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TO THE M.RIESZ THEOREM ON ABSOLUTE CONVERGENCE OF THE TRIGONOMETRIC FOURIER SERIES (THE SECOND REPORT)

Abstract

This paper is a continuation of the author's investigations in the same name paper on the extension of the known M.Riesz criterion for absolute convergence of trigonometric Fourier series of continuous functions for values $p \neq 2$. *The case of functions* $f \in L_p(T)$, $g \in L_q(T)$ *generating the convolution* $h = f * g$ *are considered, where* $1 < p, q \leq 2$. *The exact upper estimate of* $l^{r'}$ *norm of sequence of Fourier coefficients of the convolution by product of norms* $\|f\|_p \cdot \|g\|_q$, *where* $r' = pq/(2pq - p - q) \in [1, \infty)$, *as well as the upper estimate of residual series generating above mentioned* $l^{r'}$ *norm by product of the best (in metrics* $L_p(T)$ *and* $L_q(T)$, *respectively) approximations* $E_{n-1}(f)_\rho \cdot E_{n-1}(g)_q$, $n \in N$, *of these functions are obtained, and its exactness in the sence of the order in the scale of power majorants was proved.*

Let $L_p(T)$, $1 \leq p < \infty$, be the space of all measurable 2π-periodic functions $f : R \to \mathbf{C}$ with the finite norm $\|f\|_p = \left((1/2\pi)^{-1} \int_T |f(x)|^p \, dx\right)^{1/p} < \infty$, $C(T) \equiv L_\infty(T)$ be the space of all continuous 2π periodic functions, $\|f\|_\infty = \max\{|f(x)|;\ x \in T\}$, where $T = [-\pi, \pi]$. For a function $f \in L_1(T)$ with the Fourier-Lebesque series

$$f(x) \sim \sum\nolimits_{n \in Z} c_n\,(f)\, e^{inx},\ x \in T, \tag{1}$$

put $\rho_n^{(\gamma)}(f) = \left(\sum_{|\nu|=n}^{\infty} |c_\nu(f)|^\gamma\right)^{1/\gamma}$, $\gamma \in (0, \infty)$, $n \in Z_+$.

It is obvious that if $\rho_0^{(\gamma)}(f) < \infty$ then $\rho_0^{(\gamma)}(f) \downarrow 0\,(n \uparrow \infty)$; besides, it is clear that the condition $\rho_0^{(1)}(f) < \infty$ provides absolute and uniform convergence of series (1) everywhere on T, moreover $\|f(\cdot) - S_n(f;\cdot)\|_\infty \leq \rho_n^{(1)}(f;x)$, where $S_n(f;x)$ are partial sums of series (1) of order $n \in Z_+ : S_n(f;x) = \sum_{|\nu|=0}^{n} c_\nu(f) e^{i\nu x}$. It is also obvious that the absolute convergence of series (1) everywhere on T implies $\rho_0^{(1)}(f) < \infty$.

The convolution $h = f * g$ of the functions $f \in L_1(T)$ and $g \in L_1(T)$ is defined by the formula $h(x) = (f * g)\ (x) = (1/2\pi) \int_T f(x-y) g(y) dy$. It is known (see f.e. [1], v.1, § 2.1, pp. 64-65, [2], v.1, § 3.1, pp. 65-66) that the function h is determined almost everywhere, 2π periodic, measurable and $\|h\|_1 \leq \|f\|_1 \cdot \|g\|_1$, hence, in particular, it follows that $h = f * g \in L_1(T)$. The last statement is a special case of the following result known under the name of W.Young inequality (see f.e. [1], v.1, theorem (1.15), pp.67-68; [2], v.2, theorem 13.6.1, pp.176-177; [2], v.1, theorem 3.1.4, p.70, theorem 3.1.6, p.72):

Theorem A. *Let* $1 \leq p,\ q \leq \infty,\ 1/r = 1/p+1/q-1 \geq 0, f \in L_p(T),\ g \in L_q(T),$ $h = f*g$; *then* $h \in L_r(T)$ *and* $\|h\|_r \leq \|f\|_p \cdot \|g\|_q$. *When* $1/p+1/q = 1$, *i.e.* $q = p'$ *is an exponent conjugate to* p ($p' = 1$ *for* $p = \infty$ *and* $p' = \infty$ *for* $p = 1$), *the function* h *is determined everywhere, continuous and* $\|h\|_\infty \leq \|f\|_p \cdot \|g\|_{p'}$.

We also note that the Fourier coefficients $c_n(h)$ of the convolution $h = f * g$ of two functions $f \in L_1(T)$ and $g \in L_1(T)$ are calculated by the formula (see [1], v.1, theorem (1.5), p. 64; [2], v.1, p.66, formula (3.1.5))

$$c_n(h) = c_n(f * g) = c_n(f) \cdot c_n(g), n \in Z, \tag{2}$$

such that

$$h(x) \sim \sum_{n\in Z} c_n(f) \cdot c_n(g)e^{inx},\ x \in T. \tag{3}$$

Denote by $A^{(\gamma)}(T)$ the class of all functions $f \in L_1(T)$ for which $\rho_0^{(\gamma)}(f) < \infty$ $\left(A^{(1)}(T) \equiv A(T)\right)$. By virtue of M.Riesz criterion on absolute convergence of trigonometric Fourier series of continuous functions (see [4], §9.7, pp. 634-635; [1], v.1, ch.6, theorem 6 on p. 399; [5], §2.2, p.17; [2], v.1, §10.6.2, remark (4) on p.208) the convolution $h = f*g$ of any two functions $f \in L_2(T)$ and $g \in L_2(T)$ belongs the class $A(T)$. In the case $1 \leq p < 2$ the correspondity statement does not hold, more exactly, for any $p \in [1, 2)$ there exist functions $f_0(\cdot; p),\ g_0(\cdot; p) \in L_p(T)$, such that their convolution $h_0 = f_0 * g_0 \notin A(T)$ (see for example, [5], Example 1 (case $p = 1$) and Example 2 (case $1 < p < 2$)).

In the paper [6] (theorem 4 A on p. 53) the following was proved.

Theorem B. *If functions* $f \in L_p(T),\ g \in L_p(T)$ *for some* $p \in (1, 2]$, *then their convolution* $h = f * g \in A^{(p'/2)}(T)$, *where* $p' = p/(p-1)$.

In this paper [6] (p.53, theorem 5) it was proved that the statement of Theorem B is exact, namely, for each $p \in (1, 2]$ there exist the functions $f_0(\cdot; p) \in L_p(T)$, $g_0(\cdot; p) \in L_p(T)$, such that their convolution $h_0 = f_0 * g_0 \notin A^{(\gamma)}(T)$ for any number $\gamma < p'/2$, i.e. we cannot decrease the exponent $p'/2 \geq 1$ in the statement of Theorem B (see f.e. Example 3 in [5]). Consequently, since $p'/2 > 1$ at $1 < p < 2$ then a fortiory $h_0 = f_0 * g_0 \notin A(T)$ in the case $p \in (1, 2)$ (see Example 2 in [5]).

Theorem 1. *Let* $1 < p \leq 2,\ 1 < q \leq 2, f \in L_p(T),\ g \in L_q(T),\ h = f * g,$ $r = pq/(p+q-pq),\ r' = pq/(2pq-p-q) = p'q'/(p'+q')$, *where* $r \in (1, \infty]$, $1/r + 1/r' = 1/p + 1/p' = 1/q + 1/q' = 1$; *then*

1) $h \in L_r(T)$ *in the case* $r < \infty$ (*i.e. if* $1 < p \leq 2,\ 1 < q < 2$ *or* $1 < p < 2,$ $1 < q \leq 2$), *and* $\|h\|_r \leq \|f\|_p \cdot \|g\|_q$;

$h \in C(T)$ *in the case* $r = \infty$ (*i.e., if* $p = q = 2$), *and* $\|h\|_\infty \leq \|f\|_2 \cdot \|g\|_2$;

2) $h \in A^{(r')}(T)$ *and* $\rho_0^{(r')}(h) = \left(\sum_{|n|=0}^{\infty} |c_\nu(h)|^{r'}\right)^{1/r'} \leq \|f\|_p \cdot \|g\|_q$;

3) $\rho_n^{(r')}(h) = \left(\sum_{|\nu|=n}^{\infty} |c_\nu(h)|^{r'}\right)^{1/r'} \leq M(p)M(q)\cdot E_{n-1}(f)_p \cdot E_{n-1}(g)_q,\ n\in N,$

where $M(p)$ *is the constant in the known M.Riesz inequality (see f.e. [4], § 8.20, p.594; [2], v.2, § 12.10, p.120; [7], § 5.11, p.339)*

$$\|\varphi(\cdot) - S_n(\varphi;\cdot)\|_p \leq M(p)\cdot E_n(\varphi)_p,\ n\in Z_+, \tag{4}$$

$1<p<\infty,\ \varphi\in L_p(T),\ E_n(\varphi)_p$ *is the best approximation of the function* φ in $L_p(T)$ *metric by trigonometric polynomials of order* $\leq n$.

Proof. 1) The statement $h\in L_r(T)$ in the case $r<\infty$ and $h\in C(T)$ in the case $r=\infty$ is the obvious consequence of Theorem A: $\|h\|_r \leq \|f\|_p\,\|g\|_q$, $1/r = 1/p+1/q-1>0$ and $\|h\|_\infty \leq \|f\|_2\cdot\|g\|_2$, $1/r=0$ $\ (\Longrightarrow r=\infty \Longleftrightarrow p=q=2)$;

2) By virtue of equality $1/r' = 2-(1/p+1/q) = (1-1/p)+(1-1/q) = = 1/p'+1/q' = (p'+q')/p'q'$, we obtain $r' = p'q'/(p'+q')$, whence

$$\sum_{|n|=0}^{\infty} |c_n(h)|^{r'} = \sum_{|n|=0}^{\infty} |c_n(f)|^{r'}\cdot|c_n(g)|^{r'} = \sum_{|n|=0}^{\infty} |c_n(f)|^{p'\cdot r'/p'}\,|c_n(g)|^{q'\cdot r'/q'},$$

and applying the Hōlder inequality with the exponents $s = p'/r' = 1+p'/q' > 1$ and $s' = q'/r' = 1+q'/p' > 1$ $(1/s+1/s'=1)$, we obtain

$$\sum_{|n|=0}^{\infty} |c_n(h)|^{r'} \leq \left(\sum_{|n|=0}^{\infty} |c_n(f)|^{p'}\right)^{r'/p'}\cdot\left(\sum_{|n|=0}^{\infty} |c_n(g)|^{q'}\right)^{r'/q'}.$$

Hence, by virtue of the first part of Hausdorff - Young theorem (see f.e. [1], v.2, §12.2, theorem (2.3) on p.153; [2], v.2, §13.5, theorem 13.5.1 on p. 172; [4], § 2.4, p.211) we have $(1<p,\ q\leq 2)$

$$\rho_0^{(r')}(h) = \left(\sum_{|n|=0}^{\infty} |c_n(h)|^{r'}\right)^{1/r'} \leq$$

$$\leq \left(\sum_{|n|=0}^{\infty} |c_n(f)|^{p'}\right)^{1/p'}\left(\sum_{|n|=0}^{\infty} |c_n(g)|^{q'}\right)^{1/q'} \leq \|f\|_p\cdot\|g\|_q;$$

3) Fix arbitrary $n\in N$ and denote $(x\in T)$

$$f_{n-1}(x) = f(x) - S_{n-1}(f;x) \sim \sum_{|\nu|=n}^{\infty} c_\nu(f)e^{i\nu x},$$

$$g_{n-1}(x) = g(x) - S_{n-1}(g;x) \sim \sum_{|\nu|=n}^{\infty} c_\nu(g)e^{i\nu x};$$

then, by virtue of (2) and (3), we have

$$h_{n-1}(x) = f_{n-1}(x) * g_{n-1}(x) \sim \sum_{|\nu|=n}^{\infty} c_\nu(f)\cdot c_\nu(g)e^{i\nu x} = h(x) - S_{n-1}(h;x),$$

and consequently, by virtue of estimate in 2) of the present theorem and M.Riesz inequality (4), we obtain

$$\rho_n^{(r')}(h) \equiv \rho_0^{(r')}(h_n) = \left(\sum_{|\nu|=n}^{\infty} |c_\nu(f)\cdot c_\nu(g)|^{r'}\right)^{1/r'} \leq \|f_{n-1}(\cdot)\|_p\cdot\|g_{n-1}(\cdot)\|_q =$$

$$= \|f(\cdot)-S_{n-1}(f;\cdot)\|_p\cdot\|g(\cdot)-S_{n-1}(g;\cdot)\|_q\le M(p)E_{n-1}(f)_p\cdot M(q)E_{n-1}(g)_q=$$

$$=M(p)\cdot M(q)\cdot E_{n-1}(f)_p\cdot E_{n-1}(g)_q.$$

Theorem 1 is proved.

Remark 1. Theorem 1 in the case of $1<p=q\le 2\,(\Longrightarrow r'=p'/2)$ is proved by the author in [5] (Theorem 1).

Remark 2. In the proof of point 3) of Theorem 1, the equality $h(x)-S_{n-1}(h;x)= =[f(x)-S_{n-1}(f;x)]*[g(x)-S_{n-1}(g;x)]$ was established. Using the obvious identity

$$f(x)*S_{n-1}(g;x)=g(x)*S_{n-1}(f;x)=S_{n-1}(f;x)*S_{n-1}(g;x)=S_{n-1}\left(f*g;x\right),$$

we can be convinced the validity of this equality:

$$[f(x)-S_{n-1}(f;x)]*[g(x)-S_{n-1}(g;x)]=$$

$$=f(x)*g(x)-S_{n-1}(f;x)*g(x)-f(x)*S_{n-1}(g;x)+S_{n-1}(f;x)*S_{n-1}(g;x)=$$

$$=f(x)*g(x)-S_{n-1}\left(f*g;x\right)=h(x)-S_{n-1}(h;x).$$

From this equality, by virtue of Theorem A ($r>1$ at $p>1,\ q>1$) and M.Riesz inequality (4), we have

$$E_{n-1}(h)_r\le\|h\left(\cdot\right)-S_{n-1}\left(h;\cdot\right)\|_r=\|f*g(\cdot)-S_{n-1}\left(f*g;\cdot\right)\|_r=$$

$$=\|[f(\cdot)-S_{n-1}\left(f;\cdot\right)]*[g(\cdot)-S_{n-1}(g;\cdot)]\|_r\le$$

$$\le\|f(\cdot)-S_{n-1}(f;\cdot)\|_p\cdot\|g(\cdot)-S_{n-1}(g;\cdot)\|_q\le M(p)E_{n-1}(f)_p\cdot M(q)E_{n-1}(g)_q,$$

whence the estimate $E_{n-1}(h)_r\le M(p)\cdot M(q)\cdot E_{n-1}(f)_p\cdot E_{n-1}(g)_q,\ n\in N$, follows.

The estimates in 1) and 2) of Theorem 1 are exact in the following sense: without loss of statement of the theorem in the point 1) we cannot increase the exponent $r\in(1,\infty]$ in the case of $r<\infty$, and substitute by no other one in the case of $r=\infty$; we cannot decrease the exponent $r'\in[1,\infty)$ ($r'=1$ for $p=q=2$) in 2), namely the following is valid.

Theorem 2. *For any* $p,\,q\in(1,2]$ *there exist functions* $f_0\left(\cdot;p\right)\in L_p(T)$ *and* $g_0(\cdot;q)\in L_q(T)$ *such that*

1) $h_0=f_0*g_0\notin L_\theta(T)$ *for every* $\theta>r$ *in the case of* $r<\infty$ *and* $\|h_0\|_\infty=\|f_0\|_2\cdot\|g_0\|_2$ *in the case of* $r=\infty$;

2) $h_0=f_0*g_0\notin A^{(\gamma)}(T)$ *for every* $\gamma<r'$.

Proof. Put $(1<p,\ q<\infty,\ p'=p/\left(p-1\right),\ q'=q/\left(q-1\right))$

$$f_0(x;p)=\sum\nolimits_{n=2}^{\infty}\left(n^{1/p'}\ln n\right)^{-1}e^{inx},\ g_0(x;q)=\sum\nolimits_{n=2}^{\infty}\left(n^{1/q'}\ln n\right)^{-1}e^{inx};$$

since

$$c_n(f_0) \equiv \left(n^{1/p'} \ln n\right)^{-1} \downarrow 0\,(n \uparrow \infty)\,,\ c_n\,(g_0) \equiv \left(n^{1/q'} \ln n\right)^{-1} \downarrow 0\,(n \uparrow \infty)$$

and

$$\sum\nolimits_{n=2}^{\infty} n^{p-2} c_n^p\,(f_0) = \sum\nolimits_{n=2}^{\infty} n^{p-2} n^{-p/p'} (\ln n)^{-p} = \sum\nolimits_{n=2}^{\infty} n^{-1} (\ln n)^{-p} < \infty,$$

$$\sum\nolimits_{n=2}^{\infty} n^{q-2} c_n^q\,(g_0) = \sum\nolimits_{n=2}^{\infty} n^{q-2} n^{-q/q'} (\ln n)^{-q} = \sum\nolimits_{n=2}^{\infty} n^{-1} (\ln n)^{-q} < \infty,$$

then by virtue of Hardy and Littlewood theorem (see f.e. [4], §10.3, pp.657-658, [1], v.2, §12.6, lemma (6.6) on p.193; [2], v.1, § 7.3.5, pp.148-149) $f_0\,(\cdot;p) \in L_p(T)$, $g_0(\cdot;q) \in L_q(T)$, moreover

$$\|f_0\|_p \asymp \left(\sum\nolimits_{n=2}^{\infty} n^{-1}\,(\ln n)^{-p}\right)^{1/p},\ \|g_0\|_q \asymp \left(\sum\nolimits_{n=2}^{\infty} n^{-1} (\ln n)^{-q}\right)^{1/q}.$$

1) For convolution $h_0 = f_0 * g_0$ of these functions (see above (2) and (3); $c_n\,(h_0) \downarrow 0\,(n \uparrow \infty)$)

$$h_0\,(x;p,q) = f_0(x;p) * g_0(x;q) = \sum\nolimits_{n=2}^{\infty} \left(n^{1/p'+1/q'} \ln^2 n\right)^{-1} e^{inx} \tag{5}$$

in the case of $r < \infty$ for every $\theta > r$ we have $(1/r' = 1/p' + 1/q' = 1 - 1/r)$

$$\sum\nolimits_{n=2}^{\infty} n^{\theta-2} c_n^{\theta}(h_0) = \sum\nolimits_{n=2}^{\infty} n^{\theta-2} \left(n^{1/p'+1/q'} \ln^2 n\right)^{-\theta} =$$

$$= \sum\nolimits_{n=2}^{\infty} n^{\theta-2} n^{-(1-1/r)\theta} (\ln n)^{-2\theta} =$$

$$= \sum\nolimits_{n=2}^{\infty} n^{-(2-\theta/r)}\,(\ln n)^{-2\theta} = \infty, \quad \text{since} \quad \theta/r > 1 \Longrightarrow 2 - \theta/r < 1;$$

hence by virtue of above mentioned Hardy and Littlewood theorem (in the part-necessity) it follows that $h_0 \notin L_\theta(T)$. In the case of $r = \infty$ (i.e. for $p = q = 2$), putting $f_0 = g_0$, by virtue of Parseval equality, we obtain (see formula (5))

$$\|f_0\|_2 \cdot \|g_0\|_2 = \left(\sum_{n=2}^{\infty} n^{-1} (\ln n)^{-2}\right)^{1/2} \left(\sum_{n=2}^{\infty} n^{-1} (\ln n)^{-2}\right)^{1/2} =$$

$$= \sum_{n=2}^{\infty} n^{-1} (\ln n)^{-2} = h_0(0;2,2) \le \|h_0\|_\infty \le \|f_0\|_2\,\|g_0\|_2\,,$$

whence $\|h_0\|_\infty = \|f_0\|_2 \cdot \|g_0\|_2$, where $h_0 = f_0 * g_0$.

2) For every $\gamma < r$ $(\Longrightarrow \gamma/r' = \gamma\,(1/p' + 1/q') < 1)$ we have (see above formula (5))

$$\rho_0^{(\gamma)}(h_0) = (\textstyle\sum_{n=2}^{\infty} |c_n(h_0)|^{\gamma})^{1/\gamma} = \left(\textstyle\sum_{n=2}^{\infty} n^{-(1/p'+1/q')\gamma}\,(\ln n)^{-2\gamma}\right)^{1/\gamma} =$$

$$= \left(\textstyle\sum_{n=2}^{\infty} n^{-\gamma/r'} (\ln n)^{-2}\right)^{1/2} = \infty, \text{ whence it follows that } h_0 = f_0 * g_0 \notin A^{(\gamma)}(T).$$

Theorem 2 is proved.

Remark 3. The statement of point 2) of Theorem 2 in the case of $1 < p = q \leq 2 \, (\Longrightarrow r' = p'/2)$, was proved in [6] (theorem 5 on p. 53).

Remark 4. Since $r' > 1$ for $r < \infty$, i.e. in the case of $1 < p \leq 2, \ 1 < q < 2$ or $1 < p < 2, \ 1 < q \leq 2$, then the convolution $h_0 = f_0 * g_0$ of functions $f_0(\cdot; p) \in L_p(T)$ and $g_0(\cdot; q) \in L_q(T)$ taken in proof of Theorem 2 in the considered case does not belong to the class $A(T)$. We also note that $f_0(\cdot; p) \notin A(T)$, $g_0(\cdot; q) \notin A(T)$.

Remark 5. Statement 1) of Theorem 2 in the case of $r = \infty$ may be generalized by the following way. For every function $f \in L_2(T)$ with the real Fourier coefficients $\{c_n(f)\} \subset R, \ n \in Z$, by virtue of Theorem A, the convolution $h = f * f \in C(T)$ and $\|h\|_\infty = \|f * f\|_\infty \leq \|f\|_2 \|f\|_2 = \|f\|_2^2$. On the other hand, taking into account equality (2), we have

$$\|h\|_\infty = \|f * f\|_\infty = \max\{|(f * f)(x)|; \ x \in T\} \geq |(f * f)(0)| = \sum_{|n|=0}^{\infty} c_n^2(f) = \|f\|_2^2 .$$

Thus, by virtue of written out estimates, $\|f * f\|_\infty = \|f\|_2^2$.

In the following theorem it is shown that estimate 3) of Theorem 1 is exact in the sense of order in scale of power majorants of sequences of the best approximations of the functions $f \in L_p(T)$ and $g \in L_q(T)$, where $1 < p, \ q \leq 2$.

Theorem 3. *Let* $1 < p, \ q \leq 2, \ \alpha, \ \beta \in (0, \infty)$, $r' = pq/(2pq - p - q) = p'q'/(p' + q') \geq 1$; *there exist functions* $f_0(\cdot; \alpha; p) \in L_p(T), \ g_0(\cdot; \beta; q) \in L_q(T)$ *such that*

1) $E_{n-1}(f_0) \asymp n^{-\alpha}, \ E_{n-1}(g_0)_q \asymp n^{-\beta}, \ n \in N;$

2) $\rho_n^{(r')}(f_0 * g_0) = \left(\sum_{|\nu|=n}^{\infty} |c_\nu (f_0 * g_0)|^{r'}\right)^{1/r'} \asymp n^{-(\alpha+\beta)}, n \in N.$

Proof. Put $(1 < p, \ q < \infty, \ p' = p/(p-1), \ q' = q/(q-1))$

$$f_0(x; \alpha; p) = \sum_{n=1}^{\infty} n^{-(\alpha + 1/p')} e^{inx}, \quad g_0(x; \beta; q) = \sum_{n=1}^{\infty} n^{-(\beta + 1/q')} e^{inx};$$

since $c_n(f_0) = n^{-(\alpha+1/p')} \downarrow 0 \, (n \uparrow \infty)$, $c_n(g_0) = n^{-(\beta+1/q')} \downarrow 0 \, (n \uparrow \infty)$ and

$$\sum_{n=1}^{\infty} n^{p-2} c_n^p(f_0) = \sum_{n=1}^{\infty} n^{p-2} n^{-p(\alpha+1/p')} = \sum_{n=1}^{\infty} n^{-(p\alpha+1)} < \infty,$$

$$\sum_{n=1}^{\infty} n^{q-2} c_n^q(g_0) = \sum_{n=1}^{\infty} n^{q-2} n^{-q(\beta+1/q')} = \sum_{n=1}^{\infty} n^{-(q\beta+1)} < \infty,$$

then, by virtue of Hardy-Littlewood theorem, we have $f_0(\cdot; \alpha; p) \in L_p(T)$, $g_0(\cdot; \beta; q) \in L_q(T)$ and $\|f_0\|_p \asymp \left(\sum_{n=1}^{\infty} n^{-(p\alpha+1)}\right)^{1/p}$, $\|g_0\|_q \asymp \left(\sum_{n=1}^{\infty} n^{-(q\beta+1)}\right)^{1/q}$.

Further, by virtue of the obvious inequality $E_{n-1}(\varphi)_p \leq \|\varphi(\cdot) - S_{n-1}(\varphi; \cdot)\|_p$ and M.Riesz inequality (4), we obtain

$$E_{n-1}(f_0)_p \asymp \|f_0(\cdot) - S_{n-1}(f_0; \cdot)\|_p \asymp \left(\sum_{\nu=n}^{\infty} \nu^{p-2} c_\nu^p(f_0)\right)^{1/p} =$$

$$= \left(\sum_{\nu=n}^{\infty} \nu^{-(p\alpha+1)}\right)^{1/p} \asymp n^{-\alpha}, \ n \in N;$$

$$E_{n-1}(g_0)_q \asymp \|g_0(\cdot) - S_{n-1}(g_0;\cdot)\|_q \asymp \left(\sum_{\nu=n}^{\infty} \nu^{q-2} c_\nu^q(g_0)\right)^{1/q} =$$

$$= \left(\sum_{\nu=n}^{\infty} \nu^{-(q\beta+1)}\right)^{1/q} \asymp n^{-\beta}, \ n \in N.$$

Besides, it is easy to note that $f_0(\cdot;\alpha;p) \in A(T)$, $g_0(\cdot;\beta;q) \in A(T)$ for $1/p < \alpha < \infty$, $1/q < \beta < \infty$ and $f_0(\cdot;\alpha;p) \notin A(T)$, $g_0(\cdot;\beta;q) \notin A(T)$ for $0 < \alpha \leq 1/p$, $0 < \beta \leq 1/q$. Finally, by virtue of equality (2) we have $(1/p' + 1/q' = 1/r')$

$$\rho_n^{(r')}(f_0 * g_0) = \left(\sum_{\nu=n}^{\infty} |c_\nu(f_0) \cdot c_\nu(g_0)|^{r'}\right)^{1/r'} =$$

$$= \left(\sum_{\nu=n}^{\infty} \nu^{-(\alpha+1/p')r'} \cdot \nu^{-(\beta+1/q')r'}\right)^{1/r'} =$$

$$= \left(\sum_{\nu=n}^{\infty} \nu^{-(\alpha+\beta)r'} \cdot \nu^{-(1/p'+1/q')r'}\right)^{1/r'} =$$

$$\left(\sum_{\nu=n}^{\infty} \nu^{-(\alpha+\beta)r'-1}\right)^{1/r'} \asymp n^{-(\alpha+\beta)}, \ n \in N.$$

Theorem 3 is proved.

Remark 6. Theorem 3 in the case of $1 < p = q \leq 2 \, (\Longrightarrow r' = p'/2)$ was proved by the author in [5] (theorem 2).

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