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THE LOCAL LIMIT THEOREM FOR ONE CLASS OF FIRST PASSAGE MOMENTS IN MULTIDIMENSIONAL RANDOM WALK

Abstract

In the work the local limit theorem for the first passage moment in the multidimensional random walk is proved.

Introductions. Let $\xi_n, n \geq 1$ be a sequence of independent identically distributed random vectors in R^k, $k \geq 1$, with mean $\nu = E\xi_1$ and matrix of covariation B and let $\Delta(x)$, $x \in R^k$ be some numerical Borel function.

Let's suppose at $n \geq 1$ and $c \geq 0$

$$S_n = \sum_{k=1}^{n} \xi_k, \; T_n = n\Delta\left(S_n/n\right),$$

$$\tau = \tau_c = \inf\{n \geq 1 : T_n > c\} \text{ and } \chi_x = T_\tau - c.$$

The moments of stopping, often arising in the problems of sequential analysis, are obtained from τ_c at corresponding choice of function $\Delta(x)$.

In one-dimensional case (i.e. case $k = 1$) a large quantity of works (see [1], [5]) are dedicated to the study of asymptotic properties of the Markov's moment τ_c as $c \to \infty$. Multidimensional case is studied substantially less. The works [5], [6] have the series of the common results in this direction.

At an estimation of probability of the events of type $\{\tau_c \in (a, b)\}$ local limiting theorems (L.L.T)as which it is understood any statements, that under some conditions there exists a function $P(n, c)$, for which the following correlation is fulfilled, are used

$$P\left(\tau_c = n\right) = P\left(n, c\right)\left(1 + o(1)\right)$$

as $c \to \infty$.

In case one-dimensional random variables LLT for τ_c were studied in the works [1] and [2] at the different suppositions with respect to distribution of the random variable ξ_1 and function $\Delta(x)$.

Our purpose is to study LLT for τ_c in multidimensional case $(k > 1)$.

Conditions and notion. Let $a = (a_1 ... a_k) \in R^k$ and $b = (b_1 ... b_k) \in R^k$. We'll denote $a \leq b$ if $a_i \leq b_i$ for each $i = \overline{i, k}$.

Let's denote by L a class of functions $\Delta(x)$, $x \in R^k$ for which the following conditions are fulfilled: $\Delta(x)$ has continuous partial derivatives $\Delta_{x_i}(x)$, $i = \overline{i, k}$ at some neighborhood of point $x = \nu$, at that $\Delta(\nu) > 0$ and at least for one $i = \overline{i, k}$ $\Delta_{x_i}(\nu) \neq 0$.

Let's suppose

$$D = D\left(d, y\right) = \bigcup_{m} \{x \in R^k : (d, x - y) = m\},$$

where (x,y) here and further means ordinary scalar product of the vectors $x,y\in R^k$.

If there exists $y\in R^k$ and $d\in R^k$ such that $P(\xi_1\in D)=1$, then it means, that random vector ξ_1 has a lattice distribution with parameters $d\in R^k$. Otherwise, the distribution is called nonlattice.

Let's denote

$$H(x)=\frac{1}{EZ_{\tau_+}}\int_0^x P\left(Z_{\tau_+}>y\right)dy,$$

where

$$Z_n=n\Delta(\nu)+(\lambda,S_n-n\nu),$$

$$\lambda=(\Delta_{x_1}(\nu),...,\Delta_{x_k}(\nu)),$$

and

$$\tau_+=\inf\{n\geq 1: Z_n>0\}.$$

Let's denote also by $\varphi_\gamma(x)$ density of the normal distribution with parameters $(0,\gamma)$, where

$$\gamma=\sum_{i,j=1}^{k} cov\left(\eta_i,\eta_j\right)\Delta'_{x_i}(\nu)\,\Delta'_{x_j}(\nu)$$

and $\eta=(\eta_1,\eta_2,...,\eta_k)$ has k-dimensional normal distribution with covariation matrix B and $E\eta=0$.

Formulation and proof of the basic result.

Theorem. *Let $\xi_n, n\geq 1$ be a sequence of independent identically distributed nonlattice random vectors with a vector of the mathematical expectations $\nu=E\xi_1$ and matrix of covariation B and let $\Delta(x)\in L$.*

If

$$n=n(c)=\frac{c}{\Delta(\nu)}+\theta(c)\sqrt{\frac{c}{\Delta(\nu)}},$$

at that $\theta(c)\to\theta\in R$ at $c\to\infty$, then

$$P(\tau_c=n,\chi_c\leq x)\sim\frac{\Delta(\nu)}{\sqrt{n}}\varphi_\gamma(-\theta\Delta(\nu))H(x),\ c\to\infty$$

uniformly by $x, 0<\delta\leq x\leq M<\infty$ and θ from the bounded set in R.

Corollary 1. In conditions of the theorem it holds

$$P(\tau_c=n)\sim\frac{\Delta(\nu)}{\sqrt{n}}\varphi_\gamma(-\theta\Delta(\nu)),\ c\to\infty.$$

Corollary 2. At fulfilling the conditions of the theorem it holds

$$\lim_{c\to\infty}P(\chi_c\leq x|\ \tau_c=n)=H(x),\ x>0.$$

These statements for one-dimensional case have been proved in [1] and generalized in [2].

For proof of the theorem we need the following statements, stated in Lemmas.

Lemma 1. *Let $\eta_n, n \geq 1$ be the independent distributed random variables with $\sigma^2 = D\eta_1 < \infty$ and $a = E\eta_1$. In order that the correlation was fulfilled*

$$P\left(\sum_{k=1}^{n} \eta_k \in x + B\right) = \frac{\mu(B)}{\sigma\sqrt{n}} \varphi\left(\frac{x - na}{\sigma\sqrt{n}}\right) + 0(1/\sqrt{n})$$

for any bounded convex Borel set B in R and uniformly by $x \in R$, it is necessary and sufficient, that the distribution of the random variable η_1 was nonlattice. Here μ is Lebesque measure in R and $\varphi(x)$ is a density of the standard normal distribution.

Concerning this result see [7].

Lemma 2. *Let random vector ξ_1, have nonlattice distribution and $\Delta(x) \in L$. Then*

$$\lim_{c \to \infty} P(\chi_c \leq x) = H(x), \ x > 0.$$

The statement of this lemma is a particular case of theorem 4.1 from the work [1].

Lemma 3. *It holds*

$$H'(x) = \frac{1}{\Delta(\nu)} P(Z_i > x, i \geq 1), \ x > 0.$$

This lemma follows from theorem 2.7 of the work [1].

Proof of the theorem. Let's devide an interval $(0, x]$ into m equal parts and suppose.

$$l_j = \left(\frac{j-1}{m}x, \ \frac{j}{m}x\right], \ j = \overline{1, m}.$$

By the formula of total probability we have:

$$P(\tau_c = n, \chi_c \leq x) = \sum_{j=1}^{n} P(\tau_c = n, T_n \in c + l_j) =$$

$$= \sum_{j=1}^{n} P(\tau_c = n / T_n \in c + l_j) \, P(T_n \in c + l_j). \quad (1)$$

We'll estimate conditional probability

$$Q_{n,j}(c) = P(\tau_c = n / T_n \in c + l_j).$$

It is easy to see, that

$$Q_{n,j}(c) = P(\tau_c \geq n / T_n \in c + l_j) = P(T_i \leq c, 1 \leq i < n / T_n \in c + l_j) =$$

$$= P(T_n - T_i \geq T_n - c, 1 \leq i < n / T_n \in c + l_j) =$$

$$= P(T_n \leq T_{n-i} \geq T_n - c, 1 \leq i < n / T_n \in c + l_j).$$

Hence we have

$$P\left(T_n - T_{n-i} \geq \frac{j}{m}x, 1 \leq i < n / T_n \in c + l_j\right) \leq Q_{n,j}(c) \leq$$

$$\leq P\left(T_n - T_{n-i} \geq \frac{j-1}{m}x, 1 \leq i < n/T_n \in c + l_j\right).$$

Let's denote

$$\Pi_{n,j}(x,c) = P\left(T_n - T_{n-1} \geq \frac{j}{m}x, 1 \leq i < n/T_n \in c + l_j\right).$$

From (1) we obtain, that

$$\sum_{j=1}^{m}\Pi_{n,j}(x,c)P\left(T_n \in c + l_j\right) \leq P\left(\tau_c = n, \chi_c \leq x\right) \leq$$

$$\leq \sum_{j=1}^{m}\Pi_{n,j-1}(x,c)P\left(T_n \in c + l_j\right). \qquad (2)$$

Further we'll need the following lemmas:

Lemma 4. *For any $\varepsilon > 0$ there exists entire q_1, such that for sufficiently large c and for all r and x from the bounded set we have*

$$\max_{j \leq m} P\left(J_{ni} \leq r,\ \exists i \in [q_1, n)\, / T_n \in c + l_j\right) < \varepsilon,\ J_{ni} = T_n - T_{n-1}. \qquad (3)$$

Estimation (3) for the case $k = 1$ has been proved in the work [2]. In general case it is proved with the help of analogous discussions, conducted in the work [2].

Lemma 5. J*oint conditional distribution of the random variables*

$$J_{n1} = T_n - T_{n-1},\ J_{n2} = T_n - T_{n-2}, ...J_{np} = T_n - T_{n-p}$$

at condition $\frac{1}{n}S_n = y$ ($y \in R^k$ is fixed) is weakly convergent to the unconditional distribution of the random variables $Z_1, Z_1, ..., Z_p$ and it holds

$$\lim_{n\to\infty} P\left(J_{ni} \geq x, 1 \leq i \leq p/T_n \in c + l_j\right) = P\left(Z_i \geq x, 1 \leq i \leq p\right). \qquad (4)$$

At $k = 1$ lemma 5 is proved with the help of techniques from [4].

In multidimensional case it is proved with the help of reasons from the works [2] and [4], at that lemma 7 from the work [1] is used.

Let's continue the proof of the theorem.

Since $EZ_1 = \Delta(\nu) > 0$, then from the intensive law of large numbers implies, that for any $\varepsilon > 0$ there exits sufficiently large integer q_2 such that

$$P\left(Z_j \leq x, \exists i > q_2\right) < \varepsilon. \qquad (5)$$

Supposing $q = \max\left(q_1, q_2\right)$ instead of q_1 in (3) and instead of q_2 in (5) we have

$$P\left(J_{ni} \geq x, 1 \leq i \leq q/T_n \in c + l_j\right) - \varepsilon \leq P\left(J_{ni} \geq x, 1 \leq i \leq n/T_n \in c + l_j\right) \leq$$

$$\leq P\left(J_{ni} \geq x, 1 \leq i \leq q/T_n \in c + l_j\right) \qquad (6)$$

and

$$P\left(J_i \geq x, 1 \leq i \leq q\right) - \varepsilon \leq P\left(Z_i \geq x, 1 \leq i \leq q\right). \qquad (7)$$

From (4)-(6) and (7) at $p = q$ we obtain, that

$$P\left(Z_i \geq x, 1\right) - 2\varepsilon \leq P\left(J_{ni} \geq x, 1 \leq i < n/T_n \in c + l_j\right) \leq$$

$$\leq P\left(Z_i \geq x, i \geq 1\right) + 2\varepsilon. \tag{8}$$

Taking into account (8), from (4) we have:

$$\sum_{j=1}^{m}\left(P\left(Z_i \geq \frac{j}{m}x, i \geq 1\right) - 2\varepsilon\right)P\left(T_n \in c + l_j\right) \leq P\left(\tau_c = n, \chi_c \leq x\right) \leq$$

$$\leq \sum_{j=1}^{m}\left(P\left(Z_i \geq \frac{j-1}{m}x, i \geq 1\right) + 2\varepsilon\right)P\left(T_n \in c + l_j\right). \tag{9}$$

Now let's estimate probability of $P\left(T_n \in c + l_j\right)$. By virtue of lemma 1 and integral limit theorem for τ_c, announced in [3], we'll obtain

$$P(Z_n \in c + l_j) = \frac{\frac{x}{m}}{\sqrt{n}}\varphi_\gamma\left(\frac{c - n\Delta(\nu)}{\sqrt{n}}\right) + o\left(\frac{1}{\sqrt{n}}\right)$$

or

$$P(Z_n \in c + l_j) \sim \frac{\frac{x}{m}}{\sqrt{n}}\varphi_\gamma\left(-\theta\Delta\left(\nu\right)\right). \tag{10}$$

Further we'll note, that at the given conditions with respect to function $\Delta\left(x\right)$ we have"

$$T_n = Z_n + \varepsilon_n,$$

where

$$Z_n = \sum_{i=1}^{m} X_i,\ \ X_i = \Delta\left(\nu\right) + \left(\lambda, \xi_i - \nu\right),$$

$$\varepsilon_n = n\left(\frac{1}{n}S_n - \nu, \lambda_n - \lambda\right),\quad \lambda_n = \left(F'_{x_i}(\nu_n)...F'_{x_k}\left(\nu_n\right)\right)$$

and $\nu_n, n \geq 1$ is some sequence of the random points from the neighbourhood of the point ν, at that $\nu_n \xrightarrow{a} \nu,\ n \to \infty$.

Taking into account, that

$$\frac{\varepsilon_n}{\sqrt{n}} \xrightarrow{P} \text{ at } n \to \infty,$$

from (10) we obtain

$$P(T_n \in c + l_j) \sim \frac{\frac{x}{m}}{\sqrt{n}}\varphi_\gamma\left(-\theta\Delta\left(\nu\right)\right). \tag{11}$$

Substituting (11) in (9) we have

$$\varphi_\gamma\left(-\theta\Delta(\nu)\right)(1-\varepsilon)\sum_{j=1}^{m}\frac{x}{m}P\left(Z_i \geq \frac{j}{m}x, i \geq 1\right) - 2\varepsilon \leq \sqrt{n}P\left(\tau_c = n, \chi_c \leq x\right) \leq$$

$$\leq (1+\varepsilon)\,\varphi_\gamma\left(-\theta\Delta(\nu)\right)\sum_{j=1}^{m}\frac{x}{m}P\left(Z_i\geq\frac{j-1}{m}x, i\geq 1\right)+2\varepsilon.$$

According to lemma 3

$$P\left(Z_i>x, i\geq 1\right)=\frac{\Delta\left(\nu\right)}{EZ_{\tau_+}}P\left(Z_{\tau_+}>x\right).$$

Therefore from (12), choosing first of all ε arbitrary small, and then m sufficiently large, we find that

$$P\left(\tau_c=n,\ \chi_c\leq x\right)=\frac{\Delta(\nu)}{\sqrt{n}}\varphi_\gamma\left(-\theta\Delta\left(\nu\right)H(x)+o\left(\frac{1}{\sqrt{n}}\right)\right.$$

uniformly by x from the bounded set in $[\delta,\infty)$ with $\delta>0$ and by θ from the bounded set in R.

The theorem is proved.

Corollary 1 is obtained from the theorem as $x\to\infty$ and corollary 2 implies from the theorem and corollary 1.

Remark. Lemma 2 and corollary 2 show that quantities τ_c and χ_c as $c\to\infty$ are asymptotically independent (see, also [1]).

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