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ON SMOOTHNESS OF GENERALIZED SOLUTION OF ELLIPTIC EQUATIONS WITH NON-LINEAR PART

Abstract

In the present work it is studied the differential properties of generalized solution of elliptic equations of $2r$ *order containing non-linear term, depending on differential properties of coefficients of the equations. In particular it is investigated the case, when the generalized solution can be understood in the ordinary sense, too.*

Let $r \geq 1$ be a natural, $\Omega \subset R_n$ be a bounded domain of the points $x = (x_1, ..., x_n)$ with $(n-1)$ -dimensional boundary $\Gamma \in C^{(r)}$, $W_2^r(\Omega)$ be Sobolev space of functions $f(x)$ with the norm

$$\|f\|_{W_2^r(\Omega)} = \|f\|_{L_2(\Omega)} + \left(\sum_{|k|=r} \left\| f^{(k)} \right\|^2_{L_2(\Omega)} \right)^{\frac{1}{2}} \equiv \|f\|_2 + D^{\frac{1}{2}}(f) < \infty, \tag{1}$$

where
$f^{(k)} = f^{(k)}(x) = \dfrac{\partial^{|k|} f(x_1, ..., x_n)}{\partial x_1^{k_1} ... \partial x_n^{k_n}}$, $k = (k_1, ..., k_n)$, $k_j \geq 0$ integer,$|k| = k_1 + ... + k_n$.

It's known, that the function $f(x) \in W_2^r(\Omega)$ has on Γ the following traces ([1;2]):

$$\left. \frac{\partial^\lambda f(x)}{\partial N^\lambda} \right|_\Gamma = \varphi_\lambda \in L_2(\Gamma) \quad (\lambda = 0, 1, ..., r-1), \tag{2}$$

for which the inequality is valid:

$$\|\varphi_\lambda\|_{L_2(\Gamma)} \leq C \|f\|_{W_2^r(\Omega)} (\lambda = 0, 1, ..., r-1), \tag{3}$$

where N is a normal to boundary Γ (at the point x).

Here and further, generally speaking, different positive constants are denoted by C.

Sometimes we'll write $A << B$ instead of $A \leq CB$, where the positive constant C is independent of B.

Let $\overset{\circ}{W}{}_2^r(\Omega)$ be a set of all functions $f(x) \in W_2^r(\Omega)$, for which

$$\left. \frac{\partial^\lambda f(x)}{\partial N^\lambda} \right|_\Gamma = 0 \quad (\lambda = 0, 1, ..., r-1). \tag{4}$$

We give function $\Phi(x) \in W_2^r(\Omega)$ and denote it by W_ϕ a set of functions $f(x) \in W_2^r(\Omega)$ for which $f(x) - \Phi(x) \in \overset{\circ}{W}{}_2^r(\Omega)$.

For functions $f(x) \in W_2^r(\Omega)$ Poincare inequality takes place [1;2]:

$$\|f\|_{L_2(\Omega)} \leq C\left(\sum_{\lambda=0}^{r-1}\|\varphi_\lambda\|_{L_2(\Gamma)} + \sqrt{D(f)}\right). \tag{5}$$

Let for each pair of (k,l) integer non-negative n-demensional vectors with length $|k| \leq r$, $|l| \leq r$ are associated measurable bounded functions $a_{kl}(x) = a_{lk}(x)$ in domain Ω such, that

$$\sum_{\substack{|k|\leq r\\|l|\leq r}} a_{kl}(x)\,\xi_k\xi_l \geq \chi_0 \sum_{|k|=r}\xi_k^2, \tag{6}$$

is satisfied, where the constant χ_0 independent of the points $x \in \Omega$ and of the number ξ_k, ξ_l corresponding to vectors k and l.

Let $F(x) \in L_2(\Omega)$ and $a(x)$ be a measurable function in domain Ω such, that

$$0 < a_0 \leq a(x) \leq a_1, \ (x \in \Omega) \tag{7}$$

where a_0 and a_1 are constants.

The function $u(x) \in W_\Phi$ for which the identity

$$\int_\Omega \sum_{\substack{|k|\leq r\\|l|\leq r}} a_{kl}(x)\,u^{(k)}(x)v^{(l)}(x)dx + \int_\Omega a(x)\,|u(x)|^{p-1}\,sign\ u(x)v(x)dx -$$

$$-\int_\Omega F(x)v(x)dx = 0 \qquad \left(\forall v(x) \in \overset{\circ}{W}{}_2{}^r(\Omega)\right) \tag{8}$$

is satisfied we'll call generalized solution of the boundary value problem

$$\left.\begin{array}{ll} Lu = F(x) & (x \in \Omega), \\ u \in W_\Phi, & \end{array}\right\} \tag{9}$$

where

$$Lu \equiv \sum_{\substack{|k|\leq r\\|l|\leq r}} (-1)^{|k|}\left(a_{kl}(x)u^{(l)}(x)\right)^{(k)} + a(x)\,|u(x)|^{p-2}\,u(x). \qquad (1 < p \leq 2).$$

The existence of the function $u(x) \in W_\Phi$, for which the identity (8) is valid, is proved, for example, as in corresponding theorems of the works [3;4;7].

Theorem 1. *For function $u(x) \in W_\Phi$ for which the identity (8) is satisfied the following inequality is valid*

$$\|u\|_{W_2^r(\Omega)} \leq C\left(\|\Phi\|_{W_2^r(\Omega)} + \|\Phi\|_{Lp(\Omega)}^{\frac{p}{2}} + \|F\|_{L_2(\Omega)}\right) \equiv CK. \tag{10}$$

Proof. Assuming $v = u - \Phi$ in (8) we obtain:

$$\int_\Omega \sum_{\substack{|k|\le r\\|l|\le r}} a_{kl}(x)u^{(k)}(x)\,(u-\Phi)^{(l)}\,dx + \int_\Omega a(x)\,|u(x)|^{p-1}\,sign\ u(x)[u(x)-\Phi(x)]dx-$$

$$-\int_\Omega F(x)[u(x)-\Phi(x)]dx = 0.$$

Hence, by virtue of (6)-(7) we have

$$\chi_0\int_\Omega \sum_{|k|\le r}[(u(x)-\Phi(x))^{(k)}]^2dx + a_0\int_\Omega |u(x)|^p\,dx \le$$

$$\le \int_\Omega \sum_{\substack{|k|\le r\\|l|\le r}} a_{kl}(x)\Phi^{(k)}(x)\left[\Phi^{(l)}(x)-u^{(l)}(x)\right]dx+$$

$$+\int_\Omega a(x)\,|u(x)|^{p-1}\,sign\ u(x)\Phi(x)dx + \int_\Omega F(x)\,(u(x)-\Phi(x))\,dx \equiv$$

$$\equiv J_1 + J_2 + J_3. \tag{11}$$

Let's estimate integrals J_j $(j=1,2,3)$ from above. By virtue of the boundedness of the function $a_{kl}(x)$ and Hōlder inequality we obtain:

$$|J_1| \le C\sum_{\substack{|k|\le r\\|l|\le r}} \int_\Omega \left|\Phi^{(k)}\right|\left|(\Phi-u)^{(l)}\right|dx \le C\,\|\Phi\|_{W_2^r(\Omega)}\,\|u-\Phi\|_{W_2^r(\Omega)}\,.$$

Allowing for (7), using the Hōlder inequality and then the Young inequality, we have

$$|J_2| \le a_1\int_\Omega |u(x)|^{p-1}\,|\Phi(x)|\,dx \le a_1\,\|u\|^{p-1}_{L_p(\Omega)}\,\|\Phi\|_{L_p(\Omega)} \le$$

$$\le \frac{a_1^q}{N^q}\,\|u\|^p_{L_p(\Omega)} + N^p\,\|\Phi\|^p_{L_p(\Omega)}\,,$$

the number N we choose such, that $a_0 - \left(\frac{a_1}{N}\right)^q = \lambda_0 > 0$. Further, we estimate integral J_3:

$$|J_3| \le \|F\|_{L_2(\Omega)}\,\|u-\Phi\|_{L_2(\Omega)}\,.$$

Taking into account the estimates for J_1, J_2, J_3 from (11) we obtain

$$\chi_0\int_\Omega \sum_{|k|=r}\left|(u-\Phi)^{(k)}\right|^2 dx + \lambda_0\,\|u\|^p_{L_p(\Omega)} \le$$

$$\le C\left[\|\Phi\|_{W_2^r(\Omega)}\,\|u-\Phi\|_{W_2^r(\Omega)} + N^p\,\|\Phi\|^p_{L_p(\Omega)} +\right.$$

$$+\|F\|_{L_2(\Omega)}\|u-\Phi\|_{L_2(\Omega)}\Big]\le C\sqrt{D(u-\Phi)}\left(\|\Phi\|_{W_2^r(\Omega)}+\|F\|_2\right)+C\|\Phi\|_p^p.$$

Consequently

$$D^{1/2}(u-\Phi)\le C\left(\|\Phi\|_{W_2^r(\Omega)}+\|\Phi\|_p^{\frac{p}{2}}+\|F\|_2\right),$$

$$\|u\|_p^{\frac{p}{2}}\le C\left(\|\Phi\|_{W_2^r(\Omega)}+\|\Phi\|_p^{\frac{p}{2}}+\|F_2\|\right),$$

where the positive constant is indepent of Φ, F.

By virtue of (5) we have $\|u-\Phi\|_{W_2^r(\Omega)} << D^{1/2}(u-\Phi)$, since $u-\Phi\in\mathring{W}_2^r(\Omega)$, so

$$\|u-\Phi\|_{W_2^r(\Omega)}\le C\left(\|\Phi\|_{W_2^r(\Omega)}+\|\Phi\|_p^{\frac{p}{2}}+\|F\|_2\right).$$

Thus

$$\|u\|_{W_2^r(\Omega)}\le C\left(\|\Phi\|_{W_2^r(\Omega)}+\|\Phi\|_p^{\frac{p}{2}}+\|F\|_2\right). \tag{12}$$

Theorem 2. *Let coefficients $a_{kl}(x)$ of equation (8) have bounded partial derivates of order $m(m\le r)$ inclusively in domain Ω.*

$$\left|a_{kl}^{(s)}(x)\right|\le M,\ (s=(s_1,...,s_n),|s|\le m). \tag{13}$$

Then, at conditions (6) and $F(x)\in L_2(\Omega)$ generalized solution of boundary problem (9) belongs to weighted class $W^{r+m}_{2,\frac{2-p}{2p}n}(\Omega)$ $(1<p\le 2)$, besides, for it the inequality is valid (see (10)).

$$\|u\|_{W^{r+m}_{2,\frac{2-p}{2p}n}}(\Omega)\equiv\|u\|_{L_2(\Omega)}+\left(\sum_{|k|=r+m}\left\|\rho^{\frac{2-p}{2p}n}u^{(k)}\right\|^2_{L_2(\Omega)}\right)^{\frac12}<<$$

$$<<\max\left(K;K^{p-1}\right). \tag{14}$$

Proof. We fix arbitrary point $x^0=\left(x_1^0,...,x_n^0\right)\in\Omega$ and let σ,σ^1 be two balls with centers at the point x^0 whose radii respectively equal to $\frac{1}{4}\rho(x^0)$ and $\frac{1}{2}\rho(x^0)$ where $\rho(x)$ is distance from the point $x\in\Omega$ to the boundary Γ.

We give entire non negative vector $s=(s_1,...,s_n)\,(s_j\ge 0)$, denote by $\Delta_h^s f(x)$ $s-$ mixed difference of function $f(x)$ at the point x with interval h:

$$\Delta^s f(x)=\Delta_h^s f(x)=\Delta_{he_1}^{s_1}\Delta_{he_2}^{s_2}...\Delta_{he_n}^{s_n}f(x_1,x_2,...x_n),$$

where $\Delta_{he_i}^{s_i}$ is an operator of s_i-fold difference by variable x_i.

Let $\upsilon(x)\in\mathring{W}_2^r(\sigma')$. Since function $\upsilon(x)$ has support in σ', then at sufficiently small h from (8) can obtain the equality (see for example [4;6]):

$$\int_{\sigma'}\sum_{\substack{|k|\le r\\|l|\le r}}\Delta^s\left(a_{kl}(x)u^{(k)}(x)\right)\upsilon^{(l)}(x)dx+\int_{\sigma'}\Delta^s\left(a(x)\left|u(x)\right|^{p-2}u(x)\right)\times$$

$$\times\upsilon(x)dx=\int\limits_{\sigma'}\Delta^s F(x)\upsilon(x)dx. \tag{15}$$

Using the difference analogue of Leibnitz formula

$$\Delta^s\left(f(x)\varphi(x)\right)=\sum_{0\le\lambda\le s}C_s^\lambda\Delta^{s-\lambda}f(x)\Delta^\lambda\varphi[x+h\,(s-\lambda)]$$

we write the equality (15) in the following form:

$$\int\limits_{\sigma'}\sum_{|k|,|l|\le r}\sum_{0\le\lambda\le s}C_s^\lambda\frac{\Delta^\lambda a_{kl}[x+h(s-\lambda)]}{h^{|\lambda|}}\frac{\Delta^{s-\lambda}u^{(k)}(x)}{h^{|s-\lambda|}}\times$$

$$\times\upsilon^{(l)}(x)dx+\int\limits_{\sigma'}\sum_{0\le\lambda\le s}C_s^\lambda\frac{\Delta^\lambda a[x+h(s-\lambda)]}{h^{|\lambda|}}\frac{\Delta^{s-\lambda}[|u(x)|^{p-2}\,u(x)]}{h^{|s-\lambda|}}\upsilon(x)dx=$$

$$=\int\limits_{\sigma'}\frac{\Delta^\lambda F(x)}{h^{|s|}}\upsilon(x)dx. \tag{16}$$

We can construct the function $\eta(x)\in C^{(r+m)}(R_n)$, finitary in σ' and satisfying the following conditions (for example, see [5]):

1) $\eta(x)\equiv 1$ on σ,

2) $0\le\eta(x)\le 1,\ x\in\sigma'$,

3) $\eta(x)\equiv 0,\ x\notin\sigma'$;

4) $\left|\eta^{(\tau)}(x)\right|^2\le C_0\eta(x), |\tau|\le r+m$,

where the positive constant C_0 is indepent of x.

Let $\Lambda_\eta=\sup p\quad\eta(x)\quad\left(\Lambda_\eta\subset\bar\Lambda_\eta\subset\sigma'\right)$.

Assume

$$\upsilon(x)=\eta(x)\frac{\Delta^s u(x)}{h^{|s|}}.$$

It's obvious, that $\upsilon(x)\in\mathring{W}_2^r(\sigma')$. Put this function in equality (16). Then we have

$$\int\limits_{\sigma'}\left\{\sum_{\substack{|k|\le r\\|l|\le r}}\sum_{0\le\lambda\le s}C_s^\lambda\frac{\Delta^s a_{kl}[x+h(s-h)]}{h^{|\lambda|}}\frac{\Delta^{s-\lambda}u^{(k)}(x)}{h^{|s-\lambda|}}\times\right.$$

$$\times\sum_{0\le\tau\le l}C_l^\tau\eta^{(\tau)}(x)\frac{\Delta^s u^{(l-\tau)}(x)}{h^{|s|}}+\frac{\Delta^s(a(x)\,|u(x)|^{p-2}\,u(x))}{h^{|s|}}\frac{\Delta^s u(x)}{h^{|s|}}\eta(x)-$$

$$\left.-\frac{\Delta^s F(x)}{h^{|s|}}\frac{\Delta^s u(x)}{h^{|s|}}\eta(x)\right\}dx=0. \tag{17}$$

Leaving all terms of the considered multiple sum corresponding to $|k| = |l| = r$ in the left part of (17), transferring remainder terms to the right part of (17), and then using the inequality (6) to the left part, we have

$$\chi_0 X^2 \equiv \chi_0 \int\limits_{\sigma'} \sum_{|k|=r} \left(\frac{\Delta^s u^{(k)}(x)}{h^{|s|}} \right)^2 \eta(x) dx \leq$$

$$\leq J \equiv \int\limits_{\sigma'} \sum_{\substack{|k|\leq r \\ |l|\leq r}} a_{kl}(x) \frac{\Delta^s u^{(k)}(x)}{h^{|s|}} \frac{\Delta^s u^{(l)}(x)}{h^{|s|}} \eta(x) dx =$$

$$= -\sum{}' J_{\ k,l,\lambda,\tau} - J_1 + J_2, \tag{18}$$

where

$$J_{k,l,\lambda,\tau} = C_s^\lambda C_l^\tau \int\limits_{\sigma'} \frac{\Delta^\lambda a_{kl}[x + h(s-\lambda)]}{h^{|\lambda|}} \frac{\Delta^{s-\lambda} u^{(k)}(x)}{h^{|s-\lambda|}} \frac{\Delta^s u^{(l-\tau)}(x)}{h^{|s|}} \eta^{(\tau)}(x) dx,$$

$$J_1 = \int\limits_{\sigma'} \frac{\Delta^s (a(x) |u(x)|^{p-2} u(x))}{h^{|s|}} \frac{\Delta^s u(x)}{h^{|s|}} \eta(x) dx,$$

$$J_2 = \int\limits_{\sigma'} \frac{\Delta^s F(x)}{h^{|s|}} \frac{\Delta^s u(x)}{h^{|s|}} \eta(x) dx,$$

k, l, λ, τ are entire non-negative vectors, for which $|k| \leq r,\ |l| \leq r,\ |k|+|l| < 2r$, $0 \leq \lambda \leq s,\ 0 \leq \tau \leq l$.

Let's estimate integrals in the right part of (18). Assuming as yet that $u(x) \in W_2^{r+|s|-1}(\sigma')$.

1) if $\lambda = 0$ and $|k| = r$ (then $|l| < r$), we obtain

$$|J_{k,l,0,\tau}| \leq C \int\limits_{\sigma'} \left| \frac{\Delta^s u^{(k)}(x)}{h^{|s|}} \right| \left| \frac{\Delta^s u^{(l-\tau)}(x)}{h^{|s|}} \right| \left| \eta^{(\tau)}(x) \right| dx \leq$$

$$\leq C \left(\int\limits_{\sigma'} \left| \frac{\Delta^s u^{(k)}(x)}{h^{|s|}} \right|^2 \eta dx \right)^{\frac{1}{2}} \left(\int\limits_{\Lambda_\eta} \left| \frac{\Delta^s u^{(l-\tau)}(x)}{h^{|s|}} \right|^2 dx \right)^{\frac{1}{2}} \leq$$

$$\leq CX \left\| u^{(l-\tau+s)} \right\|_{L_2(\sigma')} \leq CX \|u\|_{W_2^{r+|s|-1}(\sigma')}. \tag{19}$$

By virtue of the condition $|a_{kl}(x)| \leq M \ \ (x \in \Omega)$, the first relation of this chain of inequalities is valid, the second is known, in the third it is used the condition 4) of function $\eta(x)$ and the fact, that in the present case $|s + l - \tau| \leq r + |s| - 1$. Therefore we can use the estimate of difference relation by corresponding derivative (see [6]).

2) If $\tau=0,\ |l|=r$ (then $|k|<r$), so $|k|+|s-\lambda|\leq r+|s|-1$, therefore

$$|J_{k,l,\lambda,0}|\leq C\left(\int\limits_{\Lambda_\eta}\left(\frac{\Delta^{s-\lambda}u^{(k)}(x)}{h^{|s-\lambda|}}\right)^2dx\right)^{\frac{1}{2}}\left(\int\limits_{\sigma'}\left(\frac{\Delta^{s}u^{(l)}(x)}{h^{|s|}}\right)^2\eta(x)\,dx\right)^{\frac{1}{2}}<<$$

$$<<\|u\|_{W_2^{r+|s|-1}(\sigma')}X\ . \tag{20}$$

3) In the rest cases (i.e. when $|k|<r,\ |l|<r$) we have $|s-\lambda|+|k|\leq r+|s|-1$, $|s|+|l-\tau|\leq r+|s|-1$. Estimating difference quotient for $a_{kl}^{(\lambda)}(x)$ (since by condition $|a_{kl}^{(\lambda)}(x)|\leq M,\ |\lambda|\leq m$), and function $\eta^{(\tau)}(x)$ with constant we obtain

$$|J_{k,l,\lambda,\tau}|\leq C\left(\int\limits_{\Lambda_\eta}\left(\frac{\Delta^{s-\lambda}u^{(k)}(x)}{h^{|s-\lambda|}}\right)^2dx\right)^{\frac{1}{2}}\left(\int\limits_{\Lambda_\eta}\left(\frac{\Delta^{s}u^{(l-\tau)}(x)}{h^{|s|}}\right)^2dx\right)^{\frac{1}{2}}<<$$

$$<<\|u\|^2_{W_2^{r+|s|-1}(\sigma')}\ . \tag{21}$$

By virtue of (19)-(21) we obtain

$$\left|-\sum{}'J_{k,l,\lambda,\tau}\right|\leq C_1\|u\|_{W_2^{r+|s|-1}(\sigma')}X+C_2\|u\|^2_{W_2^{r+|s|-1}(\sigma')}, \tag{22}$$

where C_1,C_2 is independent of u.

Using the formula analogous to formula of integration by parts, by virtue of the finiteness of $\eta(x)$ in σ' for integral J_1 from (18) we have

$$|J_1|=\left|\int\limits_{\sigma'}\frac{\Delta_h^s}{h^{|s|}}\left(a(x)|u(x)|^{p-1}sign\ u(x)\right)\frac{\Delta_h^s u(x)}{h^{|s|}}\eta(x)\,dx\right|=$$

$$=\left|\int\limits_{\sigma'}a(x)|u(x)|^{p-1}sign\ u(x)\frac{\Delta_{-h}^s}{h^{|s|}}\left(\frac{\Delta_h^s u(x)}{h^{|s|}}\eta(x)\right)dx\right|<<$$

$$<<\|u\|^{p-1}_{L_p(\sigma')}\left\|\left(\frac{\Delta_h^s u}{h^{|s|}}\eta\right)^{(s)}\right\|_{L_2(\delta')}<<$$

$$<<\begin{cases}\|u\|^{p-1}_{L_p(\sigma')}X+\|u\|^{p-1}_{L_p(\sigma')}\|u\|_{W_2^{r+|s|-1}(\sigma')}, & |s|=r\ ,\\ \|u\|^{p-1}_{L_p(\sigma')}\|u\|_{W_2^{r+|s|-1}(\sigma')}, & |s|<r\end{cases} \tag{23}$$

By analogy for integral J_2 from (18)we have

$$|J_2|<<\begin{cases}\|F\|_{L_2(\sigma')}X+\|F\|_{L_2(\sigma')}\|u\|_{W_2^{r+|s|-1}(\sigma')}\ , & |s|=r,\\ \|F\|_{L_2(\sigma')}\|u\|_{W_2^{r+|s|-1}(\sigma')}\ , & |s|<r,\end{cases} \tag{24}$$

From (18)-(24), in case of integer unit vector $s=(s_1,...,s_n)$ $(|s|=1)$ we obtain

$$X^2 << \left(\|u\|_{W_2^r(\sigma')} + \|u\|^{p-1}_{L_p(\sigma')} + \|F\|_{L_2(\sigma')}\right) X +$$

$$+ \|u\|^2_{W_2^r(\sigma')} + \|u\|^{2(p-1)}_{L_p(\sigma')} + \|F\|^2_{L_2(\sigma')} .$$

Consequently

$$X \le C\left(\|u\|_{W_2^r(\sigma')} + \|u\|^{p-1}_{L_p(\sigma')} + \|F\|_{L_2(\sigma')}\right) \equiv S \ , \tag{25}$$

where the positive constant c is independent of u, F and quantity h.

Recalling that $\eta(x)=1$ on σ, for $|s|=1$ we have

$$\left[\int\limits_{\sigma} \sum_{|k|=r} \left(\frac{\Delta^s u^{(k)}(x)}{h}\right)^2 dx\right]^{\frac{1}{2}} << S \ .$$

Therefore [6]:

$$\left[\int\limits_{\sigma} \sum_{|k|=r} |u^{(k+s)}(x)|^2 dx\right]^{\frac{1}{2}} << S \qquad (|k+s|=r+1) \ .$$

Consequently $u(x) \in W_2^{r+1}(\sigma)$, besides

$$\|u\|_{W_2^{r+1}(\sigma)} << \|u\|_{W_2^r(\sigma')} + \|u\|^{p-1}_{L_p(\sigma')} + \|F\|_{L_2(\sigma')} \ . \tag{26}$$

Overdenote σ, σ' by σ_m, σ_0 respectively, and introduce into consideration the system of open balls $\sigma_0, \sigma_1, ..., \sigma_m$ with center at the point x^0 strongly embedded each other

$$\sigma = \sigma_m \subset \sigma_{m-1} \subset ... \subset \sigma_1 \subset \sigma_0 = \sigma' \ .$$

After the proof of inequality by (26) taking into account the condition (13) by analogue in turn for $|s| = 2, ..., m$ we can establish the validity of the following inequalities

$$\|u\|_{W_2^{r+2}(\sigma_2)} << \|u\|_{W_2^{r+1}(\sigma_1)} + \|u\|^{p-1}_{L_p(\sigma_1)} + \|F\|_{L_2(\sigma_1)} \ ,$$

$$\|u\|_{W_2^{r+3}(\sigma_3)} << \|u\|_{W_2^{r+2}(\sigma_2)} + \|u\|^{p-1}_{L_p(\sigma_2)} + \|F\|_{L_2(\sigma_2)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\|u\|_{W_2^{r+m}(\sigma_m)} << \|u\|_{W_2^{r+m-1}(\sigma_{m-1})} + \|u\|^{p-1}_{L_p(\sigma_{m-1})} + \|F\|_{L_2(\sigma_{m-1})} .$$

From these correlations and the inequality (26) it follows

$$\|u\|_{W_2^{r+m}(\sigma)} << \|u\|_{W_2^r(\sigma')} + \|u\|^{p-1}_{L_p(\sigma')} + \|F\|_{L_2(\sigma')} . \tag{27}$$

In the following it is used the following theorem of Truazi [6]:

For arbitrary bounded measurable domain $\Omega \subset R_n$ it holds

$$\int_\Omega \left(\rho^s\left(x^0\right)\|f\|_{L_p(\sigma(x^0))}\right)^p dx^0 \sim \int_\Omega \left(\rho^{s+\frac{n}{p}}(x)\,|f(x)|\right)^p dx\ ,$$

where $1 \le p < \infty$, s is an arbitrary real number, $\sigma\left(x^0\right)$ is any ball with center x^0, strongly lying inside $\Omega, \sim$ is an equivalence sign.

We multiply both pats of (27) by $\left[\rho\left(x^0\right)\right]^{-\frac{p-1}{p}n}$ and then we square it and integrate by $x^0 \in \Omega$, then we apply to each term of the obtained inequality the Truazi theorem and if we raise result to the power $\frac{1}{2}$, then we have

$$\|u\|_{W^{r+m}_{2,\frac{2-p}{2p}n}(\Omega)} << \|u\|_{W_2^r(\Omega)} + \|u\|^{p-1}_{L_p(\Omega)} + \|F\|_{L_2(\Omega)}\ .$$

This inequality together with (10) proves (14) $(1 < p \le 2)$.

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