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ON ESTIMATION OF THE BEST APPROXIMATION BY BILINEAR FORMS IN SPACE WITH MIXED NORM

Abstract

In this paper the best approximation of multivariable functions by sums of pair product of a fewer variable functions is considered. The different choices of exact annihilator for classes of bilinear forms establishing upper and lower estimates of the best approximation by bilinear forms in space with mixed norm are used.

Consider the space with mixed norm $L_{pq}(K),\ K = I^l,\ I = [0,1],\ 0 < p, q < \infty$ of the functions $f = f(x,y),\ x = (x^{(1)},...,x^{(d)}),\ y = (x^{(d+1)},...,x^{(l)})$ for which the integral

$$\|f\|_{pq} = \left(\int\limits_{I^{l-d}} \left(\int\limits_{I^d} |f(x,y)|^p \, dx \right)^{q/p} dy \right)^{1/q}$$

exists and is finite.

Denote by $L_{\infty q}, L_{p\infty}$ and $L_{\infty\infty}$ spaces with bounded norms

$$\sup_{x\in I^d} \|f(x,y)\|_{L_q(I^{l-d})},\ \sup_{y\in I^{I-d}} \|f(x,y)\|_{L_q(I^d)} \text{ and } \sup_{(x,y)\in K} |f(x,y)|$$

respectively.

Consider a class of bilinear forms

$$B = B^{pq}_{M-1}(K) = \left\{ \beta \,\middle|\, \beta = \sum_{i=1}^{M-1} \varphi_i(x)\psi_i(y), \varphi_i \in L_p(I^d), \psi_i \in L_p(I^{I-d}) \right\}.$$

Lemma 1. *The set $B^{pq}_{M-1}(K)$ is a subset of the class $L_{pq}(K)$. For proof we need the result (see [1]) that in the case of the space $L_{pq}(K)$ is formulated by the following form.*

Lemma 2. *For any $f, g \in L_{pq}(K),\ 0 < p, q \leq \infty$,*

$$\|f - g\|^{p^*}_{pq} \leq \|f\|^{p^*}_{pq} + \|q\|^{p^*}_{pq},$$

where $p^* = \min\{1, p, q\}$.

It is clear that Lemma 2 is valid also for sum of finite number of functions from L_{pq}. Allowing for this we have

$$\left\| \sum_{i=1}^{M-1} \varphi_i(x)\psi_i(y) \right\|_{L_{pq}(K)} \leq \sum_{i=1}^{M-1} \|\varphi_i(x)\psi_i(y)\|_{L_{pq}(K)}$$

using the definition of the norm $\|\cdot\|_{pq}$ *and the fact that the functions* $\varphi_i(x)$ *depend only on the variables* $\left(x^{(1)},...,x^{(d)}\right)$ *and the functions* $\psi_i(y)$ *depend only the variables* $\left(x^{(d+1)},...,x^{(l)}\right)$ *we continue*

$$=\sum_{i=1}^{M-1}\|\varphi_i(x)\|_{L_p(I^d)}\cdot\|\psi_i(y)\|_{Lq}\left(I^{I-d}\right)<+\infty.$$

Lemma 1 is proved.

We lead the multi-variable functions representability criteria in bilinear forms in the space $L_{pq}(K)$. We compare the operator

$$\overset{M}{\nabla} f=\overset{M}{\nabla}(f,\theta)=\begin{vmatrix} f(x_1,y_1)f(x_1,y_2)...f\left(x_1,y_M\right) \\ f(x_2,y_1)f(x_2,y_2)...f\left(x_2,y_M\right) \\ .... \\ f(x_M,y_1)f(x_M,y_2)...f\left(x_M,y_M\right) \end{vmatrix}$$

to the function f at each point

$$\theta=\theta_{MI}=\{x_1,...,x_M,y_1,...,y_M\}\in I^{IM},\ x_i=\left\{x_1^{(1)},...,x_i^{(d)}\right\},$$

$$y_i=\left\{x_i^{d+1},...,x_i^{(l)}\right\},\ i=\overline{1,M}.$$

Theorem 1. *If* $f\in L_{pq}(K)$ *then*
$f=\sum\limits_{i=1}^{M-1}\varphi_i(x)\psi_i(y)$ *almost for all* $(x,y)\in I^l \iff \overset{M}{\nabla}(f,\theta)=0$ *almost for all* $\theta\in I^{lM}$.

Proof. Let $f\in L_{pq}(K)$ we act by the scheme establishing the exact annihilator for a class of bilinear forms in the space C from [2].

For any sum $\sum\limits_{i=1}^{M-1}\varphi_i(x)\psi_i(y)\in L_{pq}(K)$

$$\overset{M}{\nabla}\sum_{i=1}^{M-1}\varphi_i(x)\psi_i(y)=\begin{vmatrix} \sum\limits_{i=1}^{M-1}\varphi_i(x_1)\psi_i(y_1)...\sum\limits_{i=1}^{M-1}\varphi_i(x_1)\psi_i(y_M) \\ \sum\limits_{i=1}^{M-1}\varphi_i(x_2)\psi_i(y_1)...\sum\limits_{i=1}^{M-1}\varphi_i(x_2)\psi_i(y_M) \\ .... \\ \sum\limits_{i=1}^{M-1}\varphi_i(x_M)\psi_i(y_1)...\sum\limits_{i=1}^{M-1}\varphi_i(x_M)\psi_i(y_M) \end{vmatrix}.$$

Using the property of determinant we express the right hand side of this equality the sum $(M-1)^M$ of M-th order determinants.

Every elements of these determinants will have the form

$$\varphi_i(x_k)\psi_i(y_q),\ i=\overline{1,M-1};\ k,q=\overline{1,M}.$$

We show that each of these determinants is almost everywhere zero. Let L be an arbitrary one of these determinants since there exists $M-1$ addends in $\sum\limits_{i=1}^{M-1}\varphi_i\psi_i$

and the determinant L is of order M, then there exists a least two columns with the same product in L i,e., there exists the index $i_0 \in \{1, ..., M-1\}$ such that the functions $\varphi_{i_0}, \psi_{i_0}$ participate at least in two columns of determinant. Let they be p-th and s columns $(p < s)$. Then the considered determinant will have the form

$$L = \begin{vmatrix} ...\varphi_{i_0}(x_1)\psi_{i_0}(y_p)...\varphi_{i_0}(x_1)\psi_{i_0}(y_s)... \\ ...\varphi_{i_0}(x_2)\psi_{i_0}(y_p)...\varphi_{i_0}(x_2)\psi_{i_0}(y_s)... \\ .... \\ ...\varphi_{i_0}(x_M)\psi_{i_0}(y_p)...\varphi_{i_0}(x_M)\psi_{i_0}(y_s)... \end{vmatrix} =$$

$$= \psi_{i_0}(y_p)\psi_{i_0}(y_s) \begin{vmatrix} ...\varphi_{i_0}(x_1)...\varphi_{i_0}(x_1)... \\ ...\varphi_{i_0}(x_2)...\varphi_{i_0}(x_2)... \\ .... \\ ...\varphi_{i_0}(x_M)...\varphi_{i_0}(x_M)... \end{vmatrix}$$

since one of multipliers have the form of determinant with two same columns.

The necessity is proved.

For proof of sufficiency we shall use the mathematical induction method. Let $M = 2$

$$\overset{2}{\nabla} f = \begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) \\ f(x_2, y_1) & f(x_2, y_2) \end{vmatrix}.$$

We take the point $(a, b) \in I^l, a = (a^{(1)}, ..., a^{(d)}),\ b = (a^{(d+1)}, ..., a^{(l)})$ in which $f(a, b) \neq 0$.

Denote by $x_1 = a,\ y_1 = b,\ x_2 = x,\ y_2 = y$. Then $\overset{2}{\nabla} f = 0 \Longrightarrow f(x, y) = \dfrac{f(x, b) f(a, y)}{f(a, b)}$.

Then almost everywhere $f(x, y)$, i.e. almost everywhere has the form $\varphi(x)\psi(y)$. Let $f = 0$ hold for $M = k-1$ and the function $f(x, y)$ have the form $\sum\limits_{i=1}^{k-2} \varphi_i(x)\psi_i(y)$.

Then there exist the parts $(a_i, b_j) \in I^l, i, j = \overline{1, k-1}$ such that

$$c \overset{df}{=} \begin{vmatrix} f(a_1, b_1)...f(a_1, b_{k-1}) \\ .... \\ f(a_{k-1}, b_1)...f(a_{k-1}, b_{k-1}) \end{vmatrix} \neq 0$$

we have

$$f = \begin{vmatrix} f(x_1, y_1)...f(x_1, y_k) \\ .... \\ f(x_k, y_1)...f(x_k, y_k) \end{vmatrix} = 0.$$

Then denoting $x_i = a_i,\ y_i = b_i\ \ i = \overline{1, k-1};\ x_k = x, y_k = y,$

$$c_{ij} = \begin{vmatrix} f(a_1, b_1)...f(a_1, b_{j-1})\ f(a_1, b_{j+1})...f(a_1, b_{k-1}) \\ .... \\ f(a_{i-1}, b_1)...f(a_{i-1}, b_{j-1})\ f(a_{i-1}, b_{j+1})...f(a_{i-1}, b_{k-1}) \\ f(a_{i+1}, b_1)...f(a_{i+1}, b_{j-1}) f(a_{i+1}, b_{j+1})...f(a_{i+1}, b_{k-1}) \\ .... \\ f(a_{k-1}, b_1)...f(a_{k-1}, b_{j-1})\ f(a_{k-1}, b_{j+1})...f(a_{k-1}, b_{k-1}) \end{vmatrix}$$

we obtain $f(x,y)\cdot c=$

$$=\sum_{j=1}^{M-1}(-1)^{k+j-1}f(x,b_j)\left|\begin{array}{c} f(a_1,b_1)...f(a_1,b_{j-1})\ f(a_1,b_{j+1})...f(a_1,y)\\ f(a_2,b_1)...f(a_2,b_{j-1})\ f(a_2,b_{j+1})...f(a_2,y)\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ f(a_{k-1},b_1)...f(a_{k-1},b_{j-1})\ f(a_{k-1},b_{j+1})...f(a_{k-1},y)\end{array}\right|.$$

Opening the determinant by elements of the last column we continue the equations

$$=\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(-1)^{k-2+i+j}f(x,b_j)f(a_i,y)c_{ij},$$

hence we obtain that almost everywhere

$$f(x,y)=\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(-1)^{i+j}\frac{c_{ij}}{c}f(x,b_j)f(a_i,y)\overset{df}{=}$$

$$=\sum_{i=1}^{k-1}(-1)^{i}f(a_i,y)\varphi_i(x)\overset{df}{=}\sum_{i=1}^{k-1}\varphi_i(x)\psi_i(y).$$

Theorem 1 is proved.

M-B.A Babayev [2] gived the following notion of exact annihilator of set of functions.

Consider the metric space X and the set $G\subset R^n$.

Definition. *The family $\{\nabla_\theta\}$; $\theta\in G$ of the continuous operators $\nabla_\theta:\ X\to X$ is called exact annihilator of the set $H\subset X$, if*

$$f\in H\iff\nabla_\theta f=0\quad\forall\theta\in G.$$

According to this definition it is clear that the exact annihilator $\left\{\overset{M}{\nabla}(.,\theta)\right\}_{\theta\in I^{lM}}$ of the class of bilinear $B=\left\{\sum\limits_{i=1}^{M-1}\varphi_i(x)\psi_i(y)\right\}$ is constructed in theorem 1.

The exact annihilator of the class B is invariant with respect to a numerical multiplier independent of parameter $\theta\in I^{lM}$.

Therefore the families $\left\{\overset{M}{\nabla}_*(\cdot,\theta)\right\}_{\theta\in I^{lM}}$ and $\{\overset{M}{\nabla}_0(\cdot,\theta)\}_{\theta\in I^{lM}}$ where

$$\overset{M}{\nabla}_*(f,\theta)=\frac{\overset{M}{\nabla}(f,\theta)}{\left\|\overset{M-1}{\nabla}(f,\theta)\right\|_{L_{pq}(I^{l(M-1)})}},\ \ \overset{M}{\nabla}_0(f,\theta)=\frac{\overset{M}{\nabla}(f,\theta)}{\|f\|^{M-1}_{L_{pq}(I^l)}}$$

are also exact annihilator of the class B.

Theorem 2. *The exact annihilator $\overset{M}{\nabla}_*$ and $\overset{M}{\nabla}_0$ allow to establish lower bound of the best approximation of the function $f\in L_{pq}(K), 0<p,q\le\infty$ by the set of bilinear forms $B^{pq}_{M-1}(K)$ by the following form*

$$A_{Mpq}(f)b_{M_{pq}}\left\|\overset{M}{\nabla}_*f\right\|_{L_{pq}(I^{lM})}\le E[f,B]_{L_{pq}(K)}\qquad(1)$$

and

$$b_{Mpq}\left\|\overset{M}{\nabla}_0 f\right\|_{L_p(I^{lM})}\leq E[f,B]_{L_{pq}(K)},\tag{2}$$

where

$$A_{Mpq}(f)=\frac{\left\|\overset{M-1}{\nabla}f\right\|_{L_{pq}I^{l(M-1)}}}{\|f\|^{M-1}_{L_{pq}(K)}},\ b_{Mpq}=\left(\frac{2^{Mp^*}-1}{2^{p^*}-1}\cdot M!\right)^{-1/p^*},$$

$$p^*=\min(1,p,q).$$

Here the norm $\|\cdot\|_{L_{pq}(I^{lM})}$ *is taken on all* $x_i^{(j)}$, $y_i^{(k)}$, $i=\overline{1,M}$, $j=\overline{1,d}$; $k=\overline{d+1,l}$.

Proof. Consider the determinant

$$\Gamma_k(\beta,f)\overset{df}{=}\begin{vmatrix}\beta(x_1,y_1)...\beta(x_1,y_k)f(x_1,y_{k+1})...f(x_1,y_M)\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \beta(x_M,y_1)...\beta(x_M,y_k)f(x_M,y_{k+1})...f(x_M,y_M)\end{vmatrix},$$

where $\beta=\sum\limits_{k=1}^{M-1}\varphi_k(x)\psi_k(y)$ and denote by $A_{\Gamma_k}(x_j,y_k)$ algebraic complement of the element $\Gamma_k(\beta,f)$ of determinant being on intersection of the j-the row with the k-th column. Since $\left\{\overset{M}{\nabla}_*(f,\theta)\right\}$ is annihilator of the class B then for all

$$\theta=(x_1,...,x_M,y_1,...,y_M)\in I^{lM},$$

$$\overset{M}{\nabla}(f,\theta)=\overset{M}{\nabla}(f,\theta)-\sum_{k=1}^{M}[\Gamma_k(\beta,f)-\Gamma_k(\beta,f)]-\overset{M}{\nabla}(\beta,\theta)=$$

$$=\sum_{k=1}^{M}\begin{vmatrix}\beta(x_1,y_1)...\beta(x_1,y_{k-1})\ [f-\beta]\ (x_1,y_k)\ f(x_1,y_{k+1})...f(x_1,y_M)\\ \beta(x_2,y_1)...\beta(x_2,y_{k-1})\ [f-\beta]\ (x_2,y_k)\ f(x_2,y_{k+1})...f(x_2,y_M)\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \beta(x_M,y_1)...\beta(x_M,y_{k-1})[f-\beta](x_M,y_k)\ f(x_M,y_{k+1})...f(x_M,y_M)\end{vmatrix}.$$

Opening the determinant by the k-th column, we have

$$\left|\overset{M}{\nabla}(f,\theta)\right|\leq\sum_{k=1}^{M}\sum_{j=1}^{M}|[f-\beta]\,(x_j,y_k)|\,|A_{\Gamma_k}(x_j,y_k)|\,.$$

Denote by $p^*=\min(1,p,q)$. Integrating the last inequality by cube I^{lM} and applying lemma 2 we obtain

$$\left\|\overset{M}{\nabla}(f,\theta)\right\|^{p^*}_{L_{pq}(I^{lM})}\sum_{k=1}^{M}\sum_{j=1}^{M}\|[f-\beta]\,(x_j,y_k)\,A_{\Gamma_k}(x_j,y_k)\|^{p^*}_{L_{pq}(I^{lM})}\,.\tag{3}$$

We need one more auxiliary supposition. Let $t_i=(x_i^{(1)},...,x_i^{(l)})\in I^l$

$$\bar{t}_i=(t_1,...,t_{i-1},t_{i+1},...,t_M)\in I^{l(M-1)}.$$

Lemma 3. *For the functions $g(t)\in L_{pq}(I^l),\gamma(\bar t)\in L_{pq}(I^{l(M-1)})$ the equality*

$$\|g(t)\gamma(\bar t)\|_{L_{pq}(I^{lM})}=\|g(t)\|_{L_{pq}(I^l)}\,\|\gamma(\bar t)\|_{L_{pq}(I^{l(M-1)})}\,.$$

Is valid.

For proof lemma 3 it is sufficient to investigate the case $M=2$, $i=k=1$, $t=(x_1,y_1)$, $\bar t=(x_2,y_2)$

$$\|g(x_1,y_1)\gamma(x_2,y_2)\|_{L_{pq}(I^{2l})}=$$

$$=\left(\int\limits_{I^{l-d}}\int\limits_{I^{l-d}}\left(\int\limits_{I^d}\int\limits_{I^d}|g(x_1,y_1)\gamma(x_2,y_2)|^{p_1}\,dx_1dx_2\right)^{q_2/p_1}dy_1dy_2\right)^{1/p_2}=$$

$$=\left(\int\limits_{I^{l-d}}\int\limits_{I^{l-d}}\left(\int\limits_{I^d}|\gamma(x_2,y_2)|^{p_1}\left[\int\limits_{I^d}|g(x_1,y_1)|^{p_1}\,dx_1\right]dx_2\right)^{q_2/p_1}dy_1dy_2\right)^{1/q}=$$

$$=\left(\int\limits_{I^{l-d}}\int\limits_{I^{l-d}}\left[\int\limits_{I^d}|g(x_1,y_1)|^{q_2/p_1}\left(\int\limits_{I^d}|\gamma(x_2,y_2)|^{p_2}\,dx\right)^{q_2/p_1}dy_1dy_2\right)^{1/q}=$$

$$=\left(\int\limits_{I^{l-d}}\left[\int\limits_{I^d}|g(x_1,y_1)|^{p_1}\,dx_1\right]^{q_2/p_1}dy_1\int\limits_{I^{l-d}}\left(\int\limits_{I^d}|\gamma(x_2,y_2)|^{p_1}\,dx_2\right)^{q/p_1}dy_2\right)^{1/q}=$$

$$=\left(\int\limits_{I^{l-d}}\left[\int\limits_{I^d}|g(x_1,y_1|^{p_1}\,dx_1\right]^{q_2/p_1}dy_1\right)^{1/q}\left(\int\limits_{I^{l-d}}\left(\int\limits_{I^d}|\gamma(x_2,y_2|^{p}\,dx_2\right)^{q/p}dy_2\right)^{1/q}=$$

$$=\|g\|_{L_{p_1q_2}(I^l)}\,\|\gamma\|_{L_{p_1}(I^l)}\,.$$

Lemma 3 is proved.

Granting that $A_{\Gamma_k}(x_j,y_k)$ does not depend on x_j and y_k and applying lemma 3 we confine estimate (3)

$$\left\|\overset{M}{\nabla}(f,\theta)\right\|^{p^*}_{L_{pq}(I^{lM})}\le\sum_{k,j=1}^{M}\|[f-\beta]\,(x_j,y_k)\|^{p^*}_{L_{pq}(I^l)}\,\|A_{\Gamma_k}(x_j,y_k)\|^{p^*}_{L_{pq}(I^{l(M-1)})}\,.\qquad(4)$$

The determinant $A_{\Gamma_k}(x_j,y_k)$ represents the algebraic sum of elements of the norm

$$\eta\overset{df}{=}\beta(x_{i_1},y_{j_1})...\beta(x_{i_{k-1}},y_{j_{k-1}})f(x_{i_{k+1}},y_{j_{k+1}})...f(x_{i_M},y_{j,M}),$$

where

$$i_r\ne i_s \text{ and } j_r\ne j_s \text{ at } r\ne s;\ i_r,\ j_s=1,...,k-1,\ k+1,...,M.$$

According to lemma 3 the norm of each element satisfies the equations

$$\|\eta\|^{p^*}_{L_{pq}(I^{l(M-1)})}=\|\beta\|^{(k-1)p^*}_{L_{pq}(k)}\cdot\|f\|^{(M-k)p^*}_{L_{pq}(k)}\,.$$

Therefore

$$\|A_{\Gamma_k}(x_j,y_k)\|^{p^*}_{L_p(I^{l(M-1)})} \le (M-1)!\,\|\beta\|^{(k-1)p^*}_{L_p(k)}\;\|f\|^{(M-k)p^*}_{L_p(k)}.$$

Allowing for this in (4) we obtain

$$\left\|\overset{M}{\nabla}(f,\theta)\right\|^{p^*}_{L_{pq}(I^{lM})} \le \sum_{k,j=1}^{M}(M-1)!\,\|f-\beta\|^{p^*}_{L_p(k)}\,\|\beta\|^{(k-1)p^*}_{L_p(k)}\;\|f\|^{(M-k)p^*}_{L_p(k)} =$$

$$= M!\,\|f-\beta\|^{p^*}_{L_p(k)}\sum_{k=1}^{M}\|\beta\|^{(k-1)p^*}_{L_p(k)}\cdot\|f\|^{(M-k)p^*}_{L_p(k)}.$$

In approximation of the function f by the set B^{pq} it is sufficient to be restricted by the subset

$$B_0^p = \{\beta\in B^p\left|\|\beta\|_{L_p(k)}\right| \le 2\,\|f\|_{L_p(k)}\}, \tag{5}$$

since

$$E(f,B_0)_{L_p(k)} = E(f,B)_{L_p(k)}.$$

Let $\beta\in B_0^p$. Then using estimate (5) and applying the formula of sum of elements of geometric progression we obtain

$$\left\|\overset{M}{\nabla}(f,\theta)\right\|^{p^*}_{L_p(I^{lM})} \le M!\,\|f-\beta\|^{p^*}_{L_p(k)}\sum_{k=1}^{M}2^{(k-1)p^*}\,\|f\|^{(k-1)p^*}_{L_p(k)}\;\|f\|^{(M-k)p^*}_{L_p(k)} =$$

$$= M!\frac{2^{Mp^*-1}}{2^{p^*-1}}\,\|f-\beta\|^{p^*}\,\|f\|^{(M-1)p^*}_{L_p(k)} \Longrightarrow$$

$$\Longrightarrow \left[M!\frac{2^{Mp^*-1}}{2^{p^*-1}}\right]^{-1}\|f\|^{-(M-1)p^*}_{L_p(k)}\left\|\overset{M}{\nabla}(f,\theta)\right\|^{p^*}_{L_p(K)} \le \|f-\beta\|^{p^*}_{L_p(k)} \Longrightarrow$$

$$\Longrightarrow b_{Mp}\cdot A_{Mp}(f)\left\|\overset{M}{\nabla}_* f\right\|_{L_p(I^{lM})} \le E(f,B)_{L_p(k)}.$$

Estimate (1) is established. For proof of estimate (2) the analogous reasoning are used. Theorem 2 is proved.

We now lead the upper estimate of the best approximation.

Theorem 3. *With the help of the annihilators $\overset{M}{\nabla}_*$ and $\overset{M}{\nabla}_0$ we can establish the following estimates of the best approximation of the functions $f\in L_{pq}(k)$, $0<p,\ q\le\infty$ by the set of the bilinear forms $B^{pq}_{M-1}(k)$*

$$E(f,B)_{L_{pq}(K)} \le \left\|\overset{M}{\nabla}_* f\right\|_{L_{pq}(I^{lM})} \tag{6}$$

and

$$E(f,B)_{L_{pq}(k)} \le A^{-1}_{Mpq}\left\|\overset{M}{\nabla}_0\right\|_{L_p(I^{lM}),}$$

where the coefficients A^{-1}_{Mpq} was determined earlier.

Proof. Let $f \not\equiv \sum\limits_{k=1}^{M} \varphi_i(x)\psi_i(y)$. Then as it follows from the proof of theorem 1, there exists a subseting the cube I^{lM} of positive measure (in the sense of measure in the space $R^{l(M-1)}$) in which

$$\overset{M-1}{\nabla} f = \begin{vmatrix} f(x_2,y_2)...f(x_2,y_M) \\ .... \\ f(x_M,y_2)...f(x_M,y_M) \end{vmatrix} \neq 0.$$

Allowing for this and opening the determinant $\overset{M}{\nabla} f$ by the first row we shall have

$$\left\| \overset{M}{\nabla}_* \right\|_{L_p(K)} = \left\| \overset{M-1}{\nabla} f \right\|^{-1}_{L_p(I^{l(M-1)})} \left\| f(x_1,y_1) \overset{M-1}{\nabla} f + \sum_{k=2}^{M} (-1) f(x_1,y_k) \times \right.$$

$$\left. \times \begin{vmatrix} f(x_1,y_1)...f(x_2,y_{k-1}) f(x_2,y_{k+1})...f(x_2,y_M) \\ .... \\ f(x_M,y_1)...f(x_M,y_{k-1}) f(x_M,y_{k+1})...f(x_M,y_M) \end{vmatrix} \right\|_{L_{pq}(k)}.$$

Assuming here $x_1, = x,\ y_1 = y$, we obtain

$$\left\| \overset{M}{\nabla}_* \right\|_{L_{pq}(k)} = \left\| \overset{M-1}{\nabla}_* f \right\|^{-1}_{L_{pq}(I^{l(M-1)})} \left\| f(x,y) \overset{M-1}{\nabla}_* f - \beta^0(x,y) \right\|_{L_{pq}(k)} =$$

$$= \left\| \overset{M-1}{\nabla}_* f \right\|^{-1}_{L_p(I^{l(M-1)})} \left| \overset{M-1}{\nabla}_* f \right| \left\| f(x,y) - \beta^{00}(x,y) \right\|_{L_p(k)},$$

where

$$\beta^0(x,y) = \sum_{i=1}^{M} (-1)^{i-1} \varphi_i^0(x) \psi_i^0(y)$$

is a specific bilinear function from B^p and $\beta^{00}(x,y) \left(\left| \overset{M-1}{\nabla} f \right| \right)^{-1} \beta^0(x,y)$. Then

$$\left\| \overset{M}{\nabla}_* f \right\|_{L_{pq}(k)} \geq \left\| \overset{M-1}{\nabla} f \right\|^{-1}_{L_{pq}(I^{l(M-1)})} \left| \overset{M-1}{\nabla} f \right| E(f,B)_{L_{pq}(k)}.$$

It remains to take the norm $\|\cdot\|_{L_{pq}(I^{l(M-1)})}$ of the both sides of this inequality with respect to $x_2,...,x_M,y_2,...,y_M$ and to apply lemma 3 which reduces to upper estimate of the best approximation (6)

$$E(f,B)_{L_{pq}(k)} \leq \left\| \overset{M}{\nabla}_* f \right\|_{L_{pq}(I^{lM})}.$$

The analogous reasons are used. In the proof of the second part of theorem 3.

Theorem 3 is proved.

Joining the estimations obtained in theorems 2 on 3 we obtain two sided estimates of the best approximation of multivariable functions by bilinear forms in the space $L_{pq}(K)$ with the help of exact annihilators $\overset{M}{\nabla}_*$and $\overset{M}{\nabla}_0$. It should be noted that the coefficient $A_{Mpq}(f)$ participating in the lower estimate in theorem 2 and in upper estimate in theorem 3 doesn't allow to obtain exact order of the best approximation. However in some cases we succeeded to find the order of the best approximation by bilinear forms in the space L_{pq} by exact annihilation of classes of approximations of functions, namely to establish estimate of the best approximation with coefficients independent of approximated function.

At $M = 2$ the class B turns into the set of products of functions each of which depends on one group variables $B_1 = \{\beta\,|\beta = \varphi\,(x)\psi(x)\}$.

From theorem 2 and 3 we obtain.

Corollary 1.

$$f \in L_{pq}(k),\ 0 < p, q \le \infty \Longrightarrow$$

$$[2(2^{p^*}+1)]^{-1/p^*}\left\|\overset{2}{\nabla}_+ f\right\|_{L_{pq}(I^{2l})} \le E(f, B_1)_{L_{pq}(K)} \le \left\|\overset{2}{\nabla}_+ f\right\|_{L_{pq}(I^{2l})}.$$

Here

$$\overset{2}{\nabla}_+ f = \overset{2}{\nabla}_* f = \overset{2}{\nabla}_0 f = \frac{\overset{2}{\nabla} f}{\|f\|_{L_p(K)}},$$

at $p, q \ge 1$ the coefficient of lower estimate turns into $\frac{1}{6}$.

Consider the function $f = f(x, y) \in L_p(I^2),\ 0 < p, q \le \infty$..

Denote by

$$[a, b] = \begin{vmatrix} a(x, y) & b(x, y_1) \\ a(x_1, y) & b(x_1, y) \end{vmatrix}$$

the 2-nd order determinant

$$\|ab\|_{L_p(I^4)} \overset{df}{=} \|[a, b]\|_{L_p(I^4)},$$

where the norm is taken with respect to variables $(x, y, x_1, y_1) \in I^4$.

Consider the class

$$B^* = B^*_{B,c} = \Big\{\ \beta = \varphi_1(x)\psi_1(y) + \varphi_2(x)\psi_2(y)\}|\ \|\beta, f\|_{L_p(I^4)} \le c\,\|ff\|_{L_{pq}(I^4)},$$

$$\|\beta\beta\|_{L_p(I^4)} \le c\,\|ff\|_{L_{pq}(I^4)}\Big\},$$

where c is a constants depending an f and the set approximation $E(f, B^*)_{L_p(I^2)}$.

Theorem 4. $f \in L_p(I^2) \Longrightarrow$

$$[3(1+2c)]^{-1/p^*}\left\|\overset{3}{\nabla}_* f\right\|_{L_{pq}(I^6)} \le E(f, B^*)_{L_{pq}(I^2)} \le \left\|\overset{3}{\nabla}_* f\right\|_{L_{pq}(I^6)},$$

where $p^* = \min(1, p, q),\ c \ge 2$.

The proof of theorem 4 is led by the application of methods of proofs of theorems 2 and 3. In the case $p = q$ *from results of this paper we obtain the results in usual norm* L_p*, earlier cited in [2].*

References

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