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STONE-WEIERSTRASS APPROXIMATION THEOREM FOR PIECEWISE CONTINUOUS FUNCTIONS

Abstract

In the work the classical Stone-Weierstrass theorem is generalized for piecewise continuous functions.

Stone-Weierstrass known theorem relative to the uniform algebras of continuous fucntions is generalized in many directions [see 1, p.418]. In many cases, especially at studying completeness of systems of eigen functions of discontinuous differential operators, it is necessary to study the approximation properties of the sets of piecewise continuous functions, that form the algebra. For example, the system of functions

$$\left\{e^{i(nt+\beta(t))};\, e^{-i(nt+\beta(t))}\right\}_{n=0}^{\infty}$$

where

$$\beta(t)=\left\{\begin{array}{ll}\alpha, & at \quad -\pi\le t<0;\\ \beta, & at \quad 0<t\le\pi,\end{array}\right. \qquad (\alpha,\beta - \text{ are constants})$$

forms the system of eigen functions of the discontinuous differentiable operator

$$y'(x)=\lambda y(x), \quad x\in(-\pi,0)\cup(0,\pi) \ .$$

It is easy to note, that the linear capsule of this system is connected with some subalgebra of algebra of the piecewise continuous functions. Therefore, there arises a question on necessity to generalize the called theorem in corresponding direction. It is remarkable to note, that the similar question evidently is considered for the first time.

First of all we'll introduce some designation and concepts, used later on.

Let $c\in(a,b)$ $(-\infty<a<b<\infty)$. Let's denote by $C_R([a,b];c)$ a space with sup-norm of the real functions f, continuous on $[a,c)\cup(c,b]$ and having left and right limits at the point c, at that (not limiting a generality) $f(c)=f(c-0)$.

Similarly, let $S=\{c_1,c_2,...,c_n\}$ be a finite number of such points, that $c_i\in(a,b)$ $(i=1,2,...,n)$. We'll denote by C_R $([a,b];S)$ the space with sup-norm of the real functions f, continuos on $[a_1,c_1)\cup(c_1,c_2)\cup...\cup(c_n,b]$ and having finite left and right limits at the points c_i $i=1,2,...,n$, where $f(c_i)=f(c_i-0)$.

It is obvious, that the spaces $C_R([a,b];c)$ and $C_R([a,b];S)$ are Banach spaces.

Theorem 1. *Let A be some subalgebra of algebra $C_R([a,b];c)$ which satisfy the following conditions:*

1) A separates the points of the segment $[a,b]$, i.e.

1a) for any different points $x_1,x_2\in[a,c)\cup(c,b]$ there exists such a function $g\in A$, that $g(x_1)\neq g(x_2)$;

1b) for any point $x_1\in[a,c)\cup(c,b]$ there will be found such functions $g,\varphi\in A$ that $g(x_1)\neq g(c)$ and $\varphi(x_1)\neq\varphi(c+0)$;

1c) there exists such a function $g\in A$ that $g(c)\neq g(c+0)$

2) the set A doesn't disappear in one point of the segment $[a,b]$, i.e.
2a) $\forall x_1 \in [a,c) \cup (c,b]$ there exists such a function $h \in A$ such that $h(x_1) \neq 0$;
2b) there exists such functions $h_1, h_2 \in A,$, that $h_1(c) \neq 0, \quad h_2(c+0) \neq 0$
Then $A \equiv C_R([a,b];c)$, where $\bar{A}$ is a closure of A on norm $C_R([a,b];c)$.

For the proof of this theorem we need the following lemma.

Lemma. *Let $x_1, x_2 \in [q,c) \cup (c,b]$ and $x_1 \neq x_2$. Then for any real numbers c_1, c_2, d_1, d_2 there exists $f \in A$ such that*

$$f(x_1) = c_1, \quad f(x_2) = c_2, \quad f(c) = d_1, \quad f(c+0) = d_2 \tag{1}$$

Proof of the lemma. First of all we'll prove, that there exists the functions $u, \upsilon, \varphi, w \in A$ possessing the following properties:

$$u(x_1) = 1, \quad u(x_2) = 0, \quad u(c) = 0, \quad u(c+0) = 0; \tag{2}$$

$$\upsilon(x_1) = 0, \quad \upsilon(x_2) = 1, \quad \upsilon(c) = 0, \quad \upsilon(c+0) = 0; \tag{3}$$

$$\varphi(x_1) = 0, \quad \varphi(x_2) = 0, \quad \varphi(c) = 1, \quad \varphi(c+0) = 0; \tag{4}$$

$$w(x_1) = 0, \quad w(x_2) = 0, \quad w(c) = 0, \quad w(c+0) = 1. \tag{5}$$

Let's prove the existence of the functions u and φ (the existence of the functions υ, w is proved similarly). If there exists functions $u_1, u_2, u_3 \in A$ such, that $u_1(x_1) = 1$, $u_1(x_2) = 0$, $u_2(x_1) = 1$, $u_2(c) = 0$, $u_3(x_1) = 1$, $u_3(c+0) = 0$, then $u = u_1 u_2 u_3 \in A$ has property (2).

We'll prove the existence of the function u_3 (for u_1 and u_2 the proof is similar). From the conditions of theorem 1 it follows, that there will be found functions $g, h \in A$ such that $g(x_1) \neq g(c+0)$ and $h(x_1) \neq 0$. Let's suppose $\tilde{u}_3 = g + \lambda h$, where constant λ is chosen by the following way; if $g(x_1) \neq 0$ then $\lambda = 0$; if $g(c+0) \neq 0$ and there exists a number $\lambda \neq 0$ such, that

$$\tilde{u}_3(x_1) - \tilde{u}_3(c+0) = -g(c+0) + \lambda[h(x_1) - h(c+0)] \neq 0 \; .$$

The choice of λ shows, that $\tilde{u}_3(x_1) \neq 0$ and $\tilde{u}_3(x_1) \neq \tilde{u}_3(c+0)$. Then for the function

$$u_3(x) = \frac{\tilde{u}_3(x)}{u_3(x_1)} \, \frac{\tilde{u}_3(x) - \tilde{u}_3(c+0)}{\tilde{u}_3(x_1) - \tilde{u}_3(c+0)}$$

conditions $u_3(x_1) = 1, \quad u_3(c+0) = 0$ are fulfilled.

Similarly, if $\exists \varphi_i \in A, \quad i = \overline{1,3}$, which satisfy the conditions $\varphi_1(x_1) = 0$, $\varphi_1(c) = 1$; $\varphi_2(x_2) = 0$, $\varphi_2(c) = 1$; $\varphi_3(c+0) = 0$, $\varphi_3(c) = 1$, then function $\varphi = \varphi_1 \varphi_2 \varphi_3 \in A$ has the required property.

For an example we'll prove the existence of φ_3. From the conditions of the theorem it follows the existence of such functions $g, h \in A$ possessing the properties $g(c) \neq g(c+0)$ and $h(c) \neq 0$. Assuming $\tilde{\varphi}_3 = g + \lambda h$ we find a constant λ by the following way; if $g(c) \neq 0$, then $\lambda = 0$; if $g(c) = 0$ (then $g(c+0) \neq 0$), then $\lambda \neq 0$ is found from the condition

$$\tilde{\varphi}_3(c) - \tilde{\varphi}_3(c+0) = -g(c+0) + \lambda[h(c) - h(c+0)] \neq 0 \; .$$

Consequently, $\tilde{\varphi}_3(c) \neq 0$, $\tilde{\varphi}_3(c) \neq \tilde{\varphi}_3(c+0)$. Then the function

$$\varphi_3(x) = \frac{\tilde{\varphi}_3(x)}{\tilde{\varphi}_3(x_1)} \frac{\tilde{\varphi}_3(x) - \tilde{\varphi}_3(c+0)}{\tilde{\varphi}_3(c) - \tilde{\varphi}_3(c+0)}$$

satisfies the conditions $\varphi_3(c) = 1$, $\varphi_3(c+0) = 0$. The existence of functions $\varphi_1, \varphi_2 \in A$ is similarly proved.

So, we've shown the existence of functions $u, v, \varphi, w \in A$ is the required condition (2-5). the fucntion $f_1 = c_1 u + c_2 u + d_1 \varphi + d_2 w \in A$ is a desired funtion.

The lemma is proved.

Proof of theorem 1. Assume B is a closure of the set A on norm $C_R([a,b];c)$. Let's show, that if $f \in A$, then $|f| \in B$. Let $a = \sup\limits_{x\in[a,b]} |f(x)|$ and let $\varepsilon > 0$ be given numbers $c_1, c_2, ..., c_4$ such that for any $y \in [-a, a]$

$$\left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \varepsilon \tag{6}$$

[see 2, p.180]. The function $g = \sum\limits_{i=1}^{n} c_i y^i \in A$ and by (6) for any $x \in [a,b]$ we have

$$|g(x) - |f(x)|| < \varepsilon .$$

Since algebra B is uniformly closed, then from here it follows, that $|f| \in B$. Consequently [see 2, p.183], if $f_1, ..., f_n \in B$, then $\max\{f_1(x), ..., f_n(x)\} \in B$ and $\min\{f_1(x), ..., f_n(x)\} \in B$.

Let any $\varepsilon > 0$ and function $f \in C_R([a,b];c)$ be given.

Let's prove, that $f \in B$.

Since $A \subset B$, then from the lemma it follows, that for any $x, y \in [a,b]$ it is possible to find function $h_{xy} \in B$ such that $h_{xy}(x) = f(x)$, $h_{xy}(y) = f(y)$ (at $x = c$ or $y = c$ we have: $h_{xy}(c) = f(c)$, $h_{xy}(c+0) = f(c+0)$). From these correlations it follows, that there exists a neighbourhood $\cup_{xy}$ of point y such that $\forall t \in \cup_{xy}$ it holds $h_{xy}(t) > f(t) - \varepsilon$. It is obvious, that $[a,b] \subset \bigcup\limits_{y\in[a,b]} \cup_{xy}$. Then we have finite number $y_1, y_2, ..., y_n \in [a,b]$ such that $[a,b] \subset \bigcup\limits_{i=1}^{n} \cup_{xy}$.

Let's consider function $g_x(t) = \max\limits_{t}\{h_{xy_1}(t), ..., h_{xy}(t)\}$ belonging to the set B. It is obvious, that $g_x(x) = f(x)$ $(x \in [a,c) \cup (c,b])$ $g_x(c) = f(c)$, $g_x(c+0) = f(c+0)$. Since, any $t \in [a,b]$ belongs at least to one of the sets $\cup_{xy_i}$, $i = 1, 2, ..., n$ then for any $t \in [a,b]$ the inequality $g_x(t) = \max\limits_{t,k} h_{xy_k}(t) > f(t) - \varepsilon$ is true.

Continuing by the same way for functions g_x at all x we'll construct the system of neighborhoods $\{V_x\}$ covering $[a,b]$ on which an inequality $g_x < f(t) + \varepsilon$, $\forall t \in V_x$ is fulfilled, and using compactness of the segment $[a,b]$ we'll choose a finite number of functions $g_{x_1}(t), g_{x_2}(t), ..., g_{x_m}(t)$. Supposing $\varphi(t) = \min\{g_{x_1}(t), ..., g_{x_m}(t)\} \in B$ it is easy to show that at all $t \in [a,b]$ an inequality $f(t) - \varepsilon < \varphi(t) < f(t) + \varepsilon$ is fulfilled. This means that $f \in B$ and $B = C_R([a,b];c)$.

Theorem 1 is proved.

Remark. If algebra A contains the functions

$$X_1(x) = \begin{cases} 1, & at \ \ 0 \le x \le c; \\ 0, & at \ \ c < x \le b, \end{cases} \quad \text{and} \quad X_2(x) = \begin{cases} 0, & at \ \ a \le x < c; \\ 1, & at \ \ c < x \le b, \end{cases}$$

then using the Stone-Weierstrass theorem it is possible to simplify the proof of the theorem.

Example. Let

$$\varphi(x) = \begin{cases} \alpha, & at \quad -1 \le x \le 0; \\ \beta, & at \quad 0 < x \le 1, \end{cases}$$

$(\alpha, \beta$ are constants, $\alpha \neq \beta,\ \alpha \neq 0,\ \beta \neq 0)$. Then from theorem 1 easily follows, that the systems of functions $\{\varphi(x)\, x^n\}_{n=0}^{\infty}$ and $\{(\varphi(x)\, x^n)\}_{n=0}^{\infty}$ are complete in the space $C_R([-1,1]\,;0)$.

It is easy to remark, that the following theorem in the space $C_R([a,b]\,;S)$ is proved similarly, i.e. the following theorem is true

Theorem 2. *Let A be some subalgebra of algebra $C_R([a,b]\,;S)$ which satisfy the following conditions:*

1) A separates the points of segment $[a,b]$, i.e.;

1a) for any different points $x_1, x_2 \in [a,b] \backslash S$ there exists function $g \in A$ such that $g(x_1) \neq g(x_2)$;

1b) for any points $x_1 \in [a,b] \backslash S$ there will be found functions $g_i, \varphi_i \in A,\ i = 1,2,...,n$ such that $g_i(x_1) \neq g_i(c_i)$ and $\varphi_i(x_1) \neq \varphi_i(c_i+0)$;

1c) there exists functions $g_i \in A,\ i = 1,2,...,n$ for which $g_i(c_i) \neq g_i(c_i+0)$;

2) set A doesn't disappear in one point of segment $[a,b]$, i.e.

2a) for any point $x_1 \in [a,b] \backslash S$ there will be found functions $h \in A$, such that $h(x_1) \neq 0$;

2b) there exists functions $h_i, \psi_i \in A,\ i = 1,2,...,n$ for which $h_i(c_i) \neq 0, \psi_i(c_i+0) \neq 0$.

Then $\bar{A} \equiv C_R([a,b]\,;S)$.

For complexvalued functions there is similar [see 2, p.185]

Theorem 3. *Let A be some complex selfadjoint subalgebra $C([a,b]\,;S)$ of piecewise continuous complexvalued functions with points of discontinuity S on $[a,b]$. If A satisfies the conditions of theorem 2, then $\bar{A} \equiv (C[a,b]\,;S)$, where $\bar{A}$ is a closure of A.*

The author expresses his deep gratitude to prof. B.I.Bilalov for his attention to the work.

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Received September 18, 2003; Revised February 26, 2004.
Translated by Mamedova Sh.N.