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ON BIORTHOGONALITY OF TWO SYSTEMS OF FUNCTIONS

Abstract

In the paper the explicit form of biorthogonal system for the system of exponents $\left\{e^{i[(n+\alpha_1)t+\beta(t)]};e^{-i[(k+\alpha_2)t+\beta(t)]}\right\}_{n=0,k=1}^{\infty}$ *is obtained, where*

$$\beta(t)=\begin{cases}\beta_1, & -\pi<t<0,\\ \beta_2, & -0<t<\pi,\end{cases}\qquad \alpha_i,\beta_i\in R,\quad i=1,2$$

are real parameters.

Consider the following system of exponents:

$$\left\{e^{i[(n+\alpha_1)t+\beta(t)]};e^{-i[(k+\alpha_2)t+\beta(t)]}\right\}_{n=0,k=1}^{\infty} \tag{1}$$

where

$$\beta(t)=\begin{cases}\beta_1, & -\pi<t<0,\\ \beta_2, & -0<t<\pi,\end{cases}\qquad \alpha_i,\beta_i\in R,\quad i=1,2$$

is a real parameters which satisfy the following conditions

$$\left.\begin{array}{l} 0\le-(\alpha_1+\alpha_2)+\frac{\beta_1-\beta_2}{\pi}<1,\\ 0\le\frac{\beta_2-\beta_1}{\pi}<1\ . \end{array}\right\} \tag{2}$$

Note that earlier at $\beta(t)\equiv0$ and $\alpha_1=\alpha_2$ the basicity of system (1) in $L_p(-\pi,\pi)$, $p\in(1,+\infty)$ was considered in [1-3]. Moreover in [2,3] the explicit form of biorthogonal system for system (1) was found in the considered cases. Further the system of exponents (1) was considered in [4] in more general form, and the existence of biorthogonal systems to this system in $L_p(-\pi,\pi)$, $p\in(1,+\infty)$ was proved at definite conditions on parameters α_i,β_i, $i=1,2$.

System (1) is a model case for eigen-values problem and the function of the first order discontinuous differential operator

$$y'(x)=\lambda y(x),\quad x\in(-\pi,0)\cup(0,\pi)\ .$$

It is very important to know the explicit form of biorthogonal system. By considering many questions concerned with biorthogonal series by system (1) (for example uniform equiconvergence of biorthogonal series with trigonometric series of summable functions on a compact). In the present paper the explicit form of biorthogonal system to the system of exponents (1) is determined.

So, let

$$\begin{cases} h_n^+(\theta)=\frac{1}{2\pi}\left(e^{i\theta}+1\right)_{-1}^{-\mu_1}\left(e^{i\theta}-1\right)_{+1}^{-\mu_2}e^{-i(\alpha_1\theta+\beta(\theta)+\beta_1-\beta_2)}\times\\ \times\sum\limits_{m=0}^{n}(-1)^{n-m}C_{\mu_2}^{n-m}\sum\limits_{s=0}^{m}C_{\mu_1}^{m-s}e^{-is\theta},\quad n\ge0\\ h_k^-(\theta)=\frac{1}{2\pi}\left(e^{i\theta}+1\right)_{-1}^{-\mu_1}\left(e^{i\theta}-1\right)_{+1}^{-\mu_2}e^{-i(\alpha_1\theta+\beta(\theta)+2\beta_1-4\beta_2)}\times\\ \times\sum\limits_{m=1}^{k}(-1)^{k-m}C_{\mu_2}^{k-m}\sum\limits_{s=1}^{m}C_{\mu_1}^{m-s}e^{is\theta},\quad k\ge1 \end{cases} \tag{3}$$

where $\mu_1 = -(\alpha_1 + \alpha_2) + (\beta_1 - \beta_2)/\pi, \quad \mu_2 = (\beta_2 - \beta_1)/\pi$.

Here $(z+1)_{-1}^{\mu_1}$ and $z_{-1}^{\mu_1}$ are branches of the many-valued functions $(z+1)^{\mu_1}$ and z^{μ_1}, respectively, analytical on complex plane cut along the negative part of the real axis $(-\pi \le \arg z < \pi)$, and $(z+1)_{+1}^{\mu_2}$ and $z_{+1}^{\mu_2}$ are such branches of the functions $(z-1)^{\mu_2}$ and z^{μ_2}, respectively, analytical on complex surface, cut along the positive part of an real axis $(0 < \arg z \le 2\pi)$.

$C_\mu^n = \dfrac{\mu(\mu-1)\ldots(\mu-n+1)}{n!}$ -are binomial coefficients.

The following theorem is true.

Theorem. *Let $\mu_i,\ i = 1,2$ satisfy conditions (2). Then the system of exponent (1) the system $\left\{h_n^+(\theta), h_k^-(\theta)\right\}_{\substack{n=0\\k=1}}^{\infty}$ satisfy the following conditions*

$$\begin{array}{l} \displaystyle\int_{-\pi}^{\pi} h_n^+(t)\, e^{i[(m+\alpha_1)t+\beta(t)]}dt = \delta_{nm}, \int_{-\pi}^{\pi} h_n^+(t)\, e^{-i[(s+\alpha_2)t+\beta(t)]}dt = 0 \\ \displaystyle\int_{-\pi}^{\pi} h_k^-(t)\, e^{i[(m+\alpha_1)t+\beta(t)]}dt = 0, \int_{-\pi}^{\pi} h_k^-(t)\, e^{-i[(s+\alpha_2)t+\beta(t)]}dt = \delta_{ks} \end{array} \tag{4}$$

where $\delta_{nm}-$ is Kroneker symbol, $m,n \ge 0,\ k,s \ge 1$.

At proving the fundamental role plays the following

Lemma 1. *The following identity is true:*

$$I_n^+ \equiv \frac{1}{2\pi i}\int_L \frac{\tau^n d\tau}{(\tau+1)_{-1}^{\mu_1}(\tau-1)_{+1}^{\mu_2}(\tau-z)} = \begin{cases} z^n (z+1)_{-1}^{-\mu_1}(z-1)_{+1}^{-\mu_2}, & |z|<1, \\ 0, & |z|>1, \end{cases}$$

where $n = \overline{0,\infty}$, $L-$ is a circle of unit radius with the center in the origin which is passes round from $e^{-i\pi}$ to $e^{i\pi}$ in positive direction.

Proof of the lemma. Let l_{δ_1} be a part of circle with the center -1 and the radius δ_1 which is situated inside a unit circle, and l_{δ_2} is a part of a circle with the center +1 and radius δ_2 which is contained inside circle. L^+ and L^- are the parts of a circle L on upper and lower half-planes respectively (fig.1).

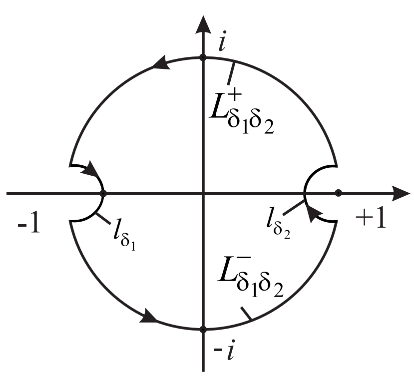

Fig. 1.

$$L^\pm_{\delta_1,\delta_2} = \left\{\tau \in L^\pm : \quad |\tau+1| > \delta_1, \quad |\tau-1| > \delta_2\right\}$$

It is clear from the definition of a singular integral that

$$I_n^+=\lim_{\substack{\delta_1\to 0+0\\ \delta_2\to 0+0}}\left[\frac{1}{2\pi i}\int\limits_{L^+_{\delta_1,\delta_2}}\frac{\tau^n d\tau}{(\tau+1)^{\mu_1}_{-1}(\tau-1)^{\mu_2}_{+1}(\tau-z)}+\frac{1}{2\pi i}\int\limits_{L^-_{\delta_1,\delta_2}}\frac{\tau^n d\tau}{(\tau+1)^{\mu_1}_{-1}(\tau-1)^{\mu_2}_{+1}(\tau-z)}\right].$$

Denote $L^*_{\delta_1,\delta_2}=l_{\delta_1}\cup L^-_{\delta_1,\delta_2}\cup l_{\delta_2}\cup L^+_{\delta_1,\delta_2}$.

Then the function $\varphi(\tau)=\dfrac{\tau^n}{(\tau+1)^{\mu_1}_{-1}(\tau-1)^{\mu_2}_{+1}}$ will be analytical inside the contour $L^*_{\delta_1,\delta_2}$. Consequently, by Cauchy theorem we have:

$$\frac{1}{2\pi i}\int\limits_{L^*_{\delta_1,\delta_2}}\frac{\tau^n d\tau}{(\tau+1)^{\mu_1}_{-1}(\tau-1)^{\mu_2}_{+1}(\tau-z)}=\begin{cases} z^n(z+1)^{-\mu_1}_{-1}(z-1)^{-\mu_2}_{+1}, & z\in Int\ L^*_{\delta_1,\delta_2}\\ 0, \quad z\in C\backslash\overline{Int\ L^*_{\delta_1,\delta_2}}\ . \end{cases}$$

It is evident that $\forall|z|<1,\ \ \exists\delta(z)>0,\ \ \forall\delta_j\in(0,\delta(z)),\ \ j=\overline{1,2}\ \ z\in IntL^*_{\delta_1,\delta_2}$, or

$$\lim_{\substack{\delta_1\to 0+0\\ \delta_2\to 0+0}}\int\limits_{L^*_{\delta_1,\delta_2}}\frac{\tau^n d\tau}{(\tau+1)^{\mu_1}_{-1}(\tau-1)^{\mu_2}_{+1}(\tau-z)}=\begin{cases} z^n(z+1)^{-\mu_1}_{-1}(z-1)^{-\mu_2}_{+1}, & |z|<1,\\ 0, \quad |z|>1\ . \end{cases}$$

It remains to prove that $\lim\limits_{\delta_j\to 0+0}\int\limits_{l_{\delta_j}}\dfrac{\tau^n d\tau}{(\tau+1)^{\mu_1}_{-1}(\tau-1)^{\mu_2}_{+1}(\tau-z)}=0,\quad j=1,2.$ The proof is carried out for the case $j=1$, and the case $j=2$ is proved analogously.

It is evident that if $\tau\in l_{\delta_1}$ then $|\tau+1|=\delta_1,\ |\tau|<1$. From the inequality of triangle it follows that

$$|z+1|\leq|\tau-z|+|\tau+1|=|\tau-z|+\delta_1,\quad or$$

$$|\tau-z|\geq|z+1|-\delta_1\ .$$

It is clear that $\forall z\neq -1$ and $\forall\delta_1\in\left(0;\frac{|z+1|}{2}\right)$

$$|\tau-z|\geq\frac{|z+1|}{2}\ .$$

It is also evident for little δ_1 when $\tau\in l_{\delta_1},\quad |\tau-1|>\dfrac{1}{2}$. Allowing for these inequalities we estimate the integral

$$\left|\int\limits_{l_{\delta_1}}\frac{\tau^n d\tau}{(\tau+1)^{\mu_1}_{-1}(\tau-1)^{\mu_2}_{+1}(\tau-z)}\right|\leq\int\limits_{l_{\delta_1}}\frac{|\tau|^n|d\tau|}{|\tau+1|^{\mu_1}|\tau-1|^{\mu_2}|\tau-z|}\leq$$

$$\leq\int\limits_{l_{\delta_1}}\frac{|d\tau|}{\delta_1^{\mu_1}\cdot\frac{1}{2}\cdot\frac{|z+1|}{2}}=\frac{4}{|z+1|}\delta_1^{-\mu_1}\int\limits_{l_{\delta_1}}|d\tau|\leq\frac{4}{|z+1|}\delta_1^{-\mu_1}\pi\delta_1=\frac{4\pi}{|z+1|}\delta_1^{1-\mu_1}\to 0\ ,$$

at $\delta_1\to 0+0$.

So, lemma 1 is proved.

Lemma 2. *The following identity is true*

$$I_n^- = \frac{1}{2\pi i}\int\limits_L \frac{\tau^{-n}d\tau}{\left(\frac{\tau+1}{\tau}\right)_{-1}^{\mu_1}\left(\frac{\tau-1}{\tau}\right)_{+1}^{\mu_2}(\tau-z)} = \begin{cases} 0, & |z|<1, \\ -z^{-n}\left(\frac{z+1}{z}\right)_{-1}^{-\mu_1}\left(\frac{z-1}{z}\right)_{+1}^{-\mu_2}, & |z|>1 \ . \end{cases}$$

where $n=\overline{1,\infty}$, *L is a circle of a unit radius with the center of origin which is passed round from $e^{-i\pi}$ to $e^{i\pi}$ in positive direction.*

Proof. Let l_{δ_1} be a part of a circle with the center at the point -1 and of radius δ_1 which is situated out of the circle. And l_{δ_2} is a part of the circle with the center at the point $+1$ and the radius δ_2, which lies out of the unit circle. L^+ and L^- are the parts of a unit circle L on upper and lower half-planes respectively (fig.2)

$$L_{\delta_1,\delta_2}^{\pm} = \left\{\tau \in L^{\pm}:\ |\tau+1|>\delta_1,\ |\tau-1|>\delta_2\right\}\ .$$

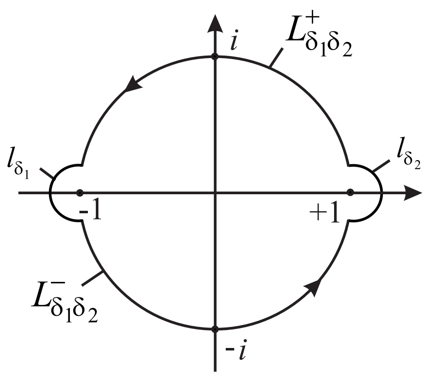

Fig. 2.

It is clear from the definition of singular integral that

$$I_n^- = \lim_{\substack{\delta_1\to 0+0\\ \delta_2\to 0+0}}\left[\frac{1}{2\pi i}\int\limits_{L_{\delta_1,\delta_2}^+}\frac{\tau^n d\tau}{\left(\dfrac{\tau+1}{\tau}\right)_{-1}^{\mu_1}\left(\dfrac{\tau-1}{\tau}\right)_{+1}^{\mu_2}(\tau-z)} + \right.$$

$$\left. +\frac{1}{2\pi i}\int\limits_{L_{\delta_1,\delta_2}^-}\frac{\tau^n d\tau}{\left(\dfrac{\tau+1}{\tau}\right)_{-1}^{\mu_1}\left(\dfrac{\tau-1}{\tau}\right)_{+1}^{\mu_2}(\tau-z)}\right].$$

Denote

$$L_{\delta_1,\delta_2}^* = l_{\delta_1}\cup L_{\delta_1,\delta_2}^- \cup l_{\delta_2}\cup L_{\delta_1,\delta_2}^+\ .$$

Then the function $\varphi(\tau) = \tau^{-n}\left(\dfrac{\tau+1}{\tau}\right)_{-1}^{-\mu_1}\left(\dfrac{\tau-1}{\tau}\right)_{+1}^{-\mu_2}$ is analytical in the outside of the contour L_{δ_1,δ_2}^*.

Consequently, by Caushy theorem, we have:

$$\int\limits_{L_{\delta_1,\delta_2}^*}\frac{\varphi(\tau)\,d\tau}{\tau-z} = \begin{cases} -\varphi(z)+\varphi(\infty), & z\in C\backslash\overline{Int\ L_{\delta_1,\delta_2}^*}, \\ \varphi(\infty),\ z\in Int\ L_{\delta_1,\delta_2}^*\ . \end{cases}$$

$|\varphi(\tau)| = \left|\frac{\tau+1}{\tau}\right|^{-\mu_1}\left|\frac{\tau-1}{\tau}\right|^{-\mu_2}|\tau|^{-n} = \left|1+\frac{1}{\tau}\right|^{-\mu_1}\left|1-\frac{1}{\tau}\right|^{-\mu_2}|\tau|^{-n} \to 0$, as $\tau\to\infty$, or $\varphi(\infty)=0$.

It is evident that $\forall\ |z|>1,\ \ \exists\ \delta(z)>0,\ \ \forall\ \delta_j\in(0;\delta(z))\,,\ j=1,2$,linebreak $z\in C\backslash\overline{Int\ L^*_{\delta_1,\delta_2}}$ or

$$\lim_{\substack{\delta_1\to 0+0\\ \delta_2\to 0+0}}\int\limits_{L^*_{\delta_1,\delta_2}}\frac{\varphi(\tau)\,d\tau}{\tau-z}=\begin{cases}-\varphi(z)\,, & |z|>1,\\ 0, & |z|<1.\end{cases}$$

It remains to prove that

$$\lim_{\delta_j\to 0+0}\int\limits_{l_{\delta_j}}\frac{\varphi(\tau)\,d\tau}{\tau-z}=0\ ,\quad j=1,2.$$

The proof is carried out for $j=1$.

It is evident that if $\tau\in l_{\delta_1}$ then $|\tau+1|=\delta_1$. From the inequality of triangle it follows that

$$|z+1|\le|\tau-z|+|\tau+1|=|\tau-z|+\delta_1\quad or$$
$$|\tau-z|\ge|z+1|-\delta_1\ .$$

Form the last it follows that $\forall z\ne -1$ and $\forall\delta_1\in\left(0,\dfrac{|z+1|}{2}\right)$

$$|\tau-z|\ge\frac{|z+1|}{2}\ .$$

For little δ_1 when $\tau\in l_{\delta_1}$, then $|\tau-1|>\frac{1}{2}$, $\ 1\le|\tau|<2$. Let's estimate the integral

$$\left|\int\limits_{l_{\delta_1}}\frac{\varphi(\tau)\,d\tau}{\tau-z}\right|\le\int\limits_{l_{\delta_1}}\frac{|\tau|^{-n}|d\tau|}{\left|\dfrac{\tau+1}{\tau}\right|^{\mu_1}\left|\dfrac{\tau-1}{\tau}\right|^{\mu_2}|\tau-z|}\le$$
$$\le\int\limits_{l_{\delta_1}}\frac{|\tau|^{\mu_1}|\tau|^{\mu_2}|d\tau|}{|\tau|^n\,|\tau+1|^{\mu_1}\,|\tau-1|^{\mu_2}\,|\tau-z|}\le\int\limits_{l_{\delta_1}}\frac{2^{\mu_1}2^{\mu_2}|d\tau|}{\delta_1^{\mu_1}\left(\frac{1}{2}\right)^{\mu_2}\dfrac{|z+1|}{2}}\le$$
$$\le\frac{16\delta_1^{-\mu_1}}{|z+1|}\int\limits_{l_{\delta_1}}|d\tau|\le\frac{16\delta_1^{-\mu_1}}{|z+1|}\pi\delta_1=\frac{16\pi\delta_1^{1-\mu_1}}{|z+1|}\to 0,\quad at\quad \delta_1\to 0+0\ .$$

allowing for the obtained inequalities

The case $j=2$ is proved analogously.

So, lemma 2 is proved.

The proof of the theorem is based on the solution of the following Riemann problem of theory of analytical functions: Find the pair of the functions $F^+(z)$ and $F^-(z)$ analytical inside and outside of a unit circle by its values on the boundary

$$\begin{cases}F^+(\tau)=G(\tau)\,F^-(\tau)+g(\tau)\,,\ \tau\in L\backslash\{\pm1\}\,,\\ F^-(\infty)=0\end{cases}\tag{5}$$

where

$$\left.\begin{array}{l}G(\tau)=e^{-i(\alpha_1\arg\tau+\alpha_2\arg\tau+2\beta(\arg\tau))}\\ g(\tau)=e^{-i(\alpha_1\arg\tau+\beta(\arg\tau))}f(\arg\tau)\end{array}\right\}\tag{6}$$

Here L is a unit circle which is passed round in positive direction, $f(t)$ is a piece-wise Hölder function which has a break at the point $t=0$, i.e., $f(t)\in KC_{p_1,p_2}[-\pi,\pi]$, where $0<p_i\leq 1$, $i=1,2$ are Hölder exponents, and $\arg z$ is a branch of the multivalued function $\arg z$ on the plane cut along the negative part of a real axis.

We'll solve this problem by the method, developed in monograph of F.D.Gakhov [5, p.436]. We have:

$$\mu_1=\frac{1}{2\pi i}\ln\frac{G(-1-0)}{G(-1+0)}=\frac{1}{2\pi i}\ln\frac{e^{-i(\alpha_1\pi+\alpha_2\pi+2\beta_2)}}{e^{-i(-\alpha_1\pi-\alpha_2\pi+2\beta_1)}}=$$

$$=\frac{1}{2\pi i}\ln e^{-(2\alpha_1\pi+2\alpha_2\pi)i}e^{-2i\beta_2+2i\beta_1}=\frac{1}{2\pi i}\ln e^{-2\pi i(\alpha_1+\alpha_2)}e^{2i(\beta_1-\beta_2)}=$$

$$=\frac{1}{2\pi i}\left(-2\pi i(\alpha_1+\alpha_2)+2i(\beta_1-\beta_2)-2\pi i æ_1\right)=-(\alpha_1+\alpha_2)+\frac{\beta_1-\beta_2}{\pi}-æ_1.$$

$$\mu_2=\frac{1}{2\pi i}\ln\frac{G(1-0)}{G(1+0)}=\frac{1}{2\pi i}\ln\frac{e^{-2\beta_1 i}}{e^{-2\beta_2 i}}=\frac{1}{2\pi i}\ln e^{2i(\beta_2-\beta_1)}=$$

$$=\frac{1}{2\pi i}\left(2i(\beta_2-\beta_1)-2\pi i æ_2\right)=\frac{\beta_2-\beta_1}{\pi}-æ_2\ .$$

We look for the solution which is bounded on a closed unit circle. From the condition of the theorem on α_i and β_i it follows that for this case $æ_1=æ_2=0$.

Further we calculate $\Gamma(z)$:

$$\Gamma(z)=\frac{1}{2\pi i}\int\limits_L\frac{\ln\left[\tau_{-1}^{-\mu_1}\tau_{+1}^{-\mu_2}G(\tau)\right]}{\tau-z}d\tau=$$

$$=\frac{1}{2\pi i}\int\limits_L\frac{\ln\left[\tau_{-1}^{\alpha_1+\alpha_2-\frac{\beta_1-\beta_2}{\pi}}\tau_{+1}^{\frac{\beta_1-\beta_2}{\pi}}e^{-i(\alpha_1\arg\tau+\alpha_2\arg\tau+2\beta(\arg\tau))}\right]}{\tau-z}d\tau=$$

$$=\frac{1}{2\pi i}\int\limits_{L^+}\frac{\ln\left[\tau_{-1}^{\alpha_1+\alpha_2-\frac{\beta_1-\beta_2}{\pi}}\tau_{-1}^{\frac{\beta_1-\beta_2}{\pi}}\tau_{-1}^{-(\alpha_1+\alpha_2)}e^{-2\beta_2 i}\right]}{\tau-z}d\tau+$$

$$+\frac{1}{2\pi i}\int\limits_{L^-}\frac{\ln\left[\tau_{-1}^{\alpha_1+\alpha_2-\frac{\beta_1-\beta_2}{\pi}}\tau_{-1}^{\frac{\beta_1-\beta_2}{\pi}}e^{2i(\beta_1-\beta_2)}\tau_{-1}^{-(\alpha_1+\alpha_2)}e^{-2\beta_1 i}\right]}{\tau-z}d\tau=$$

$$=\frac{1}{2\pi i}\int\limits_L\frac{\ln e^{-2\beta_2 i}}{\tau-z}d\tau=-\frac{2\beta_2 i}{2\pi i}\int\limits_L\frac{d\tau}{\tau-z}=\begin{cases}-2\beta_2 i\ , & |z|<1,\\ 0, & |z|>1.\end{cases}$$

From $X_1(z)=e^{\Gamma(z)}$ it follows that:

$$X_1^+(z)=e^{-2\beta_2 i}\ ,\qquad X_1^-(z)=1\ .$$

Then

$$\Psi(z)=\frac{e^{2\beta_2 i}}{2\pi i}\int\limits_L\frac{(\tau+1)_{-1}^{-\mu_1}(\tau-1)_{+1}^{-\mu_2}e^{-i(\alpha_1\arg\tau+\beta(\arg\tau))}f(\arg\tau)\,d\tau}{\tau-z}$$

or

$$\Psi(z)=\frac{e^{2\beta_2 i}}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-i(\alpha_1\theta+\beta(\theta))}f(\theta)\,d\theta}{(e^{i\theta}+1)_{-1}^{\mu_1}(e^{i\theta}-1)_{+1}^{\mu_2}(1-ze^{-i\theta})}$$

Then the solution of the Riemann problem will be the following function:

$$\left.\begin{array}{l} F^{+}(z)=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\frac{e^{-i(\alpha_1\theta+\beta(\theta))}f(\theta)\,d\theta}{(e^{i\theta}+1)_{-1}^{\mu_1}(e^{i\theta}-1)_{+1}^{\mu_2}(1-ze^{-i\theta})}(z+1)_{-1}^{\mu_1}(z-1)_{+1}^{\mu_2} \\ F^{-}(z)=\dfrac{e^{2\beta_2 i}}{2\pi}\displaystyle\int_{-\pi}^{\pi}\frac{e^{-i(\alpha_1\theta+\beta(\theta))}f(\theta)\,d\theta}{(e^{i\theta}+1)_{-1}^{\mu_1}(e^{i\theta}-1)_{+1}^{\mu_2}(1-ze^{-i\theta})}\left(\frac{z+1}{z}\right)_{-1}^{\mu_1}\left(\frac{z-1}{z}\right)_{+1}^{\mu_2} \end{array}\right\} \quad (7)$$

It is absolutely clear that at $|z|<1$

$$\frac{1}{1-ze^{-i\theta}}=\sum_{k=0}^{\infty}e^{-i\theta k}z^k$$

$$(z+1)_{-1}^{\mu_1}=\sum_{k=0}^{\infty}C_{\mu_1}^{k}z^k\ , \qquad (z-1)_{+1}^{\mu_2}=e^{i(\beta_2-\beta_1)}\sum_{k=0}^{\infty}(-1)^k C_{\mu_2}^{k}z^k$$

and at $|z|>1$

$$\frac{1}{1-ze^{-i\theta}}=-\sum_{k=1}^{\infty}e^{i\theta k}z^{-k}$$

$$\left(\frac{z+1}{z}\right)_{-1}^{\mu_1}=\left(1+\frac{1}{z}\right)_{-1}^{\mu_1}=\sum_{k=0}^{\infty}C_{\mu_1}^{k}z^{-k}$$

$$\left(\frac{z-1}{z}\right)_{+1}^{\mu_2}=\left(1-\frac{1}{z}\right)_{+1}^{\mu_2}=e^{2i(\beta_2-\beta_1)}\sum_{k=0}^{\infty}(-1)^k C_{\mu_2}^{k}z^{-k}\ .$$

Then

$$F^{+}(z)=\sum_{n=0}^{\infty}\int_{-\pi}^{\pi}\frac{\left(e^{i\theta}+1\right)_{-1}^{-\mu_1}\left(e^{i\theta}-1\right)_{+1}^{-\mu_2}}{2\pi}e^{-i(\alpha_1\theta+\beta(\theta)+\beta_1-\beta_2)}\times$$

$$\times\sum_{m=0}^{\infty}(-1)^{n-m}C_{\mu_2}^{n-m}\sum_{s=0}^{m}C_{\mu_1}^{m-s}e^{-is\theta}f(\theta)\,d\theta\ z^n$$

$$F^{-}(z)=\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}\frac{-\left(e^{i\theta}+1\right)_{-1}^{\mu_1}\left(e^{i\theta}-1\right)_{+1}^{\mu_2}}{2\pi}e^{-i(\alpha_1\theta+\beta(\theta)+2\beta_1-4\beta_2)}\times$$

$$\times\sum_{m=1}^{n}(-1)^{n-m}C_{\mu_2}^{n-m}\sum_{s=1}^{m}C_{\mu_1}^{m-s}e^{is\theta}f(\theta)\,d\theta\ z^{-n}\ .$$

Further grouping the corresponding coefficients for $h_n^{+}(\theta)$ and $h_k^{-}(\theta)$ $\quad(n\geq 0,k\geq 1)$ we'll obtain the following expressions:

$$h_n^{+}(\theta)=\frac{1}{2\pi}\left(e^{i\theta}+1\right)_{-1}^{\mu_1}\left(e^{i\theta}-1\right)_{+1}^{\mu_2}e^{-i(\alpha_1\theta+\beta(\theta)+\beta_1-\beta_2)}\sum_{m=0}^{n}(-1)^{n-m}C_{\mu_2}^{n-m}\sum_{s=0}^{m}C_{\mu_1}^{m-s}e^{-is\theta}$$

$$h_k^-(\theta)=\frac{1}{2\pi}\left(e^{i\theta}+1\right)_{-1}^{\mu_1}\left(e^{i\theta}-1\right)_{+1}^{\mu_2}e^{-i(\alpha_1\theta+\beta(\theta)+2\beta_1-4\beta_2)}\sum_{m=1}^{k}(-1)^{k-m}C_{\mu_2}^{k-m}\sum_{s=1}^{m}C_{\mu_2}^{m-s}e^{is\theta}.$$

Then

$$\left.\begin{array}{ll}F^+(z)=\sum\limits_{n=0}^{\infty}\int\limits_{-\pi}^{\pi}h_n^+(\theta)f(\theta)\,d\theta z^n, & |z|<1\\ F^-(z)=-\sum\limits_{k=1}^{\infty}\int\limits_{-\pi}^{\pi}h_k^-(\theta)f(\theta)\,d\theta z^{-k}, & |z|>1\end{array}\right\}\tag{8}$$

Now let's prove the validity of relations (4). Put in (6) $f(\theta)=e^{i[(m+\alpha_1)\theta+\beta(\theta)]}$, $m\geq 0$. In this case solution of the Riemann problem will be:

$$F^+(z)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{e^{i(m+1)\theta}d\theta}{\left(e^{i\theta}+1\right)_{-1}^{\mu_1}\left(e^{i\theta}-1\right)_{+1}^{\mu_2}\left(e^{i\theta}-z\right)}(z+1)_{-1}^{\mu_1}(z-1)_{+1}^{\mu_2}$$

$$F^-(z)=\frac{e^{2\beta_2 i}}{2\pi}\int\limits_{-\pi}^{\pi}\frac{e^{i(m+1)\theta}d\theta}{\left(e^{i\theta}+1\right)_{-1}^{\mu_1}\left(e^{i\theta}-1\right)_{+1}^{\mu_2}\left(e^{i\theta}-z\right)}\left(\frac{z+1}{z}\right)_{-1}^{\mu_1}\left(\frac{z-1}{z}\right)_{+1}^{\mu_2}.$$

After the substitution $\tau=e^{i\theta}$ these integrals have the following form:

$$F^+(z)=\frac{1}{2\pi i}\int\limits_{L}\frac{\tau^m d\tau}{(\tau+1)_{-1}^{\mu_1}(\tau-1)_{+1}^{\mu_2}(\tau-z)}(z+1)_{-1}^{\mu_1}(z-1)_{+1}^{\mu_2}$$

$$F^-(z)=\frac{e^{2\beta_2 i}}{2\pi}\int\limits_{L}\frac{\tau^m d\tau}{(\tau+1)_{-1}^{\mu_1}(\tau-1)_{+1}^{\mu_2}(\tau-z)}\left(\frac{z+1}{z}\right)_{-1}^{\mu_1}\left(\frac{z-1}{z}\right)_{+1}^{\mu_2}.$$

Based on lemma we obtain that

$$F^+(z)=\begin{cases}z^m, & |z|<1,\\ 0, & |z|>1,\end{cases}\qquad F^-(z)=\begin{cases}e^{2\beta_2 i}z^m z_{-1}^{-\mu_1}z_{+1}^{-\mu_2},\\ 0, \quad |z|>1.\end{cases}$$

Comparing the obtained expressions $F^{\pm}(z)$ with (8) we have:

$$\int\limits_{-\pi}^{\pi}h_n^+(\theta)e^{i[(m+\alpha_1)\theta+\beta(\theta)]}d\theta=\delta_{nm},\qquad\int\limits_{-\pi}^{\pi}h_k^-(\theta)e^{i[(m+\alpha_1)\theta+\beta(\theta)]}d\theta=0$$

where $n,m\geq 0$, $k\geq 1$.

Let's prove now the second pair of relations (4).

If we put in (6) $f(\theta)=e^{-i[(s+\alpha_2)\theta+\beta(\theta)]}$, $s\geq 1$ then

$$F^+(z)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{e^{-i[\alpha_1\theta+\alpha_2\theta+2\beta(\theta)]}e^{-is\theta}d\theta}{\left(e^{i\theta}+1\right)_{-1}^{\mu_1}\left(e^{i\theta}-1\right)_{+1}^{\mu_2}\left(1-ze^{-i\theta}\right)}(z+1)_{-1}^{\mu_1}(z-1)_{+1}^{\mu_2}$$

$$F^-(z)=\frac{e^{2\beta_2 i}}{2\pi}\int\limits_{-\pi}^{\pi}\frac{e^{-i[\alpha_1\theta+\alpha_2\theta+2\beta(\theta)]}e^{-is\theta}d\theta}{\left(e^{i\theta}+1\right)_{-1}^{\mu_1}\left(e^{i\theta}-1\right)_{+1}^{\mu_2}\left(1-ze^{-i\theta}\right)}\left(\frac{z+1}{z}\right)_{-1}^{\mu_1}\left(\frac{z-1}{z}\right)_{+1}^{\mu_2}.$$

is the solution of the Riemann problem (5).

At first let's calculate $F^{-}(z)$. For this we'll make the substitution $\tau = e^{i\theta}$ in the integral. Then

$$F^{-}(z) = \frac{e^{2\beta_2 i}}{2\pi i}\int\limits_L \frac{\tau_{-1}^{-\alpha_1}\tau_{-1}^{-\alpha_2}e^{-2i\beta(\arg\tau)}\tau^{-s}d\tau}{(\tau+1)_{-1}^{\mu_1}(\tau-1)_{+1}^{\mu_2}(\tau-z)}\left(\frac{z+1}{z}\right)_{-1}^{\mu_1}\left(\frac{z-1}{z}\right)_{+1}^{\mu_2} =$$

$$= \frac{e^{2\beta_2 i}}{2\pi i}\int\limits_L \frac{\tau_{-1}^{-(\alpha_1+\alpha_2)}e^{-2i\beta(\arg\tau)}\tau^{-s}d\tau}{\tau_{-1}^{\mu_1}\left(\frac{\tau+1}{\tau}\right)_{-1}^{\mu_1}\tau_{+1}^{\mu_2}\left(\frac{\tau-1}{\tau}\right)_{+1}^{\mu_2}(\tau-z)}\left(\frac{z+1}{z}\right)_{-1}^{\mu_1}\left(\frac{z-1}{z}\right)_{+1}^{\mu_2} =$$

$$= (J_1+J_2)\left(\frac{z+1}{z}\right)_{-1}^{\mu_1}\left(\frac{z-1}{z}\right)_{+1}^{\mu_2}$$

where

$$J_1 = \frac{e^{2\beta_2 i}}{2\pi i}\int\limits_{L^-} \frac{\tau_{-1}^{-(\alpha_1+\alpha_2)}e^{-2i\beta(\arg\tau)}\tau^{-s}d\tau}{\tau_{-1}^{\mu_1}\left(\frac{\tau+1}{\tau}\right)_{-1}^{\mu_1}\tau_{+1}^{\mu_2}\left(\frac{\tau-1}{\tau}\right)_{+1}^{\mu_2}(\tau-z)},$$

$$J_2 = \frac{e^{2\beta_2 i}}{2\pi i}\int\limits_{L^+} \frac{\tau_{-1}^{-(\alpha_1+\alpha_2)}e^{-2i\beta(\arg\tau)}\tau^{-s}d\tau}{\tau_{-1}^{\mu_1}\left(\frac{\tau+1}{\tau}\right)_{-1}^{\mu_1}\tau_{+1}^{\mu_2}\left(\frac{\tau-1}{\tau}\right)_{+1}^{\mu_2}(\tau-z)},$$

here L^- and L^+ are parts of a circle L in the lower and upper half-planes, respectively. Note that $-(\alpha_1+\alpha_2) = \mu_1+\mu_2$.

If $\tau \in L^-$ then $\beta(\arg\tau) = \beta_1$, $\tau_{+1}^{\mu_2} = e^{2(\beta_2-\beta_1)i}\tau_{-1}^{\mu_2}$.
Therefore

$$J_1 = \frac{e^{2\beta_2 i}}{2\pi i}\int\limits_{L^-} \frac{\tau_{-1}^{\mu_1+\mu_2}e^{-2i\beta_1}\tau^{-s}d\tau}{\tau_{-1}^{\mu_1}\left(\frac{\tau+1}{\tau}\right)_{1}^{\mu_1}e^{2(\beta_2-\beta_1)i}\tau_{-1}^{\mu_2}\left(\frac{\tau-1}{\tau}\right)_{+1}^{\mu_2}(\tau-z)} =$$

$$= \frac{1}{2\pi i}\int\limits_{L^-} \frac{\tau^{-s}d\tau}{\left(\frac{\tau+1}{\tau}\right)_{-1}^{\mu_1}\left(\frac{\tau-1}{\tau}\right)_{+1}^{\mu_2}(\tau-z)}.$$

If $\tau \in L^+$ then $\beta(\arg\tau) = \beta_2$, $\tau_{+1}^{\mu_2} = \tau_{-1}^{\mu_1}$.
Therefore

$$J_2 = \frac{e^{2\beta_2 i}}{2\pi i}\int\limits_{L^+} \frac{\tau_{-1}^{\mu_1+\mu_2}e^{-2i\beta_1}\tau^{-s}d\tau}{\tau_{-1}^{\mu_1}\left(\frac{\tau+1}{\tau}\right)_{-1}^{\mu_2}\tau_{-1}^{\mu_2}\left(\frac{\tau-1}{\tau}\right)_{+1}^{\mu_2}(\tau-z)} =$$

$$= \frac{1}{2\pi i}\int\limits_{L^+} \frac{\tau^{-s}d\tau}{\left(\frac{\tau+1}{\tau}\right)_{-1}^{\mu_1}\left(\frac{\tau-1}{\tau}\right)_{+1}^{\mu_2}(\tau-z)}.$$

Hence we obtain that

$$F^{-}(z) = \frac{1}{2\pi i}\int\limits_{L} \frac{\tau^{-s}d\tau}{\left(\frac{\tau+1}{\tau}\right)_{-1}^{\mu_1}\left(\frac{\tau-1}{\tau}\right)_{+1}^{\mu_2}(\tau-z)}\left(\frac{z+1}{z}\right)_{-1}^{\mu_1}\left(\frac{z-1}{z}\right)_{+1}^{\mu_2}.$$

Based on lemma 2 we obtain that

$$F^{-}(z)=\begin{cases} -z^{-s}, & |z|>1 \\ 0\ , & |z|<1. \end{cases}$$

From the relation $F^{+}(z)=e^{-2\beta_2 i}z_{-1}^{\mu_1}z_{+1}^{\mu_2}F^{-}(z)$ it follows that

$$F^{+}(z)=\begin{cases} -e^{-2\beta_2 i}z_{-1}^{\mu_1}z_{+1}^{\mu_2}z^{-s}, & |z|>1 \\ 0\ , & |z|<1. \end{cases}$$

Comparing the obtained expressions $F^{\pm}(z)$ with (8) we have:

$$\int_{-\pi}^{\pi} h_n^{+}(\theta)\, e^{-i[(s+\alpha_2)\theta+\beta(\theta)]}d\theta=0$$

$$\int_{-\pi}^{\pi} h_k^{-}(\theta)\, e^{-i[(s+\alpha_2)\theta+\beta(\theta)]}d\theta=\delta_{ks},\ n\geq 0,\quad k,s\geq 1\ .$$

The theorem is proved.

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