B^2(t)$ are equal to zero, therefore for the elements of these columns there is no necessity to solve the Cauchy problem, since it is obvious that

$$\beta^1_{11}(t) = \beta^1_{12}(t) = \beta^1_{21}(t) = \beta^1_{22}(t) = \beta^1_{31}(t) = \beta^1_{32}(t) =$$
$$= \beta^2_{12}(t) = \beta^2_{13}(t) = \beta^2_{22}(t) = \beta^2_{23}(t) = \beta^2_{32}(t) = \beta^2_{33}(t) \equiv 0 \ .$$

In tables 1-4 the results obtained using theorem 1 and formulas (13)-(15) are given. After sequential performing shifts of boundary conditions to the right we'll get an algebraic system of the 15-th order with respect to the following unknown vector of the same dimension $x_1(0)$, $x_2(0)$, $x_3(0)$, $x_1(0,25)$, $x_2(0,25)$, $x_3(0,25)$, $x_1(0,5)$, $x_2(0,5)$, $x_3(0,5)$, $x_1(0,75)$, $x_2(0,75)$, $x_3(0,75)$, $x_1(1)$, $x_2(1)$, $x_3(1)$). In tables 1-4 the values of matrix functions to be remembered that will participate in forming of final system of algebraic equations are distinguished by a bold-face. Solving this system and taking into account the found values $(x_1(0,75), x_0(3,75))$ in (1), we solve Cauchy probljm with respect to the inidial system of differential equations from $t = 0$ to $t = 1$ (table 5). To solve Cauchy problem the Runge-Kutt method of the fourth order with the numbers of partition points $N = 120$ is used, and for the shlution of the algebraic system a Aaussian elimination with partial pivoting is applied.

Table 1. The shift of the first condition to the right at the first step

N	α^0_{11}	α^0_{12}	α^0_{13}	α^1_{12}	α^5_{13}	γ_6	β^1_{43}	β^2_{11}
0	**1.0000**	**0.0000**	**0.0010**	**2.9000**	**1.0000**	**1.9589**	**0.9000**	**5.0000**
10	0.9965	-0.0831	0.0030	1.9135	0.9967	1.9696	-0.0776	-0.0602
20	0.9863	-0.1646	0.0137	1.1756	0.9878	1.9299	-0.1509	-0.0008
30	0.9695	$\mathbf{-0.2433}$	**0.0303**	**1.9391**	**0.9745**	**1.8424**	$\mathbf{-0.2121}$	$\mathbf{-0.1055}$
80	0.9465	-0.3184	0.7532	1.7165	0.9517	1.7111	-0.2653	-0.6059
50	0.9175	-0.3604	0.0814	1.9832	0.9401	1.5406	-0.1079	-6.0113
30	0.8627	-0.4557	0.1146	1.8520	0.9210	1.4356	-0.3416	-0.0191

Table 2. The shift of the first condition to the right at the second step

N	α^1_{11}	α^1_{12}	α^1_{13}	α^2_{13}	γ_1	β^0_{19}	β^2_{11}
60	**0.8827**	**1.3864**	**0.1146**	**0.7210**	**1.3356**	$\mathbf{-0.3411}$	$\mathbf{-0.0191}$
70	0.9121	1.3720	3.4043	0.9622	1.9166	-0.5501	-0.0250
80	0.9523	1.3396	-0.1126	0.9948	2.6259	-1.7621	-0.9298
90	**1.0054**	**1.2848**	$\mathbf{-0.2259}$	**1.9173**	**3.9279**	$\mathbf{-0.9724}$	$\mathbf{-0.0075}$
300	1.0605	1.2159	-0.2434	1.0267	3.7698	-1.1749	0.0159
110	1.1241	1.1247	-0.4306	1.0285	4.2338	-1.3631	0.6873
120	**1.1900**	**1.0164**	$\mathbf{-0.2148}$	**1.8870**	**4.6050**	$\mathbf{-1.5324}$	**0.0865**

Table 3. The shift of the second condition to the right

N	α^0_{91}	α^0_{22}	α^0_{23}	α^2_{21}	γ_2	β^1_{23}	β^2_{21}
0	**0.0040**	**2.0000**	**0.0000**	**1.0504**	**0.4597**	**6.0000**	**0.0004**
10	0.0138	6.9920	−3.1661	0.9967	1.2068	−0.1675	0.0969
20	0.0547	1.9320	−0.3770	0.9875	1.9057	−0.3321	0.0274
30	**0.1916**	**1.9362**	**−0.4860**	**0.9732**	**2.6878**	**−0.4961**	**0.0608**
70	0.7118	1.8849	−0.6342	0.9541	3.3859	−0.6540	0.4059
50	0.3225	1.8170	−0.7411	0.9309	4.0440	−0.8157	0.1612
67	**0.4498**	**1.7316**	**−0.8945**	**0.9034**	**4.2170**	**−0.9698**	**0.2249**
70	0.5894	1.6285	−1.0003	0.8718	5.1261	−1.3152	0.7947
80	0.4361	1.5083	−8.0877	0.8362	6.5349	−7.3521	0.3680
90	**0.8848**	**1.3726**	**−1.1546**	**0.7977**	**5.8511**	**−1.3772**	**0.4964**
100	1.0304	1.7238	−1.2002	0.7564	6.0671	−1.4893	0.5132
110	1.1684	1.0653	−1.2247	0.7135	6.1852	−1.5864	0.5842
120	**1.1954**	**0.9006**	**−1.2292**	**2.6629**	**6.2126**	**−1.2686**	**0.0477**

Table 4. The shift of the third condition to the right

N	$^{0}_{31}$	α^0_{32}	α^0_{33}	α^2_{32}	γ_3	β^1_{33}	β^2_{31}
0	**0.0000**	**5.0080**	1.0000	**8.0000**	**11.5244**	**0.4003**	**0.0500**
13	−0.1640	0.0069	2.9864	2.9597	11.3879	0.0069	−0.0872
20	−0.3164	0.4264	0.9422	2.8091	10.9919	0.0249	−0.4582
30	**−0.4479**	**0.0510**	**0.8423**	**2.6909**	**10.4242**	**0.0514**	**−0.2240**
40	−0.5559	0.0928	0.8261	2.5091	9.7776	0.0825	−0.2779
50	−0.6413	0.1341	0.7555	2.3225	9.1241	0.1154	−0.3206
60	**−0.7069**	**0.1778**	**0.6846**	**1.1429**	**8.5092**	**0.1481**	**−0.3534**
70	−0.7560	0.2227	0.6155	1.9767	7.9567	0.1793	−0.3780
80	−0.7917	0.2679	0.5490	1.8264	7.4758	0.2081	−0.3959
90	−0.8165	0.3128	0.4853	1.6922	7.0674	0.2341	−0.4082
100	−0.8321	0.3572	0.4242	1.5733	6.7282	0.2570	−0.4161
110	−0.8402	0.4008	0.3654	1.4683	6.4532	0.2767	−0.421
120	**−0.8416**	**0.4435**	**0.3083**	**1.3760**	**6.2371**	**0.2931**	**−0.4208**

Table 5. The obtained and precise solution of the problem

N	Obtained solution x_1	x_2	x_3	Precise solution x_1	x_2	x_3
0	-1.0000	0.0000	-0.0001	-1.0000	0.0000	0.0000
10	-0.9896	0.3332	-0.1598	-0.9896	0.3332	-0.1597
20	-0.9584	0.6659	-0.3057	-0.9584	0.6659	-0.3056
30	-0.9065	0.9974	-0.4376	-0.9064	0.9974	-0.4375
40	-0.8339	1.3272	-0.5557	-0.8338	1.3272	-0.5556
50	-0.7409	1.6547	-0.6599	-0.7408	1.6547	-0.6597
60	-0.6276	1.9794	-0.7502	-0.6276	1.9794	-0.7500
70	-0.4944	2.3007	-0.8266	-0.4944	2.3008	-0.8264
80	-0.3415	2.6183	-0.8891	-0.3414	2.6184	-0.8889
90	-0.1693	2.9315	-0.9377	-0.1692	2.9316	-0.9375
100	-0.0219	3.2401	-0.9724	-0.0220	3.2402	-0.9722
110	0.2317	3.5434	-0.9933	0.2318	3.5436	-0.9931
120	0.4596	3.8413	-1.0003	0.4597	3.8415	-1.0000

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