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INTERPOLATION THEOREM OF BESOV-MORREY TYPE SPACES AND SOME ITS APPLICATIONS

Abstract

In the work are proved differential property and satisfy repeately condition Holder for function f from $\bigcap\limits_{\mu=1}^{N} B^{l^{\mu}}_{p_\mu,\theta_\mu,a,\chi,\tau}(G)$. The smothness of solution of one class quasielliptic equations are studied.

In this paper both differential and difference-differential properties of functions belonging to intersection of the spaces $B^{l^{\mu}}_{p_\mu,\theta_\mu,a,\chi,\tau}(G,\lambda)$, $\mu=1,2,...,N$ and the smoothness of solution of one class of quasielliptic equations are studied.

Let $G\subset R^n$ be a domain satisfying the condition of flexible λ-horn introduced by O.V.Besov in the paper [1] and let

$$p_\mu\in[1,\infty)^n;\quad \theta_\mu\in[1,\infty];\ \chi,l^{\mu}\in(0,\infty)^n,\ \mu=1,2,...,N;$$

$$a\in[0,1]^n;\ \tau\in[1,\infty];\ [t]_1=\min\{1,t\}.$$

Assume for any $x\in R^n$ and $t>0$

$$I_{t^\chi}(x)=\{y:|y_j-x_j|<t^{\chi_j},\quad j=1,2,...,n\},\quad G_{t^\chi}(x)=G\cap I_{t^\chi}(x)\ .$$

Note that the Besov-Morrey type space $B^{l}_{p,\theta,a,\chi,\tau}(G,\lambda)$ $\left(p\in[1,\infty)^n,\ \theta\in[1,\infty]\right.$ $\left.l\in(0,\infty)^n\right)$ is determined and studied in the paper [2] with the finite norm $(m_i>l_i-k_i>0,\ i=1,2,...,n)$:

$$\|f\|_{B^{l}_{p,\theta,a,\chi,\tau}(G,\lambda)}=\|f\|_{p,a,\chi,\tau,G}+\sum_{i=1}^{n}\left\{\int_0^{h_0}\left[\frac{\left\|\Delta_i^{m_i}(h,G,\lambda)D_i^{k_i}f\right\|_{p,a,\chi,\tau}}{h^{l_i-k_i}}\right]^{\theta}\frac{dh}{h}\right\}^{\frac{1}{\theta}},\qquad(1)$$

here $\left\|\Delta_i^{m_i}(h,G,\lambda)D_i^{l_i}f\right\|_{p,a,\chi,\tau}=\left\|\Delta_i^{m_i}(h,G_{h^\lambda})D_i^{l_i}f\right\|_{p,a,\chi,\tau}=\left\|\Delta_i^{m_i}(h)D_i^{l_i}f\right\|_{p,a,\chi,\tau;G_{h^\lambda}}$,

$$\Delta_i^{m_i}(h,G_{h^\lambda})f(x)=\begin{cases}\Delta_i^{m_i}(h)f(x), & [x,x+m_ihe_i]\subset G_{h^\lambda}\\ 0, & [x,x+m_ihe_i]\not\subset G_{h^\lambda}\end{cases},$$

$$\Delta_i^{m_i}(h)f(x)=\sum_{j=0}^{m_i}(-1)^{m_i-j}C^{j}_{m_i}f(x+jhe_i)\ ,$$

and the space $L_{p,a,\chi,\tau}(G)$ will be called Morrey type space with the finite norm

$$\|f\|_{p,a,\chi,\tau;G}=\|f\|_{L_{p,a,\chi,\tau}(G)}=\sup_{x\in G}\left\{\int_0^{t_0}\left[[t]_1^{-\sum\limits_{j=1}^{n}\frac{\chi_ja_j}{p_j}}\|f\|_{p,G_{t^\chi}(x)}\right]^{\tau}\frac{dt}{t}\right\}^{\frac{1}{\tau}},\qquad(2)$$

where

$$\|f\|_{p,G_{t^\chi}(x)}=\left\{\int\limits_{G_{t^{\chi_n}}(x_n)}\left[\ldots\left\{\int\limits_{G_{t^{\chi_2}}(x_2)}\left(\int\limits_{C_{t^{\chi_1}}(x_1)}|f(y)|^{p_1}\,dy_1\right)^{\frac{p_2}{p_1}}dy_2\right\}^{\frac{p_3}{p_2}}\ldots\right]^{\frac{p_n}{p_{n-1}}}dy_n\right\}^{1/p_n}$$

and h_0, t_0 are fixed positive numbers.

When $\tau=\infty,\ p=(p,...,p),\ a=(a,...,a)$ the Morrey spaces $L_{p,a,\chi,\infty}(G)\equiv L_{p,a,\chi}(G)$ and Sobolev-Morrey spaces $W^l_{p,a,\chi}(G)$ are introduced and studied by V.P.Il'in in [1]. As well as when $\theta=\infty$ the spaces $B^l_{p,\infty,a,\chi,\tau}(G,\lambda)\equiv H^l_{p,a,\chi,\tau}(G,\lambda)$ with the finite norm

$$\|f\|_{H^l_{p,a,\chi,\tau}(G,\lambda)}=\|f\|_{p,a,\chi;G}+\sum_{i=1}^{n}\sup_{0<h<h_0}\frac{\left\|\Delta_i^{m_i}(h,G,\lambda)\,D_i^{k_i}f\right\|_{p,a,\chi,\tau}}{h^{l_i-k_i}}$$

will be called the Nicolskii-Morrey type space. Note that the Nicolskii-Morrey spaces $H^l_{p,a,\chi,\infty}(G,\lambda)\equiv H^l_{p,a,\chi}(G,\lambda)$ are investigated in [3].

The spaces $B^l_{p,\theta,\alpha,\chi,\tau}(G)$ are determined as the spaces $B^l_{p,\theta,\alpha,\chi,\tau}(G,\lambda)$, but by substitution $\Delta_i^{m_i}(h,G,\lambda)$ by $\Delta_i^{m_i}(h,G)\,f$. The Besov-Morrey spaces $B^l_{p,\theta,\alpha,\chi,\infty}(G)\equiv B^l_{p,\theta,\alpha,\chi}(G)$ are determined and studied in [4].

Note that the problem on smoothness of solutions of quasielliptic equations was also considered in the paper [5], too, but it should be noted that in the present paper unlike [5].

1) Hōlder "factor" is more than in the paper [5]:

2) f_{α^μ} for $\left|\alpha^\mu,\dfrac{1}{l^\mu}\right|=1,\ \mu=1,...,N$ belongs to the larger class, i.e., $f_{\alpha^\mu}\in L_{2,\alpha,\chi}(G)$

3) besides $\nu\neq 0$ the amount of vectors ν is also increased.

Let $\beta_\mu\geq 0,\ \mu=1,2,...,N,\ \sum\limits_{\mu=1}^{N}\beta_\mu=1, l=\sum\limits_{\mu=1}^{N}\beta_\mu l^\mu,\ \dfrac{1}{p}=\sum\limits_{\mu=1}^{N}\dfrac{\beta_\mu}{p_\mu},$ $\dfrac{1}{q}=\sum\limits_{\mu=1}^{N}\dfrac{\beta_\mu}{q_\mu},\ \dfrac{1}{\theta}=\sum\limits_{\mu=1}^{N}\dfrac{\beta_\mu}{\theta_\mu}$ and for any $x\in G,\ T\in(0,\infty)$ there exist the way (see [1]):

$\rho\left(t^\lambda,x\right)=\left(\rho_1(t^{\lambda_1},x),...,\rho_n(t^{\lambda_n},x)\right),\ 0\leq t\leq T$ with the following properties:

1) for any $i=1,...,n$ the functions $\rho_i(u,x)$ are absolutely continuos on $[0,T^{\lambda_i}]$; $|\rho_i'(u,x)|\leq 1$ for almost all $u\in[0,T^{\lambda_i}]$ where $\rho_i'(u,x)=\dfrac{\partial}{\partial u}\rho_i(u,x)$;

2) $\rho(0,x)=0$.

At $\delta\in(0,1]$ each of the sets

$$V(\lambda,x,\delta)=\bigcup_{0<t\leq T}\left[\rho\left(t^\lambda,x\right)+t^\lambda\delta^\lambda I\right],\ x+V(\lambda,x,\delta)$$

is called flexible λ-norm, and the point x is called a vertex of $x+V(\lambda,x,\delta)$.

Let also $\xi_i\in C_0^\infty(R);\Omega,M\in C^\infty\left(R^n\times R^n\right),\ \Omega(\cdot,z)$ and $M(\cdot,z)$ be finite uniformly with respect to z from arbitrary compact

Lemma 1. *Let* $1 \le p_\mu \le q_\mu \le r_\mu \le \infty,\ 0 < \chi \le \lambda, 0 < t \le T \le 1, 0 < \rho < \infty,$ $\Delta_i^{m_i} f \in L_{p_\mu,\alpha,\chi,\tau}(G), 1 \le \tau \le \infty,\ 0 < \eta \le T, \nu = (\nu_1, ..., \nu_n),\ \nu_j \ge 0$ *are integers,* $j = 1, ..., n,$

$$\varepsilon_i = \lambda_i \sum_{\mu=1}^{N} \beta_\mu l_i^\mu - \sum_{j=1}^{n} \left[\nu_j \lambda_j + (\lambda_j - \chi_j a_j) \left(\frac{1}{p_j} - \frac{1}{q_j} \right) \right] \tag{3}$$

$$F(x,t) = \int_{R^n} f(x+y+z) \Omega \left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{2t^\lambda} \right) \Omega^{(\nu)} \left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{2t^\lambda} \right) dy dz$$

$$B_{i,\eta}(x) = \int_0^\eta t^{-1-|\lambda|-\lambda_i-|\nu,\lambda|} F_i(x,t)\, dt\ ,\ B_{i,\eta T}(x) = \int_\eta^T t^{-1-|\lambda|-\lambda_i-|\nu,\lambda|} F_i(x,t)\, dt$$

$$F_i(x,t) = \int_{R^n} \int_{-\infty}^{\infty} M \left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda} \right) \xi_i \left(\frac{u}{t^{\lambda_i}}, \frac{\rho_i(t^{\lambda_i}, x)}{t^{\lambda_i}}, \frac{1}{2} \rho_i(t^{\lambda_i}, x) \right) \times$$

$$\times \Delta_i^{m_i} \left(\delta^{\lambda_i} u \right) f(x+y+ue_i) dy du,$$

where $|\nu, \lambda| = \sum_{j=1}^{n} \nu_j \lambda_j.$

Then the following inequalities hold

$$\sup_{x \in U} \|F(\cdot,t)\|_{q,U_{\rho\chi}(\bar{x})} \le C \prod_{\mu=1}^{N} \left\{ \|f\|_{p_\mu,a,\chi,\tau G_{T\chi}(U)} \right\}^{\beta_\mu} \times$$

$$\times t^{\sum_{j=1}^{n} \left[2\lambda_j - (\lambda_j - \chi_j a_j) \left(\frac{1}{p_j} - \frac{1}{q_j} \right) \right]} [\rho]_1^{\sum_{j=1}^{n} \chi_j a_j \frac{1}{r_j}} \rho^{\sum_{\mu=1}^{n} \chi_j a_j \left(\frac{1}{q_j} - \frac{1}{r_j} \right)} \tag{4}$$

$$\sup_{x \in U} \|B_{i,\eta}\|_{q,U_{\rho\chi}(\bar{x})} \le C_1 \prod_{\mu=1}^{N} \left\{ \left\| t^{-\lambda_i l_i^\mu} \Delta_i^{m_i} \left(t^{\lambda_i}, G_{T^\chi}(U), \lambda \right) f \right\|_{p_\mu,a,\chi,\tau} \right\}^{\beta_\mu} \times$$

$$\times [\rho]_1^{\sum_{j=1}^{n} \chi_j a_j \frac{1}{r_j}} \eta^{\varepsilon_i},\quad \varepsilon_i > 0 \tag{5}$$

$$\sup_{x \in U} \|B_{i,\eta T}\|_{q,U_{\rho\chi}(\bar{x})} \le C_2 \prod_{\mu=1}^{N} \left\{ \left\| t^{-\lambda_i l_i^\mu} \Delta_i^{m_i} \left(t^{\lambda_i}, G_{T^\chi}(U), \lambda \right) f \right\|_{p_\mu,a,\chi,\tau)} \right\}^{\beta_\mu} \times$$

$$\times [\rho]_1^{\sum_{j=1}^{n} \chi_j a_j \frac{1}{r_j}} \begin{cases} T^{\varepsilon_i}, & \varepsilon_i > 0 \\ \ln \frac{T}{\eta}, & \varepsilon_i = 0 \\ \eta^{\varepsilon_i}, & \varepsilon_i < 0 \end{cases}, \tag{6}$$

Here $U_{\rho^x}(\bar{x}) = \left\{ x : |x_j - \bar{x}_j| < \frac{1}{2} \rho^{\chi_j},\quad j = 1, 2, ..., n \right\}.$

Lemma 2. *Let* $1 \le p_\mu \le r_\mu \le \infty,\quad 0 < \chi \le \lambda,\quad 0 < t \le T \le 1,\quad 1 \le \tau_1 \le \tau_2 \le \infty,\ \nu = (\nu_1, ..., \nu_n),\ \nu_j \ge 0$ *are integers,* $j = 1, 2, ..., n$ *and* $\varepsilon_i > 0$ *then the inequality*

$$\sup_{x \in U} \|B_{i,T}\|_{q,b,\chi,\tau_2;U} \le C \prod_{\mu=1}^{N} \left\{ \left\| t^{-\lambda_i l_i^\mu} \Delta_i^{m_i} \left(t^{\lambda_i}, G_{T^\chi}(U), \lambda \right) f \right\|_{p_\mu,a,\chi,\tau_1} \right\}^{\beta_\mu} \tag{7}$$

holds. Where $B_{i,T}$ *was defined in lemma 1.*

Theorem 1. *Let the open set* $G \subset R^n$ *satisfy the condition of flexible* λ*-horn,* $1 \le p_\mu \le q_\mu \le \infty$, $\mu = 1, 2, ..., N$, $1 \le \theta \le \theta_1 \le \infty$, $\bar{\chi} = c\chi$, *where* $\frac{1}{c} = \max\limits_{1\le j\le n} \frac{\chi_j}{\lambda_j}$, $\nu = (\nu_1, ..., \nu_n)$, $\nu_j \ge 0$ *are integers,* $j = 1, ..., n$; $1 \le \tau_1 \le \tau_2 \le \infty$, $\varepsilon_i > 0$, $i = 1, ..., n$ *and let* $f \in \bigcap\limits_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,a,\chi,\tau_1}(G, \lambda)$.

Then the following inequalities hold

$$\|D^\nu f\|_{q,G} \le C_1 A_1(T) \prod_{\mu=1}^{N} \left\{ \|f\|_{B^{l^\mu}_{p_\mu,a,\chi,\tau_1}(G_k)} \right\}^{\beta_\mu}, \tag{8}$$

$$\|D^\nu f\|_{q,b,\chi,\tau_2;G} \le C_2 \prod_{\mu=1}^{N} \left\{ \|f\|_{B^{l^\mu}_{p_\mu,a,\chi,\tau_1}(G_k)} \right\}^{\beta_\mu}, \quad p_\mu \le q_\mu < \infty \tag{9}$$

and if $\varepsilon_i - \lambda_j l^1_j > 0$ *at* $i, j = 1, ..., n$, *then*

$$\|D^\nu f\|_{B^{l^1}_{q,\theta_1}(G_h)} \le C_3 A_3(T) \prod_{\mu=1}^{N} \left\{ \|f\|_{B^{l^\mu}_{p_\mu,a,\chi,\tau_1}(G_h)} \right\}^{\beta_\mu}, \tag{10}$$

$$\|D^\nu f\|_{B^{l^1}_{q,b,\chi,\tau_2}(G_h)} \le C_4 \prod_{\mu=1}^{N} \left\{ \|f\|_{B^{l^\mu}_{p_\mu,a,\chi,\tau_1}(G_h)} \right\}^{\beta_\mu}, \quad p_\mu \le q_\mu < \infty, \tag{11}$$

where

$$A_1(T) = \sum_{i=0}^{n} T^{\varepsilon_i},\ A_2(T) = \sum_{i=0}^{n} T^{\varepsilon_i - \lambda_j l^l_j},\ \varepsilon_0 = -\sum_{j=1}^{n} \left[v_j\lambda_j + (\lambda_j - \chi_j a_j)\left(\frac{1}{p_j} - \frac{1}{q_j}\right) \right].$$

Moreover $0 < T \le \min\{1, T_0\}$, C_1, C_2, C_3, C_4 *are the constants independent on* f, *and* C_1, C_3 *independent on* T.

In particular, if $\varepsilon_{i,0} = \lambda_i \sum\limits_{\mu=1}^{N} \beta_\mu l^\mu_l - \sum\limits_{j=1}^{n} \left[\nu_j\lambda_j + (\lambda_j - \chi_j a_j) \frac{1}{p_j} \right] > 0$, $i = 1, ..., n$, *then* $D^\nu f$ *is continuous on* G, *i.e.*

$$\sup_{x\in G} |D^\nu f| \le C_1 \sum_{i=0}^{n} T^{\varepsilon_{i,0}} \prod_{\mu=1}^{N} \left\{ \|f\|_{B^{t^\mu}_{p_\mu,a,\chi,\tau_1}(G_h)} \right\}^{\beta_\mu}. \tag{12}$$

Proof. First of all, we note that since $\bar{\chi} = c\chi$, $c > 0$ on the basis of the properties $\|f\|_{p,a,c\chi,\tau;G} = \frac{1}{c^\tau} \|f\|_{p,a,\chi,\tau;G}$ we can assume that $f \in \bigcap\limits_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,a,\bar{\chi},\tau_1}(G, \lambda)$ and we can substitute everywhere in inequalities (8)-(12) and ε_i, χ by $\bar{\chi}$. Namely, we shall prove such inequalities (the more χ, the more ε_i).

Let $f \in \bigcap\limits_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,a,\bar{\chi},\tau_1}(G, \lambda) \to \bigcap\limits_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu,a,\bar{\chi}}(G, \lambda) \to \bigcap\limits_{\mu=1}^{N} B^{l^\mu}_{p_\mu,\theta_\mu}(G, \lambda)$, then the existence of generalized derivative under the conditions of our theorem follows from theorem 18.4 (see [1]). Then for almost each point $x \in G$ the integral identity obtained by O.V.Besov (see [1], p.91) is valid.

$$D^\nu f = f^{(\nu)}_{T^\lambda}(x) + (-1)^{|\nu|} \int_0^T \sum_{i=1}^{n} \int_{R^n} \int_{-\infty}^{\infty} t^{-1-|\lambda|-\lambda_i-|\nu,\lambda|} \varphi^{(\nu)}_i \left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda} \right) \times$$
$$\times \zeta_i \left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^{\lambda_i}}, \frac{1}{2}\rho_i(t^{\lambda_i}, x) \right) \Delta^{m_i}_i (\delta^{\lambda_i} u, G, \lambda) f(x + y + ue_i)\, dudydt\,, \tag{13}$$

$$f_{T^\lambda}^{(v)}(x) = (-1)^{|v|}\, T^{-2|\lambda|-|v,\lambda|}\times$$

$$\times \int\limits_{R^n} f(x+y+z)\,\Omega\left(\frac{y}{T^\lambda}, \frac{\rho(t^\lambda,x)}{2T^\lambda}\right)\Omega^{(\nu)}\left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda,x)}{2T^\lambda}\right) dydz\ .$$

Here $\varphi_i(\cdot,z) \in C_0^\infty(R^n)$ and its support is contained in I_1, therefore the support of representation (13) is contained in the flexible horn $x + V(\lambda, x, \delta, \delta^{-1}T_0)$. The parameter of representation $\delta > 0$ is assumed sufficiently small, therefore $\Delta_i^{m_i}(\delta^{\lambda_i}u, G, \lambda) f = \Delta_i^{m_i}(\delta^{\lambda_i}u) f$. On the basis of the Minkowsky inequality we have

$$\|D^v f\|_{q,G} \le \left\|f_{T^\lambda}^{(\nu)}\right\|_{q,G} + \sum_{i=1}^n \|B_{i,T}\|_{q,G} \tag{14}$$

Allowing for $1 \le \theta \le \infty$, hence with the help of inequality (4) and (5) at $U = G,\ t = T,\ \rho \to \infty$ we obtain inequality (8). To prove inequalities (10) and (11) we estimate $\left\|\Delta_j^{M_j}(h^{\lambda_j}, G, \lambda) D^v f\right\|_{q,G}$. After several transformations from identity (13) we obtain the following inequality

$$\left|\Delta_j^M\left(h^{\lambda_j}, G, \lambda\right) D^v f\right| \le C_1 T^{-2|\lambda|-|\nu,\lambda|} \frac{h^{M\lambda_j}}{T^{M\lambda_j}} \int\limits_{R^n} \Omega\left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda,x)}{2T^\lambda}\right)\times$$

$$\times \Omega^{(\nu+Me_j)}\left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda,x)}{2T^\lambda}\right)\int\limits_0^1 \left|f\left(x+y+z+Mh^{\lambda_j}\xi e_j\right)\right| d\xi dydz +$$

$$+C_2\sum_{i=1}^n \int\limits_0^H t^{-1-|\lambda|-\lambda_i-|\nu,\lambda|} \int\limits_{R^n}\int\limits_{-\infty}^{\infty} \varphi_i^{(\nu)}\left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda,x)}{t^\lambda}\right)\zeta_i\left(\frac{u}{t^{\lambda_i}}, \frac{\rho(t^{\lambda_i},x)}{t^{\lambda_i}}, \frac{1}{2}\rho_i(t^{\lambda_i},x)\right)\times$$

$$\times \left|\Delta_j^M(h^{\lambda_j}, G, \lambda)\, \Delta_i^{m_i}\left(\delta^{\lambda_i}u, G, \lambda\right) f(x+y+ue_i)\right| dudydt + C_3\sum_{i=1}^n h^{M\lambda_j}\times$$

$$\times \int\limits_H^T t^{-1-|\lambda|-\lambda_i-|v,\lambda|} \int\limits_{R^n}\int\limits_{-\infty}^{\infty} \left|\varphi_i^{(\nu+Me_j)}\left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda,x)}{t^\lambda}\right)\right|\left|\zeta_i\left(\frac{u}{t^{\lambda_i}}, \frac{\rho(t^{\lambda_i},x)}{t^{\lambda_i}}, \frac{1}{2}\rho_i(t^{\lambda_i},x)\right)\right|\times$$

$$\times \int\limits_0^1 \left|\Delta_i^{m_i}\left(\delta^{\lambda_i}u, G, \lambda\right) f\left(x+y+ue_i+Mh^{\lambda_j}\xi e_j\right)\right|\ d\xi dudydt = C_1 B_0(x,t) +$$

$$+C_2\sum_{i=1}^n B_{i,H}(x,t) + C_3\sum_{i=1}^n B_{i,HT}(x,t)\ .$$

$$\left\|\Delta_j^M\left(h^{\lambda_j}, G, \lambda\right) D^v f\right\|_{q,U_{\rho\chi}(\bar{x})} \le C_1\|B_0\|_{q,U_{\rho\chi}(\bar{x})} +$$

$$+C_2\sum_{i=1}^n \|B_{i,H}\|_{q,U_{\rho\chi}(\bar{x})} + C_3\sum_{i=1}^n \|B_{i,HT}\|_{q,U_{\rho\chi}(\bar{x})} \tag{15}$$

With the help of inequalities (4)-(6) at $U = G,\ \rho \to \infty,\ 0 < l_j^1 < M,\ \ j = 1, ..., n$ we obtain

$$\frac{\left\|\Delta_j^M(h^{\lambda_j}, G, \lambda) D^v f\right\|_q}{h^{\lambda_j l_l^1}} \le C_1 T^{\varepsilon_0-\lambda_j l_l^1} \prod_{\mu=1}^N \left\{\|f\|_{p_\mu,a,\bar{\chi},\tau_1;G}\right\}^{\beta_\mu} +$$

$$+C_5 \sum_{i=1}^{n} T^{\varepsilon_i-\lambda_j l_l^1} \prod_{\mu=1}^{N} \left\{ \left\| t^{-\lambda_i l_i^{\mu}} \Delta_i^{m_i} \left(t^{\lambda_i}, G, \lambda \right) f \right\|_{p_\mu, a, \bar{\chi}, \tau_1} \right\}^{\beta_\mu} \leq$$

$$\leq C_6 A_2 (T) \prod_{\mu=1}^{N} \left\{ \|f\|_{p_\mu, a, \bar{\chi}, \tau_1; G} + \sum_{i=1}^{n} \left\| t^{-\lambda_i l_i^{\mu}} \Delta_i^{m_i} \left(t^{\lambda_i}, G, \lambda \right) f \right\|_{p_\mu, a, \bar{\chi}, \tau_1} \right\}^{\beta_\mu},$$

and hence consequently, under the condition $1 \leq \theta \leq \theta_1 \leq \infty$ we obtain inequality (10). Estimations (9) and (11) are established analogously on the basis of inequalities (4) and (7).

Let now $\varepsilon_{i,0} > 0, \ \ i = 1, ..., n$. We show that then $D^v f$ is continuous on G. On the basis of identity (13) and inequality (8) at $q \equiv \infty, \ \ \varepsilon_i = \varepsilon_i = \varepsilon_{i,0} > 0, \ \ i = 1, ..., n$ we have

$$\|D^v f - D^v f_{T^\lambda}\|_{\infty, G} \leq \sum_{i=1}^{n} T^{\varepsilon_{i,0}} \prod_{\mu=1}^{N} \left\{ \left[\int_0^{h_0} \left\| t^{-\lambda_i l_i^{\mu}} \Delta_i^{m_i} \left(t^{\lambda_i}, G, \lambda \right) f \right\|_{p_\mu, a, \bar{\chi}, \tau_1}^{\theta} \frac{dt}{t} \right]^{\frac{1}{\theta_\mu}} \right\}^{\beta_\mu}$$

$\lim\limits_{T \to 0} \|D^v f - D^v f_{T^\lambda}\|_{\infty, G} \to 0$. Since $D^v f_{T^\lambda}$ is continuous on G, the convergence of $L_\infty (G)$ coincides in the given case with the uniformity and consequently $D^v f$ is continuous on G.

The theorem is proved.

Theorem 2. *Let all the continuous of theorem 1 be satisfied, then at* $\varepsilon_i > 0$, $i = 1, ..., n$ *the derivative* $D^v f$ *satisfies on* G *the Hölder condition in the metric* L_q *with the factor* σ*, more precisely*

$$\|\Delta (\gamma, G) D^v f\|_{q, G} \leq C \prod_{\mu=1}^{N} \left\{ \|f\|_{B^{l^\mu}_{p_\mu}, a, \chi, \tau_1; (G, \lambda)} \right\}^{\beta_\mu} |\gamma|^{\sigma^1}, \tag{16}$$

where σ *is any number, satisfying the inequalities*

$$\begin{aligned} 0 \leq \sigma < \sigma^1 \leq 1, \quad & if \quad \frac{\varepsilon^0}{\lambda_0} > 1, \\ 0 \leq \sigma < \sigma^1 < 1, \quad & if \quad \frac{\varepsilon^0}{\lambda_0} = 1, \\ 0 \leq \sigma < \sigma^1 \leq \frac{\varepsilon^0}{\lambda_0}, \quad & if \quad \frac{\varepsilon^0}{\lambda_0} < 1. \end{aligned} \tag{17}$$

Here $\varepsilon^0 = \min \varepsilon_i, \quad \lambda_0 = \max \lambda_j, \quad i, j = 1, ..., n$, and σ is also Hölder exponent but by the substitution in (16) the spaces $B^{l^\mu}_{p_\mu, \theta_\mu, a, \chi, \tau_1} (G, \lambda)$ by $B^{l^\mu}_{p_\mu, \theta_\mu, a, \chi} (G, \lambda)$.

If $\varepsilon_{i,0} > 0, \quad i = 1, ..., n$, then

$$\sup_{x \in G} |\Delta (\gamma, G) D^v f| \leq C \prod_{\mu=1}^{N} \left\{ \|f\|_{B^{l^\mu}_{p_\mu}, a, \chi, \tau_1; (G, \lambda)} \right\}^{\beta_\mu} |\gamma|^{\sigma^0}, \tag{18}$$

where σ^0 satisfies the same conditions σ but by the substitution ε_i by $\varepsilon_{i,0}$.

Granting that in the case when l is a non-integral vector, and $p_1 = p_2 = ... = p_n = p \in (1, \infty)$ as well as for any $l \in (0, \infty)^n$ and at $p_1 = p_2 = ... = p_n = 2$, the spaces $W^l_p (G)$ and $B^l_{p,p} (G)$ coincide, then it is easily proved that for such l and p the spaces $W^l_{p,a,\chi,\tau} (G)$ and $B^l_{p,p,a,\chi,\tau} (G)$ also coincide.

Let us consider the following equation in the bounded domain $G\subset R^n$

$$\sum_{\mu=1}^{N}\sum_{\substack{|\alpha^\mu,\frac{1}{l^\mu}|\le 1\\ |\delta^\mu,\frac{1}{l^\mu}|\le 1}}\left(D^{\alpha^\mu}a_{\alpha^\mu\delta^\mu}(x)\,D^{\delta^\mu}u(x)\right)=\sum_{\mu=1}^{N}\sum_{|\alpha^\mu,\frac{1}{l^\mu}|\le 1}D^{\alpha^\mu}f_{\alpha^\mu}\ , \tag{19}$$

where $\alpha^\mu=(\alpha_1^\mu,...,\alpha_n^\mu)$, $\delta^\mu=(\delta_1^\mu,...,\delta_n^\mu)$, $\mu=1,2,...,N$. Let $p_1=p_2=...=p_n=\theta=2$, $l\in N^n$, $\lambda_j^{-1}=\sum\limits_{\mu=1}^{N}\beta_\mu l_j^\mu$, $j=1,...,n$ and the coefficients $a_{\alpha^\mu\delta^\mu}(x)\equiv a_{\delta^\mu\alpha^\mu}(x)$ $(\mu=1,2,...,N)$, $a_{\alpha^\mu\delta^\mu}(x)$ for all μ $(\mu=1,2,...,N)$ be bounded, measurable in G, $\xi\in R^n$ and

$$\sum_{\mu=1}^{N}\sum_{\substack{|\alpha^\mu,\frac{1}{l^\mu}|=1\\ |\delta^\mu,\frac{1}{l^\mu}|=1}}^{N}(-1)^{|\alpha^\mu|}a_{\alpha^\mu\delta}(x)\,\xi_{\alpha^\mu}\xi_{\delta^\mu}\ge$$

$$\ge c_0\sum_{\mu=1}^{N}\sum_{|\alpha^\mu,\frac{1}{l^\mu}|=1}|\xi_{\alpha^\mu}|^2,\ c_0=const>0. \tag{20}$$

The function G is called a generalized solution of equation (19) in the domain $u(x)\in\bigcap\limits_{\mu=1}^{N}W_2^{l^\mu}(G)$ for all μ $(\mu=1,2,...,N)$ such that

$$\sum_{\substack{|\alpha^\mu,\frac{1}{l^\mu}|\le 1\\ |\delta^\mu,\frac{1}{l^\mu}|\le 1}}(-1)^{|\alpha^\mu|}\int_G a_{\alpha^\mu\delta}(x)D^{\alpha^\mu}u(x)D^{\alpha^\mu}\varsigma(x)dx=$$

$$=\sum_{|\alpha^\mu,\frac{1}{l^\mu}|\le 1}(-1)^{|\alpha^\mu|}\int_G f_{\alpha^\mu}D^{\alpha^\mu}\varsigma(x)dx \tag{21}$$

for any function $\varsigma(x)\in\bigcap\limits_{\mu=1}^{N}\mathring{W}_2^{l^\mu}(G)$ the space $\mathring{W}_2^{l^\mu}(G)$ is complement of $C_0^\infty(G)$ in the metric $W_2^l(G)$. We assume that $f_{\alpha^\mu}\in L_2(G)$ for $\left|\alpha^\mu,\dfrac{1}{l^\mu}\right|<1$,, and $f_{\alpha^\mu}\in L_{2,\alpha,\chi}(G)$ for $\left|\alpha^\mu,\dfrac{1}{l^\mu}\right|=1$ for all $\mu(\mu=1,2,...,N)$.

Theorem 3. *If* $\dfrac{|\lambda|}{2}+|\nu,\lambda|\le 1$, *any generalized solution of equation (19) from* $\bigcap\limits_{\mu=1}^{N}W_2^{l^\mu}(G)$ *belongs to the space* $C_{\nu+\sigma^1}(G^d)$, $\overline{C^d}\subset G$.

Proof. Let at first all $a_{\alpha^\mu\delta^\mu}(x)$ $(\mu=1,2,...,N)$ except for ones for which $\left|\alpha^\mu,\dfrac{1}{l^\mu}\right|=\left|\delta^\mu,\dfrac{1}{l^\mu}\right|=1$ $(\mu=1,2,...,N)$ and all $f_{\alpha^\mu}=0$ $(\mu=1,2,...,N)$. Let $x_0\in G$, $\Pi_b(x_0)$

be a parallelepiped in R^n.

$$\Pi_b(x_0)=\left\{x:|x_j-x_{j,0}|<b^{\lambda_j}, j=1,2,...,n\right\},$$

and G^d be a subdomain of the domain G such that $(0<d<1, d=const)$:

$$G^d=\left\{y:|y_j-x_j|<d^{\lambda_j},\quad j=1,2,...,n,\quad x\in\partial G\right\}$$

we shall assume that $b\le d$. For all $\mu=1,2,...,N$ from variational principle it follows that

$$\int\limits_{\Pi_b(x_0)}\sum_{\left|\alpha^\mu,\frac{1}{l^\mu}\right|=\left|\delta^\mu,\frac{1}{l^\mu}\right|=1}(-1)^{|\alpha^\mu|}\quad a_{\alpha^\mu\delta^\mu}(x)D^{\delta^\mu}(\theta(x)(u-p(x)))D^{\alpha^\mu}(\theta(x)(u-p(x)))dx\ge$$

$$\ge\int\limits_{\Pi_b(x_0)}\sum_{\left|\alpha^\mu,\frac{1}{l^\mu}\right|=\left|\delta^\mu,\frac{1}{l^\mu}\right|=1}(-1)^{|\alpha^\mu|}\quad a_{\alpha^\mu\delta^\mu(x)}D^{\delta^\mu}(u-p(x))D^{\alpha^\mu}(u-p(x))\,dx=$$

$$=A_\mu(u-p(x),\Pi_b(x_0)),\tag{22}$$

for any $\theta(x)\in C^\infty(\Pi_b(x_0))$ such that $\theta(x)\equiv1$ in the neighbourhood of $\partial\Pi_b(x_0)$, in any polynomial $p(x)$ of the form $p(x)=\sum\limits_{\left|\alpha^\mu,\frac{1}{l^\mu}\right|=1}C_{\alpha^\mu}x^{\alpha^\mu}$ and at arbitrary solution $u(x)$ of equation (19).

Assume in (22)

$$\theta(x)=1-\prod_{i=1}^n\bar\omega_i\left(\frac{x_i-x_{i,0}}{b^{\lambda_i}}\right),$$

where $\bar\omega_i(t)\in C^\infty(R^1), \bar\omega_i\equiv1$ at $|t|<2^{-\lambda_i}$, $\bar\omega_i\equiv0$ at $|t|\ge1; 0\le\bar\omega_i(t)\le1$. It is clear that $\theta(x)\equiv0$ in $\Pi_{\frac{b}{2}}(x_0)$, $\theta(x)\equiv1$ in the neighbourhood of $\partial\Pi_b(x_0)$, we select the coefficients $p(x)$ as $\int\limits_{\Pi_b(x_0)\backslash\Pi_{\frac{b}{2}}(x_0)}(u-p(x))x^{\alpha^\mu}dx=0$. From inequality (22) with the help of inequality (8) we obtain

$$A_\mu(u-p(x),\Pi_b(x_0))\le A_\mu\left(u-p(x),\Pi_b(x_0)\backslash\Pi_{\frac{b}{2}}(x_0)\right)+$$

$$+c_1A_\mu\left(u-p(x),\Pi_b(x_0)\backslash\Pi_{\frac{b}{2}}(x_0)\right)\le\zeta A_\mu\left(u-p(x),\Pi_b(x_0)\backslash\Pi_{\frac{b}{2}}(x_0)\right).\tag{23}$$

Since $A_\mu(u-p(x),G)=A_\mu(u,G)$ then $A_\mu\left(u,\Pi_{\frac{b}{2}}(x_0)\right)\le\left(1-\frac{1}{\varsigma}\right)A_\mu\left(u,\Pi_b(x_0)\right)$ hence by induction we obtain that

$$A_\mu\left(u,\Pi_{\frac{b}{2}}(x_0)\right)\le\left(1-\frac{1}{\varsigma}\right)^kA_\mu\left(u,\Pi_b(x_0)\right).$$

Let $0<\xi<\frac{b}{2^k}$. It follows that $\Pi_\xi(x_0)\subset\Pi_{\frac{b}{2}}(x_0)$. Further $k\ln2<\ln\frac{b}{\xi}$, we take $k=\left[\frac{\ln\frac{b}{\xi}}{\ln2}\right]$, $\omega=1-\frac{1}{\varsigma}$ then

$$A_\mu\left(u,\Pi_b(x_0)\right)\le\omega^kA_\mu(u,G)<\omega^{\frac{\ln\frac{b}{\xi}}{\ln2}-1}A_\mu(u,G)=e^{\frac{\ln\frac{b}{\xi}}{\ln2}\ln\omega-\ln\omega}A_\mu(u,G)=$$

$$= e^{\left(\frac{\ln\omega}{\ln 2}-\frac{\ln\omega}{\ln\frac{b}{\xi}}\right)\ln\frac{b}{\xi}} A_\mu(u,G) = \left(e^{\ln\frac{b}{\xi}}\right)^{\left(\frac{\ln\omega}{\ln 2}-\frac{\ln\omega}{\ln\frac{b}{\xi}}\right)} A_\mu(u,G) =$$

$$= \left(\frac{b}{\xi}\right)^{\left(\frac{\ln\omega}{\ln 2}-\frac{\ln\omega}{\ln\frac{b}{\xi}}\right)} A_\mu(u,G) = \left(\frac{\xi}{b}\right)^{\left|\frac{\ln\omega}{\ln 2}-\frac{\ln\omega}{\ln\frac{b}{\xi}}\right|} A_\mu(u,G) \le \left(\frac{\xi}{b}\right)^{\left|\frac{\ln\omega}{\ln 2}-\frac{\ln\omega}{\ln\frac{b}{\xi}}\right|} A_\mu(u,G),$$

for any $x_0 \in G^d, b \le d$. Denote by $\eta_1 = \left|\frac{\ln\omega}{\ln 2}\right| = \sum\limits_{j=1}^{n}\chi_j a_j$ and $\eta_2 = \left|\frac{\ln\omega}{\ln\frac{b}{\xi}}\right|$ then

$$A_\mu\left(u,\Pi_\xi(x_0)\right) \le \left(\frac{\xi}{b}\right)^{\eta_1-\eta_2} A_\mu\left(u,G\right). \tag{24}$$

It is obvious that $0<\eta_1,\eta_2<1$ assuming $A_\mu\left(u,G\right)=N_f, t=\zeta$ then

$$\int\limits_0^1\left[\zeta^{-\eta_1}\int\limits_{\Pi_\zeta(x_0)}u^2dx\right]^{\frac{1}{2}}\frac{d\zeta}{\zeta}\le C\int\limits_0^1\frac{db}{b^{1-\frac{1}{2}\eta_2}}. \tag{25}$$

It means that $u(x)\in L_{2,\alpha.\chi,1}(G^d)\subset L_{2,\alpha.\chi,r}(G^d)$ and also $D_j^{l^\mu}u(x)\in L_{2,\alpha.\chi,r}(G^d)$, $j=1.,,,.n$ for all $\mu(\mu=1,2,...,N)$. If we check the conditions of theorems 1 and 2 it turns out to be that $\varepsilon_i>0,\varepsilon_{i,0}>0,\ i=1,...,n$. Thus by theorem 1 $D^\nu u(x)$ is continuous on G^d and by theorem 2 $D^\nu u(x)$ satisfies the Hōlder condition, i.e. $u(x)\in C_{\nu+\sigma^1}(G^d)$.

Now we consider nonhomogeneous quasielliptic equation

$$\sum_{\mu=1}^{n}\sum_{\substack{\left|\alpha^\mu,\frac{1}{l^\mu}\right|=1\\ \left|\delta^\mu,\frac{1}{l^\mu}\right|=1}}\left(D^{\alpha^\mu}a_{\alpha^\mu\delta^\mu}(x)D^{\delta^\mu}u(x)\right)=\sum_{\mu=1}^{n}\sum_{\left|\alpha^\mu,\frac{1}{l^\mu}\right|\le 1}D^{\alpha^\mu}f_{\alpha^\mu}$$

where $a_{\alpha_\mu\delta^\mu}(x)$ satisfies earlier imposed restrictions, inequality (20) is satisfied, $f_{\alpha^\mu}\in L_2(G)$ for $\left|\alpha^\mu,\frac{1}{l^\mu}\right|<1$, and $f_{\alpha^\mu}\in L_{2,\alpha,\chi}(G)$ for $\left|\alpha^\mu,\frac{1}{l^\mu}\right|=1$ for all $\mu(\mu=1,2,...,N)$.

We again consider some substitution G^d. Let $x_0\in G^d, b\le d$ and u_{b,x_0} be a solution of equation (19) in $\Pi_b(x_0)$ from $\overset{\circ}{W}{}_2^{l^\mu}\left(\Pi_b(x_0)\right)$. The existence of such solution is obtained by the functional method on the basis of Riesz theorem.

Assuming $\varsigma\equiv u_{b,x_0}$ in (21), by virtue of (20) we obtain

$$\int\limits_{\Pi_b(x_0)}\sum_{\left|\alpha^\mu,\frac{1}{l^\mu}\right|\le 1}\left(D^\mu u_{b,x_0}\right)^2dx\le\sum_{\left|\alpha^\mu,\frac{1}{l^\mu}\right|<1}b^{2-2|\nu,\lambda|}\int\limits_{\Pi_b(x_0)}f_{\alpha^\mu}^2dx+$$

$$+\sum_{\left|\alpha^\mu,\frac{1}{l^\mu}\right|=1}\int\limits_{\Pi_b(x_0)}f_{\alpha^\mu}^2dx\le C_1b^s, \tag{26}$$

here $s=\min\limits_{\left|\alpha^\mu,\frac{1}{l^\mu}\right|\le 1}\left\{\left(2-2\left|\nu,\lambda\right|\right);\left|\chi,a\right|\right\}>0$. C_1 and s do not depend on $u(x)$ and x_0. Since $\bar{u}(x)=u(x)-u_{b,x_0}$ is a solution of equation (19) when the right hand side is zero, therefore for it

$$A_\mu\left(\bar{u},\Pi_\xi(x_0)\right)\le C_2\left(\frac{\xi}{b}\right)^{\eta_1-\eta_2}A_\mu(u,G), \tag{27}$$

is valid for any $\xi<b$ if $x_0\in G^b$.

Then form (26) and (27) we obtain

$$A_\mu\left(u,\Pi_\xi(x_0)\right)\le C_3A_\mu\left(\bar{u},\Pi_\xi(x_0)\right)+C_4A_\mu(u_{b,x_0},\Pi_\xi(x_0))\le C_5\left(\frac{\xi}{b}\right)^{\eta_1-\eta_2}A_\mu(u,G).$$

Further, we again apply theorems 1, 2 and in this case we obtain the required results.

Finally, we consider equation (19) whose all the coefficients different from zero exist for small derivatives of solution. Then we transfer such members to the right hand side of the equation and obtain the required result.

The theorem is proved.

The following theorem on smoothness of solution up to the boundary holds when the generalized solution satisfies the Dirichlet boundary condition.

Theorem 4. *Let the domain G be so that there exists $\varsigma=const>0$ such that whatever be point $x_0\in\partial G$ and the number $\omega<1$ we can find a parallelepiped $\Pi_{\varsigma\omega}(x^1)$ such that $\Pi_{\varsigma\omega}(x^1)\subset\Pi_\omega(x_0)\cap(R^n\backslash G)$ and $u(x)$ is a solution of equation (19) from $\bigcap\limits_{\mu=1}^{N}\overset{\circ}{W}{}_2^{l^\mu}(G)$. If $\frac{|\lambda|}{2}+|\nu,\lambda|\le 1$, then $u(x)$ belong to the space $C_{\nu+\sigma^1}(\overline{C})$.*

The author sincerely thanks to prof. A.D.Jabrailov, prof. V.S.Guliyev and reviewer for their attention to the paper and done remarks.

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Received November 10, 2003; Revised May 24, 2004.
Translated by Mirzoyeva K.S.