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ON GENERALIZED FRACTIOAL INTEGRALS, ASSOCIATED WITH THE BESSEL DIFFERENTIAL EXPANSIONS

Abstract

The properties of generalized fractional integrals, associated with the Bessel differential expansions are studied.

The important properties of Riesz potentials

$$I_\alpha f(x) = \int_{R^n} |x-y|^{\alpha-n} f(y)dy, \qquad 0 < \alpha < n$$

and its generalizations

$$T_K f(x) = \int_{R^n} K(|x-y|) f(y)dy$$

were studied by many authors ([1]-[6], see also [7], [8]). It is known that the fractional integral I_α is bounded from $L_p(R^n)$ to $L_q(R^n)$, when $0 < \alpha < n$, $1 < p < n/\alpha$, and $1/q = 1/p - \alpha/n$ as the Hardy-Littlewood-Sobolev theorem. It is also known (see [7]), that the modified fractional integral

$$\widetilde{I}_\alpha f(x) = \int_{R^n} \left(|x-y|^{\alpha-n} - |y|^{\alpha-n} \chi_{R^n \setminus E_+(0,1)}(y) \right) f(y)dy,$$

is bounded from $L_p(R^n)$ to BMO, when $0 < \alpha < n$, $p = n/\alpha$.

Suppose that R^n is an n-dimensional Euclidean space, $x = (x_1, ..., x_n) = (x', x_n)$ are vectors in R^n, $|x|^2 = \sum\limits_{i=1}^n x_i^2$, $R_+^n = \{x = (x_1, \dots x_n) : \ x_n > 0\}$, $E_+(x,r) = \{y \in R_+^n : |x-y| < r\}$, $|E_+(0,r)|_\gamma = \int\limits_{E_+(0,r)} x_n^\gamma dx = Cr^{n+\gamma}$.

The Bessel differential expansion B_n is defined by

$$B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n} \frac{\partial}{\partial x_n}, \gamma > 0.$$

The $L_{p,\gamma}(R_+^n)$, $1 \le p < \infty$ spaces are defined as the set of all measurable functions $f(x)$, $x \in R_+^n$ on R_+^n with finite norm

$$\|f\|_{L_{p,\gamma}(R_+^n)} = \left(\int_{R_+^n} |f(x)|^p x_n^\gamma dx \right)^{1/p}, \quad 1 \le p < \infty.$$

At $p = \infty$ the spaces $L_\infty(R_+^n)$ are defined by means of usual modification

$$\|f\|_{L_{\infty,\gamma}(R_+^n)} = \|f\|_{L_\infty(R_+^n)} = \operatorname*{esssup}_{x \in R_+^n} |f(x)|.$$

The operator of generalized shift (B_n-shift operator) is defined by the following way (see [9],[10]):

$$T^y f(x) = C_\gamma \int_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha_n + y_n^2}\right) \sin^{\gamma-1} \alpha d\alpha.$$

Note that this shift operator is closely connected with B_n–Bessel's singular differential expansions (see [9], [10]).

Let $1 \le p \le \infty$, $f \in L_{p,\gamma}(R_+^n)$. Then for all $x \in R_+^n$, $T^x f$ belongs $L_{p,\gamma}(R_+^n)$ and

$$\|T^x f\|_{L_{p,\gamma}\left(R_+^n\right)} \le \|f\|_{L_{p,\gamma}\left(R_+^n\right)}. \tag{1}$$

A locally integrable function f will be said to belong to $BMO_\gamma(R_+^n)$ (see [11]), if the norm

$$\|f\|_{BMO_\gamma(R_+^n)} = \sup_{x \in R_+^n, r>0} |E_+(0,r)|_\gamma^{-1} \int_{E_+(0,r)} |T^y f(x) - f_{E_+(0,r)}(x)| y_n^\gamma dy,$$

is finite; here

$$f_{E_+(0,r)}(x) = |E_+(0,r)|_\gamma^{-1} \int_{E_+(0,r)} T^y f(x) y_n^\gamma dy$$

denotes the mean value of f over the ball $E_+(0,r)$.

We denote by M_{B_n} the Hardy-Littlewood maximal operator on R_+^n

$$M_{B_n} f(x) = \sup_{t>0} |E_+(0,t)|_\gamma^{-1} \int_{E_+(0,t)} T^y |f(x)| y_n^\gamma dy.$$

For a function $K : (0,+\infty) \to (0,+\infty)$, let

$$T_{K,\gamma} f(x) = \int_{R_+^n} T^y K(|x|) f(y) y_n^\gamma dy.$$

If $K(t) = t^{\alpha-n-\gamma}$, $0 < \alpha < n+\gamma$, then $T_{K,\gamma}$ is the fractional integral, associated with the Bessel differential operator or the Riesz–Bessel potential denoted by

$$I_{\alpha,\gamma} f(x) = \int_{R_+^n} T^y |x|^{\alpha-n-\gamma} f(y) y_n^\gamma dy.$$

We also consider the modified fractional integral, associated with the Bessel differential expansion B_n

$$\widetilde{I}_{\alpha,\gamma} f(x) = \int_{R_+^n} \left(T^y |x|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma} \chi_{R_+^n \setminus E_+(0,1)}(y)\right) f(y) y_n^\gamma dy.$$

and the modified generalized fractional integral, associated with the Bessel differential expansion B_n

$$\widetilde{T}_{K,\gamma} f(x) = \int_{R_+^n} \left(T^y K(x) - K(y) \chi_{R_+^n \setminus E_+(0,1)}(y)\right) f(y) y_n^\gamma dy.$$

We consider the following conditions on K :

$(K_{1,\gamma})$ $0 \le K(t)$ is decreasing on $(0,\infty)$, $\lim\limits_{t\to 0} K(t) = \infty$;

$(K_{2,\gamma})$ $\exists C_1 > 0,\ \exists \sigma > 0\ \ \forall R > 0\ \ \int\limits_0^R K(t)t^{n+\gamma-1}dt \le C_1 R^{\sigma}$;

$(K_{3,\gamma})$ $\exists C_2 > 0,\ \exists \gamma(p) > 0\ \ \forall R > 0\ \ \left(\int\limits_R^{\infty} K^{p'}(t)t^{n+\gamma-1}dt\right)^{1/p'} \le C_2 R^{-\gamma(p)}$;

$(K_{4,\gamma})$ $\exists C_3 > 0,\ |K(r) - K(s)| \le C_3|r - s|\frac{K(r)}{r},\ \ \frac{1}{2} \le \frac{s}{r} \le 2$;

$(K_{5,\gamma})$ $\exists C_4 > 0,\ \forall R > 0\ \ \left(\int\limits_R^{\infty} K^{p'}(t)t^{n+\gamma-p'-1}dt\right)^{1/p'} \le C_4 R^{-1}$.

Remark 1. Note that conditions $(K_{1,0}) - (K_{3,0})$ consider by A.D.Gadzhiev in [3], $(K_{4,0})$ consider by Nakai E. and H.Sumitomo in [5] and $(K_{5,0})$ consider by V.S.Guliev, R.Ch.Mustafayev in [6].

For generalized fractional integrals was proved by A.D.Gadzhiev [3] the following variant Hardy-Littlewood-Sobolev theorem.

Theorem A.D.Gadzhiev [3]. *Let $f \in L_p(R^n)$, $1 \le p < \infty$ and the kernel K satisfies $(K_{1,0})$, $(K_{2,0})$ and $(K_{3,0})$. Then*

i) The integral $T_K f(x)$ converges absolutly for almost every x;

ii) If $q = \left(1 + \frac{\sigma}{\gamma(p)}\right)p$, where σ is a number form $(K_{2,0})$, then $T_K f$ is of weak-type (p,q);

iii) if $1 < p < r$ and

$$\frac{1}{q} = \frac{1}{p}\left[\frac{r-p}{r-1}\frac{\gamma(1)}{\sigma+\gamma(1)} + \frac{p-1}{r-1}\frac{\gamma(r)}{\sigma+\gamma(r)}\right].$$

Then $T_K f \in L_q(R^n)$ and

$$\|T_K f\|_{L_q(R^n)} \le C\|f\|_{L_p(R^n)}.$$

The following theorem is valid.

Theorem 1 [10]. *Let $0 < \alpha < n+\gamma$, $1 \le p < \frac{n+\gamma}{\alpha}$, and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+\gamma}$.*
a) If $f \in L_{p,\gamma}(R_+^n)$, then the integral $I_{\alpha,\gamma} f$ converges absolutly for almost all $x \in R_+^n$.
b) If $1 < p < \frac{n+\gamma}{\alpha}$, then

$$\|I_{\alpha,\gamma} f\|_{L_{q,\gamma}(R_+^n)} \le C\|f\|_{L_{p,\gamma}(R_+^n)},$$

c) If $f \in L_{1,\gamma}(R_+^n)$, $\frac{1}{q} = 1 - \frac{\alpha}{n+\gamma}$, then

$$\left|\left\{x \in R_+^n :\ I_{\alpha,\gamma} f(x) > \beta\right\}\right|_{\gamma} \le \left(\frac{C}{\beta}\cdot\|f\|_{L_{1,\gamma}(R_+^n)}\right)^q,$$

where C is independent of f.

Theorem 2 [11]. *Let $0 < \alpha < n+\gamma$, $p = \frac{n+\gamma}{\alpha}$, $f \in L_{p,\gamma}(R_+^n)$. Then $\widetilde{I}_{\alpha,\gamma} f \in BMO_{\gamma}(R_+^n)$ and*

$$\left\|\widetilde{I}_{\alpha,\gamma} f\right\|_{BMO_{\gamma}(R_+^n)} \le C_p\,\|f\|_{L_{p,\gamma}(R_+^n)},$$

where C_p *- is dependent only of* p, γ *and* n.

Theorem 3 [11]. *Let* $f \in L_{1,\gamma}(R_+^n)$, *then for* $\alpha > 0$

$$\left|\left\{x \in R_+^n : M_{B_n} f(x) > \alpha\right\}\right|_\gamma \leq \frac{C}{\alpha} \int\limits_{R_+^n} |f(y)| y_n^\gamma dy, \tag{2}$$

where C *is independent of* f.

Let $f \in L_{p,\gamma}(R_+^n)$, $1 < p \leq \infty$, *then* $M_{B_n} f(x) \in L_{p,\gamma}(R_+^n)$ *and*

$$\|M_{B_n} f\|_{p,\gamma} \leq C_p \|f\|_{p,\gamma}, \tag{3}$$

where C_p *- is dependent only of* p, γ *and* n.

Remark 2. In the one dimensional case theorem 3 was proved by K.Stempak [12].

Analog theorema A.D.Gadzhiev for B_n–generalized fractional integrals is valid.

Theorem 4. *Let* $f \in L_{p,\gamma}(R_+^n)$, $1 \leq p < \infty$ *and the kernel* K *satisfies* $(K_{1,\gamma})$, $(K_{2,\gamma})$ *and* $(K_{3,\gamma})$. *Then*

i_γ) *The integral* $T_{K,\gamma} f(x)$ *converges absolutly for almost every* x;

ii_γ) *If* $q = \left(1 + \frac{\sigma}{\gamma(p)}\right) p$, *where* σ *is a number form* $(K_{2,\gamma})$, *then* $T_{K,\gamma} f$ *is of weak-type* (p, q);

iii_γ) *if* $1 < p < r$ *and*

$$\frac{1}{q} = \frac{1}{p}\left[\frac{r-p}{r-1}\frac{\gamma(1)}{\sigma+\gamma(1)} + \frac{p-1}{r-1}\frac{\gamma(r)}{\sigma+\gamma(r)}\right].$$

Then $T_{K,\gamma} f \in L_{q,\gamma}(R_+^n)$ *and*

$$\|T_{K,\gamma} f\|_{L_{q,\gamma}(R_+^n)} \leq C \|f\|_{L_{p,\gamma}(R_+^n)}.$$

Proof. Idea of proved the theorem analogously of the theorem A.D.Gadzhiev [3]. Fixing any $t > 0$ we have

$$|T_{K,\gamma} f(x)| \leq \int\limits_{E_+(0,t)} K(|y|) T^y |f(x)| y_n^\gamma dy +$$

$$+ \int\limits_{R_+^n \backslash E_+(0,t)} K(|y|) T^y |f(x)| y_n^\gamma dy = A(x,t) + C(x,t).$$

Let's estimate $A(x,t)$. Taking into account $(K_{2,\gamma})$ we obtain

$$A(x,t) = \sum_{k=-\infty}^{-1} \int\limits_{2^k t \leq |y| \leq 2^{k+1} t} K(|y|) T^y |f(x)| y_n^\gamma dy \leq$$

$$\leq \sum_{k=-\infty}^{-1} K(2^k t) \int\limits_{2^k t \leq |y| \leq 2^{k+1} t} T^y |f(x)| y_n^\gamma dy \leq$$

$$\leq CM_{B_n}f(x)\sum_{k=-\infty}^{-1}K(2^k t)(2^k t)^{n+\gamma}\leq$$

$$\leq CM_{B_n}f(x)\int_0^t K(\tau)\tau^{n+\gamma-1}d\tau\leq Ct^{\sigma}M_{B_n}f(x).$$

There for

$$A(x,t)\leq Ct^{\sigma}M_{B_n}f(x), \tag{4}$$

where C does not depend of f, x and t.

This means that integral $A(x,t)$ converges almost everywhere.

On the other hand applying the Hőlder's inequality and using $(K_{3,\gamma})$ we get

$$C(x,t)\leq\left(\int\limits_{R_+^n\setminus E_+(0,t)}T^y|f(x)|^p y_n^{\gamma}dy\right)^{1/p}\left(\int\limits_{R_+^n\setminus E_+(0,t)}K(|y|)^{p'}y_n^{\gamma}dy\right)^{1/p'}\leq$$

$$\leq C\|f\|_{L_{p,\gamma}(R_+^n)}\left(\int_t^{\infty}K(\tau)^{p'}\tau^{n+\gamma-1}d\tau\right)^{1/p'}\leq Ct^{-\gamma(p)}\|f\|_{L_{p,\gamma}(R_+^n)}.$$

Consequently

$$C(x,t)\leq C_1t^{-\gamma(p)}\|f\|_{L_{p,\gamma}(R_+^n)}. \tag{5}$$

Then the integral $T_{K,\gamma}f(x)$ absolutly converges at almost every $x\in R_+^n$ for function $f\in L_{p,\gamma}(R_+^n)$, $1\leq p<\infty$.

The proof of $i_{\gamma})$ is completed.

Proof of $ii_{\gamma})$.

Let $f\in L_{p,\gamma}(R_+^n)$. Now, for any $\beta>0$ we have

$$\left|\left\{x\in R_+^n:|T_{K,\gamma}f(x)|>\beta\right\}\right|_{\gamma}\leq\left|\left\{x\in R_+^n:A(x,t)>\frac{\beta}{2}\right\}\right|_{\gamma}+$$

$$+\left|\left\{x\in R_+^n:C(x,t)>\frac{\beta}{2}\right\}\right|_{\gamma}. \tag{6}$$

From (4) and the theorem 3 we get

$$\left|\left\{x\in R_+^n:A(x,t)>\frac{\beta}{2}\right\}\right|_{\gamma}\leq\left|\left\{x\in R_+^n:M_{B_n}f(x)>\frac{\beta}{Ct^{\sigma}}\right\}\right|_{\gamma}\leq$$

$$\leq\left(\frac{Ct^{\sigma}}{\beta}\right)^p\int\limits_{R_+^n}(M_{B_n}f(x))^p x_n^{\gamma}dx\leq C\frac{t^{\sigma p}}{\beta^p}\int\limits_{R_+^n}|f(x)|^p x_n^{\gamma}dx.$$

Hence for $t=\left\{2C_1\|f\|_{L_{p,\gamma}(R_+^n)}\right\}^{\frac{1}{\gamma(p)}}\beta^{-\frac{1}{\gamma(p)}}$ we have

$$\left|\left\{x\in R_+^n:A(x,t)>\frac{\beta}{2}\right\}\right|_{\gamma}\leq C\|f\|_{L_{p,\gamma}(R_+^n)}^{p+\frac{\sigma p}{\gamma(p)}}\cdot\beta^{-\frac{\sigma p}{\gamma(p)}-p}.$$

Take into account

$$\frac{1}{q}=\frac{1}{q}-\frac{\sigma}{p(\gamma(p)+\sigma)}=\frac{\gamma(p)}{p(\gamma(p)+\sigma)}\Rightarrow q=\left(1+\frac{\sigma}{\gamma(p)}\right)p$$

we get

$$\left|\left\{x\in R_{+}^{n}:A(x,t)>\frac{\beta}{2}\right\}\right|_{\gamma}\leq\left(\frac{\|f\|_{L_{p,\gamma}(R_{+}^{n})}}{\beta}\right)^{q}. \tag{7}$$

On the other hand, (5) implies

$$|C(x,t)|\leq C_{1}t^{-\gamma(p)}\|f\|_{L_{p,\gamma}(R_{+}^{n})}=\frac{\beta}{2},$$

and therefore

$$\left|\left\{x\in R_{+}^{n}:C(x,t)>\frac{\beta}{2}\right\}\right|_{\gamma}=0. \tag{8}$$

From the inequalities (7), (8) and (6) it follows that

$$\left|\left\{x\in R_{+}^{n}:|T_{k,\gamma}|>\beta\right\}\right|_{\gamma}\leq C\left(\frac{\|f\|_{L_{p,\gamma}(R_{+}^{n})}}{\beta}\right)^{q},\ \ q=\left(1+\frac{\sigma}{\gamma(p)}\right)p.$$

$ii_{\gamma})$ has been proved.

Proof of $iii_{\gamma})$.

Applying the $ii_{\gamma})$ with $p=1$ and $p=r$ we see that the operator $T_{k,\gamma}$ has weak type

$$\left(1,1+\frac{\sigma}{\gamma(1)}\right)_{\gamma}\ \text{ and; }\left(r,\left(1+\frac{\sigma}{\gamma(1)}\right)r\right)_{\gamma}.$$

Using Marcinkiewic interpolation theorem with measure $d\mu(x)=d\nu(x)=x_{n}^{\gamma}dx$ (see [7]) with $p_0=1$, $q_0=1+\frac{\sigma}{\gamma(1)}$ and $p_1=r$, $q_1=\left(1+\frac{\sigma}{\gamma(1)}\right)r$ we obtain $iii_{\gamma})$.

It is valid

Theorem 5. *a) Let $f\in L_{p,\gamma}(R_{+}^{n})$, $1\leq p<\infty$ and the kernel K satisfies $(K_{1,\gamma})$, $(K_{2,\gamma})$ and $(K_{3,\gamma})$. Then the integral $T_{K,\gamma}f(x)$ absolutly converges a.e. in R_{+}^{n}.*

b) Let $1<p<\infty$, $\frac{1}{p}-\frac{1}{q}=\frac{\sigma}{p(\gamma(p)+\sigma)}$ and the kernel K satisfies $(K_{1,\gamma})$, $(K_{2,\gamma})$ and $(K_{3,\gamma})$. Then $T_{K,\gamma}f\in L_{q,\gamma}(R_{+}^{n})$ and

$$\|T_{K,\gamma}f\|_{L_{q,\gamma}(R_{+}^{n})}\leq C\|f\|_{L_{p,\gamma}(R_{+}^{n})}.$$

c) Let $1-\frac{1}{q}=\frac{\sigma}{n+\gamma}$, the kernel K satisfies $(K_{1,\gamma})$, $(K_{2,\gamma})$.
Then

$$\left|\left\{x\in R_{+}^{n}:\ |T_{K,\gamma}|\,f(x)>\beta\right\}\right|_{\gamma}\leq\left(\frac{C}{\beta}\cdot\|f\|_{L_{1,\gamma}(R_{+}^{n})}\right)^{q},$$

where C is independent of f.

Proof. Let's prove b).

From (4) and (5) we get

$$|T_{K,\gamma}(x)|\leq C\left(t^{\sigma}M_{B_{n}}f(x)+t^{-\gamma(p)}\|f\|_{L_{p,\gamma}(R_{+}^{n})}\right). \tag{9}$$

Minimizing on t at $\left(\frac{\gamma(p)\|f\|_{L_{p,\gamma}(R_+^n)}}{\sigma M_{B_n}f(x)}\right)^{\frac{1}{\sigma+\gamma(p)}}$ we have

$$|T_{K,\gamma}f(x)| \leq C(\sigma,\gamma(p))\left(M_{B_n}f(x)\right)^{\frac{\gamma(p)}{\gamma(p)+\sigma}}\|f\|_{L_{p,\gamma}(R_+^n)}^{\frac{\sigma}{\gamma(p)+\sigma}}. \tag{10}$$

Therefore by (2)

$$\|T_{K,\gamma}f\|_{L_{q,\gamma}(R_+^n)} \leq C\|f\|_{L_{p,\gamma}(R_+^n)}^{\frac{\sigma}{\gamma(p)+\sigma}}\left(\int\limits_{R_+^n}\left(M_{B_n}f(x)\right)^{\frac{\gamma(p)q}{\gamma(p)+\sigma}}dx\right)^{1/q} =$$

$$= C\|f\|_{L_{p,\gamma}(R_+^n)}^{\frac{\sigma}{\gamma(p)+\sigma}}\left(\int\limits_{R_+^n}\left(M_{B_n}f(x)\right)^{p}dx\right)^{1/q} \leq$$

$$\leq C\|f\|_{L_{p,\gamma}(R_+^n)}^{\frac{\sigma}{\gamma(p)+\sigma}}\cdot\|f\|_{L_{p,\gamma}(R_+^n)}^{\frac{p}{q}} = C\|f\|_{L_{p,\gamma}(R_+^n)}.$$

The proof of (b) is completed.

Note that is (10) follows that the integral $T_{K,\gamma}f(x)$ absolutly converges at almost every $x \in R_+^n$ for function $f \in L_{p,\gamma}(R_+^n)$, $1 \leq p < \infty$.

Proof of c).

Let $f \in L_{1,\gamma}(R_+^n)$. It is enough to show the inequality

$$|\{x \in R_+^n : |T_{K,\gamma}f(x)| > \beta\}|_\gamma \leq C\left(\frac{\|f\|_{L_{1,\gamma}(R_+^n)}}{\beta}\right)^q$$

with 2β instead of β on the left hand of it. Then

$$|\{x \in R_+^n : |T_{K,\gamma}f(x)| > 2\beta\}|_\gamma \leq |\{x \in R_+^n : A(x,t) > \beta\}|_\gamma +$$

$$+|\{x \in R_+^n : C(x,t) > \beta\}|_\gamma.$$

By (1) and (4) we get

$$\beta|\{x \in R_+^n : A(x,t) > \beta\}|_\gamma \leq \beta\left|\left\{x \in R_+^n : M_{B_n}f(x) > \frac{\beta}{Ct^\sigma}\right\}\right|_\gamma \leq$$

$$\leq \beta\cdot\frac{Ct^\sigma}{\beta}\int\limits_{R_+^n}|f(x)|x_n^\gamma dx = Ct^\sigma\|f\|_{L_{1,\gamma}(R_+^n)}.$$

As well

$$C(x,t) \leq CK(t)\int\limits_{R_+^n\setminus E_+(0,t)}T^y|f(x)|y_n^\gamma dy \leq CK(t)\|f\|_{L_{1,\gamma}(R_+^n)}.$$

From $(K_{1,\gamma})$ and $(K_{2,\gamma})$ it follows that

$$K(t) \le C_2 t^{-n-\gamma} \int_0^t K(\tau)\tau^{n+\gamma-1} d\tau \le C_2 t^{-n-\gamma+\sigma}.$$

Thus

$$C(x,t) \le C_0 t^{\sigma-n-\gamma} \|f\|_{L_{1,\gamma}(R_+^n)}. \qquad (11)$$

If $C_0 t^{\sigma-n-\gamma}\|f\|_{L_{1,\gamma}(R_+^n)} = \beta$, then $C(x,t) \le \beta$, and, consequently, $|\{x \in R_+^n : C(x,t) > \beta\}|_\gamma = 0$. Thus

$$|\{x \in R_+^n : |T_{K,\gamma} f(x)| > 2\beta\}|_\gamma \le \frac{C}{\beta} t^\sigma \|f\|_{L_{1,\gamma}(R_+^n)} \le$$

$$\le C\beta^{\frac{n+\gamma}{\sigma-n-\gamma}} \|f\|_{L_{1,\gamma}(R_+^n)}^{\frac{n+\gamma}{n+\gamma-\sigma}} =$$

$$= C\left(\frac{\|f\|_{L_{1,\gamma}(R_+^n)}}{\beta}\right)^{\frac{n+\gamma}{n+\gamma-\sigma}} = C\left(\frac{\|f\|_{L_{1,\gamma}(R_+^n)}}{\beta}\right)^q.$$

(c) has been proved.

Theorem 6. *Let $f \in L_{p,\gamma}(R_+^n)$, $1 < p < \infty$, $\sigma = \frac{n+\gamma}{p}$ and the kernel K satisfies $(K_{1,\gamma})$, $(K_{2,\gamma})$, $(K_{4,\gamma})$ and $(K_{5,\gamma})$. Then $\widetilde{T}_{K,\gamma} \in BMO_\gamma(R_+^n)$ and*

$$\|\widetilde{T}_{K,\gamma}\|_{BMO_\gamma(R_+^n)} \le C\|f\|_{L_{p,\gamma}(R_+^n)}.$$

Proof. Let $f \in L_{p,\gamma}(R_+^n)$. For any fixed $t > 0$ we put

$$f_1(x) = f(x)\chi_{E_+(0,2t)}(x), \quad f_2(x) = f(x) - f_1(x).$$

Then

$$\widetilde{T}_{K,\gamma} f(x) = \widetilde{T}_{K,\gamma} f_1(x) + \widetilde{T}_{K,\gamma} f_2(x) = F_1(x) + F_2(x),$$

where

$$F_1(x) = \int_{E_+(0,2t)} \left[T^y K(|x|) - K(|y|)\chi_{R_+^n \setminus E_+(0,1)}(y)\right] f(y) y_n^\gamma dy,$$

$$F_2(x) = \int_{R_+^n \setminus E_+(0,2t)} \left[T^y K(|x|) - K(|y|)\chi_{R_+^n \setminus E_+(0,1)}(y)\right] f(y) y_n^\gamma dy.$$

Note that the function f_1 has compact support and that's why

$$a_1 = -\int_{E_+(0,2t)\setminus E_+(0,\min\{1,2t\})} K(|y|) f(y) y_n^\gamma dy$$

is finite.

Taking into account that

$$E_+(0,2t)\cap\left(R_+^n\setminus E_+(0,1)\right)=\begin{cases}\emptyset, & 2t<1\\ E_+(0,2t)\setminus E_+(0,1), & 2t>1\end{cases}=$$

$$=E_+(0,2t)\setminus E_+(0,min\{1,2t\}),$$

we get

$$F_1(x)-a_1=\int\limits_{E_+(0,2t)}\left[T^yK(|x|)-K(|y|)\chi_{R_+^n\setminus E_+(0,1)}(y)\right]f(y)y_n^\gamma dy+$$

$$+\int\limits_{E_+(0,2t)\setminus E_+(0,\min\{1,2t\})}K(|y|)f(y)y_n^\gamma dy=\int\limits_{E_+(0,2t)}T^yK(|x|)f(y)y_n^\gamma dy-$$

$$-\int\limits_{E_+(0,2t)\cap R_+^n\setminus E_+(0,1)}K(|y|)f(y)y_n^\gamma dy+\int\limits_{E_+(0,2t)\setminus E_+(0,\min\{1,2t\})}K(|y|)f(y)y_n^\gamma dy=$$

$$=\int\limits_{E_+(0,2t)}T^yK(|x|)|f(y)|y_n^\gamma dy.$$

Then

$$F_1(x)-a_1=T_{K,\gamma}f_1(x),$$

$$T^yF_1(x)-a_1=T^yF_1(x)-T^ya_1=T^y(F_1(x)-a_1)=T^y(T_{K,\gamma}f_1)(x)$$

and

$$|T^yF_1(x)-a_1|=|T^{-x}F_1(-y)-a_1|\le T^{-x}\left(\int\limits_{E_+(0,2t)}T^zK(|-y|)|f(z)|z_n^\gamma dz\right)=$$

$$=T^{-x}\left(\int\limits_{R_+^n}T^zK(|-y|)\left(f\chi_{E_+(0,2t)}\right)(z)z_n^\gamma dz\right)=$$

$$T^{-x}\left(\int\limits_{R_+^n}K(|z|)T^z\left(f\chi_{E_+(0,2t)}\right)(-y)z_n^\gamma dz\right).$$

Note that

$$T^z\left(f\chi_{E_+(0,2t)}\right)(-y)=C_\gamma\int\limits_0^\pi f\left(-y'-z',\sqrt{y_n^2+2y_nz_n\cos\alpha+z_n^2}\right)\times$$

$$\times\chi_{E_+(0,2t)}\left(-y'-z',\sqrt{y_n^2+2y_nz_n\cos\alpha+z_n^2}\right)\sin^{\gamma-1}\alpha d\alpha,$$

$$|y-\overline{z}|=|(y'+z',y_n-z_n)|\le$$

$$\left|\left(-y'-z',\sqrt{y_n^2+2y_nz_n\cos\alpha+z_n^2}\right)\right|\le\left|\left(y'+z',y_n+z_n\right)\right|=|y+z| \tag{12}$$

From (12) implies, that for $|y+z|\ge 2t$

$$T^z\left(f\chi_{E_+(0,2t)}\right)(-y)=0.$$

Then

$$T^{-x}\left(\int_{R_+^n}K(|z|)T^z\left(f\chi_{E_+(0,2t)}\right)(-y)z_n^\gamma dz\right)=$$

$$=T^{-x}\left(\int_{\{z:|y+z|<2t\}}K(|z|)T^z\left(f\chi_{E_+(0,2t)}\right)(-y)z_n^\gamma dz\right),$$

Taking into account, that from $|y|<t, |y+z|<2t$, we get $|z|\le|y+z|+|y|<3t$,

$$|T^yF_1(x)-a_1|\le T^{-x}\left(\int_{\{z:|y+z|<2t\}}K(|z|)T^z\left|f(-y)\right|z_n^\gamma dz\right)=$$

$$=\int_{\{z:|y+z|<2t\}}K(|z|)T^{-x}T^z\left|f(-y)\right|z_n^\gamma dz\le$$

$$\le\int_{E_+(0,3t)}K(|z|)T^{-x}T^z\left|f(-y)\right|z_n^\gamma dz=$$

$$=\int_{E_+(0,3t)}K(|z|)T^z\left(T^{-x}\left|f(-y)\right|\right)z_n^\gamma dz$$

Therefore

$$\int_{E_+(0,3t)}K(|z|)T^z(T^{-x}|f(-y)|)z_n^\gamma dz=$$

$$=\sum_{k=-\infty}^{-1}\int_{3^k\cdot 3t\le|z|<3^{k+1}\cdot 3t}K(|z|)T^z(T^{-x}|f(-y)|)z_n^\gamma dz\le$$

$$\le C\sum_{k=-\infty}^{-1}K\left(3^k\cdot 3t\right)\int_{3^k\cdot 3t\le|z|<3^{k+1}\cdot 3t}T^z(T^{-x}|f(-y)|)z_n^\gamma dz\le$$

$$\le CM_{B_n}(T^{-x}|f(-y)|)\sum_{k=-\infty}^{-1}K\left(3^k\cdot 3t\right)\left(3^{k+1}\cdot 3t\right)^{n+\gamma}\le$$

$$\le CM_{B_n}(T^{-x}|f(-y)|)\int_0^{3t}K(\tau)\tau^{n+\gamma-1}d\tau\le$$

$$\leq Ct^{\sigma}M_{B_n}(T^{-x}|f(-y)|)=Ct^{\sigma}M_{B_n}(T^{y}|f(x)|).$$

Using (3) at $\sigma=\frac{n+\gamma}{p}$ we obtain

$$\frac{1}{|E_+(0,t)|}\int\limits_{E_+(0,t)}|T^yF_1(x)-a_1|y_n^{\gamma}dy\leq Ct^{-n-\gamma}\int\limits_{E_+(0,t)}t^{\sigma}M_{B_n}\left(T^y|f(x)|\right)y_n^{\gamma}dy=$$

$$=Ct^{\sigma-n-\gamma}\int\limits_{E_+(0,t)}M_{B_n}(T^y|f(x)|)y_n^{\gamma}dy\leq$$

$$\leq Ct^{\sigma-n-\gamma}\left(\int\limits_{E_+(0,t)}M_{B_n}\left(T^y|f(x)|\right)^p y_n^{\gamma}dy\right)^{1/p}\left(\int\limits_{E_+(0,t)}y_n^{\gamma}dy\right)^{1/p'}\leq$$

$$\leq Ct^{-n-\gamma+\sigma+\frac{n+\gamma}{p'}}\|M_{B_n}(T^{\cdot}|f(x)|)\|_{L_{p,\gamma}(R_+^n)}\leq Ct^{\sigma-\frac{n+\gamma}{p}}\|T^{\cdot}f(x)\|_{L_{p,\gamma}(R_+^n)}\leq$$

$$\leq Ct^{\sigma-\frac{n+\gamma}{p}}\|f\|_{L_{p,\gamma}(R_+^n)}=C\|f\|_{L_{p,\gamma}(R_+^n)},$$

where $\|T^{\cdot}f(x)\|_{L_{p,\gamma}(R_+^n)}=\|T^yf(x)\|_{L_p(R_+^n,y_n^{\gamma}dy)}$.

Denote by

$$a_2=\int\limits_{E_+(0,\max\{1,2t\})\setminus E_+(0,2t)}K(|y|))f(y)y_n^{\gamma}dy.$$

Taking into account, that

$$\left(R_+^n\setminus E_+(0,2t)\right)\cap\left(R_+^n\setminus E_+(0,1)\right)=\begin{cases}R_+^n\setminus E_+(0,2t), & 2t\geq 1,\\ R_+^n\setminus E_+(0,1), & 2t<1\end{cases}$$

and

$$E_+(0,max\{1,2t\})\setminus E_+(0,2t)=\begin{cases}\emptyset, & 2t\geq 1,\\ E_+(0,1)\setminus E_+(0,2t), & 2t<1,\end{cases}$$

then

$$F_2(x)-a_2=\int\limits_{R_+^n\setminus E_+(0,2t)}T^yK(|x|)f(y)y_n^{\gamma}dy-$$

$$-\left(\int\limits_{R_+^n\setminus E_+(0,1)}+\int\limits_{E_+(0,1)\setminus E_+(0,2t)}\right)K(|y|)f(y)y_n^{\gamma}dy=$$

$$=\int\limits_{R_+^n\setminus E_+(0,2t)}T^yK(|x|)f(y)y_n^{\gamma}dy-\int\limits_{R_+^n\setminus E_+(0,2t)}K(|y|)f(y)y_n^{\gamma}dy$$

$$=\int\limits_{R_+^n\setminus E_+(0,2t)}\left[T^yK(|x|)-K(|y|)\right]f(y)y_n^{\gamma}dy.$$

Let's estimate $|T^yF_2(x)-a_2|$.

$$|T^yF_2(x)-a_2| = |T^{-x}F_2(y)-a_2| =$$

$$= T^{-x}\int\limits_{R^n_+\backslash E_+(0,2t)} \left[T^zK(|-y|)-K(|z|)\right]f(z)z_n^\gamma dz.$$

Therefore

$$|T^zK(|-y|)-K(|z|)| =$$

$$= C_\gamma\int\limits_0^\pi \left|K\left(\left|\left(-y'-z',\sqrt{y_n^2+2y_nz_n\cos\alpha+z_n^2}\right)\right|\right)-K(|z|)\right|\sin^{\gamma-1}\alpha d\alpha.$$

Taking into account $(K_{4,\gamma})$ we get

$$C_\gamma\int\limits_0^\pi \left|K\left(\left|\left(-y'-z',\sqrt{y_n^2+2y_nz_n\cos\alpha+z_n^2}\right)\right|\right)-K\left(\left|(z',z_n)\right|\right)\right|\sin^{\gamma-1}\alpha d\alpha \leq$$

$$\leq C\cdot C_\gamma\int\limits_0^\pi |y|\frac{K(|z|)}{|z|}\sin^{\gamma-1}\alpha d\alpha = C|y|\frac{K(|z|)}{|z|} \leq C\cdot t\frac{K(|z|)}{|z|}.$$

In fact,

$$\left|K\left(\left|\left(-y'-z',\sqrt{y_n^2+2y_nz_n\cos\alpha+z_n^2}\right)\right|\right)-K(|z|)\right| = |K(s)-K(r)|.$$

where $s=\left(-y'-z',\sqrt{y_n^2+2y_nz_n\cos\alpha+z_n^2}\right)$ and $r=|z|=|(z',z_n)|=|\overline{z}|$. Note, that

$$|r-s|\leq|y|\leq t$$

and taking into account, that $|z|>2t$, $|y|<t$ we get

$$\frac{s}{r}\leq\frac{|y+z|}{|z|}\leq 1+\frac{|y|}{|z|}<1+\frac{1}{2}<2,$$

$$\frac{s}{r}\geq\frac{|y+\overline{z}|}{|z|}\geq\frac{|\overline{z}|-|y|}{|z|}=\frac{|z|-|y|}{|z|}=1-\frac{|y|}{|z|}\geq 1-\frac{1}{2}=\frac{1}{2}.$$

$$|K(s)-K(r)|\leq Ct\frac{K(|z|)}{|z|}.$$

$$|T^{-x}F_2(-y)-a_2|\leq T^{-x}\int\limits_{R^n_+\backslash E_+(0,2t)} Ct\cdot\frac{K(|z|)}{|z|}f(z)z_n^\gamma dz.$$

From Hőlder's inequality it follows that

$$Ct\cdot\int\limits_{R^n_+\backslash E_+(0,2t)}\frac{K(|z|)}{|z|}f(z)z_n^\gamma dz\leq Ct\|f\|_{L_{p,\gamma}(R^n_+)}\times$$

$$\times\left(\int_{2t}^{\infty} K(\tau)^{p'}\tau^{n+\gamma-p'-1}d\tau\right)^{1/p'} \le C\|f\|_{L_{p,\gamma}(R_+^n)},$$

for $y\in E_+(0,1)$.

Obviously, we obtain

$$|T^yF_2(x)-a_2|\le T^{-x}\left(C\|f\|_{L_{p,\gamma}(R_+^n)}\right)=C\|f\|_{L_{p,\gamma}(R_+^n)}.$$

Denote by

$$a_f=a_1+a_2.$$

Finally,

$$\sup_{x,t}\frac{1}{|E_+(0,t)|}\int_{E_+(0,t)}\left|T^y\left(\widetilde{T}_{K,\gamma}f(x)\right)-a_f\right|y_n^{\gamma}dy\le C\|f\|_{L_{p,\gamma}(R_+^n)}.$$

Hence

$$\|\widetilde{T}_{K,\gamma}f\|_{BMO_\gamma(R_+^n)}\le C\|f\|_{L_{p,\gamma}(R_+^n)}.$$

Corollary 1. *Let $1<p<\infty$, $\sigma=\frac{n+\gamma}{p}$, $f\in L_{p,\gamma}(R_+^n)$ and the kernel K satisfies $(K_{1,\gamma})$, $(K_{2,\gamma})$, $(K_{4,\gamma})$ and $(K_{5,\gamma})$. If $T_{K,\gamma}f$ absolutely converges a.e. in R_+^n, then $T_kf\in BMO_\gamma(R_+^n)$ and*

$$\|T_{K,\gamma}f\|_{BMO_\gamma(R_+^n)}\le\|f\|_{L_{p,\gamma}(R_+^n)}.$$

Remark 3. Let $0<\alpha<n+\gamma$, $\sigma=\alpha$, $\gamma(p)=\frac{n+\gamma}{p}-\alpha$. Then the conditions $(K_{1,\gamma})$, $(K_{2,\gamma})$, $(K_{3,\gamma})$, $(K_{4,\gamma})$ and $(K_{5,\gamma})$ are valid for the kernel $K(t)=t^{\alpha-n-\gamma}$ and therefore theorem 2 and theorem 3 are valid.

Corollary 2. *Let $0<\alpha<n+\gamma$, $1\le p<\frac{n+\gamma}{\alpha}$, and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n+\gamma}$*

a) If $f\in L_{p,\gamma}(R_+^n)$, then the integral

$$J_{\alpha,\gamma}f(x)=\int_{R_+^n}|y|^{\alpha-n-\gamma}\ln\left(1+\frac{1}{|y|}\right)T^yf(x)y_n^{\gamma}dy$$

converges absolutly for almost all $x\in R_+^n$.

b) If $1<p<\frac{n+\gamma}{\alpha}$, then

$$\|J_{\alpha,\gamma}f\|_{L_{q,\gamma}(R_+^n)}\le C_p\|f\|_{L_{p,\gamma}(R_+^n)},$$

c) If $f\in L_{1,\gamma}(R_+^n)$, $\frac{1}{q}=1-\frac{\alpha}{n+\gamma}$, then

$$\left|\left\{x\in R_+^n:\ J_{\alpha,\gamma}f(x)>\beta\right\}\right|_\gamma\le\left(\frac{C}{\beta}\cdot\|f\|_{L_{1,\gamma}(R_+^n)}\right)^q.$$

d) If $p=\frac{n+\gamma}{\alpha}$, $f\in L_{p,\gamma}(R_+^n)$, then the $\widetilde{J}_{\alpha,\gamma}f\in BMO_\gamma(R_+^n)$ and

$$\left\|\widetilde{J}_{\alpha,\gamma}f\right\|_{BMO_\gamma(R_+^n)}\le C_p\|f\|_{L_{p,\gamma}(R_+^n)}.$$

Proof of Corollary 2. Let $0<\sigma<\alpha<n+\gamma,\ \sigma=\alpha-1,\ 0<\gamma(p)=\frac{n+\gamma}{p}+1-\alpha$ and $K_1(t)=t^{\alpha-n-\gamma}\ln(1+\frac{1}{t})$. Then we shall prove the conditions $(K_{1,\gamma})$, $(K_{2,\gamma})$, $(K_{3,\gamma})$, $(K_{4,\gamma})$ and $(K_{5,\gamma})$ are valid for the kernel K_1. Therefore Corollary 2 is valid.

Let's us check $(K_{2,\gamma})$:

$$\int\limits_0^R K(t)t^{n+\gamma-1}dt=\int\limits_0^R t^{\alpha-1}\ln\left(1+\frac{1}{t}\right)dt\le\int\limits_0^R t^{\alpha-2}dt=CR^{\alpha-1}=CR^{\sigma}.$$

Then

$$\left(\int\limits_R^\infty K^{p'}(t)t^{n+\gamma-1}dt\right)^{1/p'}=\left(\int\limits_R^\infty t^{(\alpha-n-\gamma)p'+n+\gamma-1}\ln^{p'}\left(1+\frac{1}{t}\right)dt\right)^{1/p'}\le$$

$$\le\left(\int\limits_R^\infty t^{(\alpha-n-\gamma-1)p'+n+\gamma-1}dt\right)^{1/p'}=CR^{\alpha-\frac{n+\gamma}{p}-1}=CR^{-\gamma(p)}.$$

In order to prove $(K_{4,\gamma})$ we apply mean-value theorem

$$|K(r)-K(s)|=|K'(\xi)||r-s|=$$
$$=\left|\xi^{\alpha-n-\gamma-1}\left((\alpha-n+\gamma)\ln\left(1+\frac{1}{\xi}\right)-\frac{1}{1+\xi}\right)\right||r-s|.$$

Since

$$\frac{1}{1+\xi}\le\ln\left(1+\frac{1}{\xi}\right),$$

then we get

$$|K(r)-K(s)|\le C\frac{K(r)}{r}|r-s|\quad\text{for}\quad\frac{1}{2}\le\frac{s}{r}\le 2.$$

Finally, taking into account that $\alpha-1=\frac{n+\gamma}{p}$ we have

$$\left(\int\limits_R^\infty K^{p'}(t)t^{n+\gamma-p'-1}dt\right)^{1/p'}=\left(\int\limits_R^\infty t^{(\alpha-n+\gamma-1)p'+n+\gamma-1}\ln^{p'}\left(1+\frac{1}{t}\right)dt\right)^{1/p'}\le$$

$$\le\left(\int\limits_R^\infty t^{(\alpha-n-\gamma-1)p'-p'+n+\gamma-1}dt\right)^{1/p'}=\left(\int\limits_R^\infty t^{-p'-1}dt\right)^{1/p'}=CR^{-1}.$$

Corollary 3. *Let* $1<p<\infty$, $0<\alpha<n+\gamma$, $0<\beta$, $0<\frac{n+\gamma}{p}-\alpha+\beta$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n+\gamma+p\beta}$ *and* $K(t)=\frac{t^{\alpha-n-\gamma}}{(1+t)^{\beta}}$. *Then for* $f\in L_{p,\gamma}(R^n_+)$

$$I_{\alpha,\beta}f=\int\limits_{R^n_+}\frac{|y|^{\alpha-n-\gamma}}{(1+|y|)^{\beta}}T^yf(x)y_n^{\gamma}dy$$

belong to $L_{q,\gamma}(R_+^n)$ *and*

$$\|I_{\alpha,\beta}f\|_{L_{q,\gamma}(R_+^n)} \leq C\|f\|_{L_{p,\gamma}(R_+^n)}.$$

Proof. It's obvious that the condition $(K_{1,\gamma})$ is filfilled. Let's check the conditions $(K_{2,\gamma})$ and $(K_{3,\gamma})$.

$$\int_0^R K(t)t^{n+\gamma-1}dt = \int_0^R \frac{t^{\alpha-1}}{(1+t)^\beta}dt \leq \int_0^R t^{\alpha-1}dt = CR^\alpha \leq CR^\sigma, \quad \text{if} \quad \sigma=\alpha.$$

$$\left(\int_R^\infty K^{p'}(t)t^{n+\gamma-1}dt\right)^{1/p'} = \left(\int_R^\infty \frac{t^{(\alpha-n-\gamma)p'+n+\gamma-1}}{(1+t)^{\beta p'}}dt\right)^{1/p'} \leq$$

$$\leq \left(\int_R^\infty t^{(\alpha-n-\gamma)p'+n+\gamma-\beta p'-1}dt\right)^{1/p'} \overset{0<\frac{n+\gamma}{p}-\alpha+\beta}{=} R^{-(\frac{n+\gamma}{p}-\alpha+\beta)} \leq CR^{-\gamma(p)}$$

$$\text{if} \quad \gamma(p) = \frac{n+\gamma}{p} - \alpha + \beta.$$

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