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EMBEDDING THEOREMS IN ANISOTROPIC WEIGHT TYPE B_n-SOBOLEV SPACE

Abstract

At the paper the anisotropic weight type B_n-Sobolev spaces $W^{l_1,...,l_1}\Gamma_{p,\theta,\gamma}\left(R_+^n,\varphi\right)$, $W^{l_1,...,l_1}\Gamma^*_{p,\theta,\gamma}\left(R_+^n\varphi\right)$ are constructed and some embedding theorems in these spaces are obtained. By means of B_n-Riesz potential a priori estimations are obtained.

A series of mathematical physics problems leads to the consideration of the differential operators with the singularity on a boundary manifold. Example for such an operator is a Bessel operator $B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}$, $\gamma > 0$ with the property $x_n = 0$. The scale of Hilbert spaces constructed for such operators was studied in [1] by the Fourier-Bessel thransformation method and there the corresponding imbeddings theorems were proved.

At the given paper an anisotropic weight type B_n -Sobolev spaces $W^{l_1,...,l_1}\Gamma_{p,\theta,\gamma}\left(R_+^n,\varphi\right)$, $W^{l_1,...,l_1}\Gamma^*_{p,\theta,\gamma}\left(R_+^n\varphi\right)$ were constructed and some imbedding theorems in these spaces were obtained.

Let R_+^n denote a half-space $x_n > 0$ of Euclidean n-dimensional space of the points $x = (x', x_n) = (x_1, ..., x_{n-1}, x_n)$. Denote by $C_{e,0}^{\infty}\left(\overline{R_+^n}\right)$ the set of infinitely differentiable functions even by the variable x_n and having in R_+^n a compact support.

Let $a = (a_1, a_{n-1}, a_n) = (a', a_n)$ $a_i > 0$ $(i = 1, 2, ..., n)$, and the function $\rho(x) = \left(\sum_{i=1}^n |x_i|^{\frac{2}{a_i}}\right)^{\frac{1}{2}}$ be an anisotropic distance and the parameter $\vartheta \in (0, r]$. Suppose for the numbers $l_i > 0$, $\nu_i \geq 0$ $(i = 1, 2, \ldots, n)$, $|a| = \sum\limits_{i=1}^n a_i$, $|a|_\gamma = |a| + \gamma a_n$, $(a, \nu) = \sum\limits_{i=1}^{n-1} a_i\nu_i + 2a_n\nu_n$, $\lambda_0 = |a|_\gamma + (a, \nu)$, $\lambda_i = 1 + |a|_\gamma - l_i a_i + (a, \nu)$ $(i = 1, 2, \ldots, n-1)$, $\lambda_n = 1 + |a|_\gamma - 2l_n a_n + (a, \nu)$, $\vartheta^a = (\vartheta^{a_1}, \ldots, \vartheta^{a_n})$, $\frac{x}{\vartheta^a} = \left(\frac{x_1}{\vartheta^{a_1}}, \ldots, \frac{x_n}{\vartheta^{a_n}}\right)$ and accept $D_i = \frac{\partial}{\partial x_i}$, $D_{x'}^{\nu'} = D_1^{v_1}...D_{n-1}^{v_n-1}$, $D_{B_n}^{v} = D_{x'}^{\nu'}B_n^{v_n}$, where $D_i^{v_i}$, $B_n^{v_n}-$are iterations of corresponding differential operators.

Suppose $E_+(0,r) = \left\{y \in R_+^n : \ \rho(y) < r\right\}$, $|E_+(0,r)|_\gamma = \int\limits_{E_+(0,r)} x_n^\gamma dx$ and $E_+^*(0,r) = R_+^n \backslash E_+(0,r)$. Note, that $|E_+(0,r)|_\gamma = Cr^{|a|_\gamma}$.

Consider received in [2] at $\gamma \neq 1, 3, ..., 2l_{n-1}$ integral representation of the functions $f \in C_{e,0}^{\infty}\left(\overline{R_+^n}\right)$.

$$D_{B_n}^{v} f(x) = \frac{c_0}{r^{\lambda_0}} \int\limits_{R_+^n} T^x f(y)\, N(y,r)\, y_n^\gamma dy +$$

$$+\sum_{i=1}^n c_i \int\limits_0^r \frac{d\vartheta}{\vartheta^{\lambda_i}} \int\limits_{R_+^n} T^x g_i(y)\, M_i\left(\frac{y}{\vartheta^a}\right) y_n^\gamma dy = J_0 + \sum_{i=1}^n J_1, \qquad (1)$$

where $N(y,\tau)$, $M_i\left(\frac{y}{\tau^a}\right)$ are finite smooth in R_+^n functions and

$$g_i(x)=D_i^{l_i}f(x),\quad i=1,2,...,n-1,\quad g_n(x)=B_n^{l_n}f(x). \qquad (2)$$

Definition 1. *Let the function* $f\in L_p^\gamma\left(E_+^*(0,\tau)\right)$ *at any* τ, $0<\tau<\infty$. *Suppose*

$$\Omega_{p,\gamma}(f,\tau)=\left(\int\limits_{E_+^*(0,\tau)}|f(x)|^p x_n^\gamma dx\right)^{\frac{1}{p}},\quad \tau>0.$$

Definition 2. *Let the function* $f\in L_p^\gamma(E_+(0,\tau))$ *at any* τ, $0<\tau<\infty$. *Suppose*

$$\Omega_{p,\gamma}^*(f,\tau)=\left(\int\limits_{E_+(0,\tau)}|f(x)|^p x_n^\gamma dx\right)^{\frac{1}{p}},\quad \tau>0.$$

In terms of the characteristics $\Omega_{p,\gamma}(f,\tau)$, $\Omega_{p,\gamma}^*(f,\tau)$ the spaces $\Gamma_{p\theta,\gamma}\left(R_+^n,\varphi\right)$, $\Gamma_{p\theta,\gamma}^*\left(R_+^n,\varphi\right)$ were investigated, which as is shown at $\theta=p$ coincide with some weight spaces $L_p^\gamma\left(R_+^n,\omega\right)=L_p\left(R_+^n,\omega(\rho(x))x_n^\gamma dx\right)$ ([3], [4]).

Let φ be a positive measurable function on $(0,\infty)$. Denote by $\Gamma_{p\theta,\gamma}\left(R_+^n,\varphi\right)$, $\Gamma_{p\theta,\gamma}^*\left(R_+^n,\varphi\right)$, $1\le p<\infty$, $1\le\theta\le\infty$, the set of measurable functions f in R_+^n with a finite norm ([3], [4]).

$$\|f\|_{\Gamma_{p\theta,\gamma}(R_+^n,\varphi)}=\left(\int\limits_0^\infty\left(\Omega_{p,\gamma}(f,t)\right)^\theta\varphi(t)dt\right)^{1/\theta},\quad 1\le\theta<\infty$$

$$\|f\|_{\Gamma_{p\theta,\gamma}(R_+^n,\varphi)}=\sup_{t>0}\Omega_{p,\gamma}(f,t)\varphi(t),\quad \theta=\infty,$$

$$\|f\|_{\Gamma_{p\theta,\gamma}^*(R_+^n,\varphi)}=\left(\int\limits_0^\infty\left(\Omega_{p,\gamma}^*(f,t)\right)^\theta\varphi(t)dt\right)^{1/\theta},\quad 1\le\theta<\infty,$$

$$\|f\|_{\Gamma_{p\theta,\gamma}^*(R_+^n,\varphi)}=\sup_{t>0}\Omega_{p,\gamma}^*(f,t)\varphi(t),\quad \theta=\infty.$$

Note, that the corresponding spaces $\Gamma_{p\theta}(X,\varphi),\Gamma_{p\theta}^*(X,\varphi)$ in case, when X-is a homogeneous group in Folland-Stein sense, were introduced and studied relative to singular integral operator and integral operator of potential type in [5].

Definition 3. *We'll say, that the function* f *determined on* R_+^n *belongs to the anisotropic weight space* $W^{l_1,...,l_n}\Gamma_{p,\theta,\gamma}\left(R_+^n,\varphi\right)$ $\left(W^{l_1,...,l_1}\Gamma_{p,\theta,\gamma}^*\left(R_+^n\varphi\right)\right)$, *if* f *has on* R_+^n *generalized by the S.L. Sobolev derivatives* $D_i^{l_i}$, $i=1,2,...,n-1$, $B_n^{l_n}f$ *and the norms*

$$\|f\|_{W^{l_1,...,l_n}\Gamma_{p,\theta,\gamma}(R_+^n,\varphi)}=\|f\|_{\Gamma_{p\theta,\gamma}(R_+^n,\varphi)}+$$
$$+\sum_{i=1}^{n-1}\left\|D_i^{l_i}f\right\|_{\Gamma_{p\theta,\gamma}(R_+^n,\varphi)}+\left\|B_n^{l_n}f\right\|_{\Gamma_{p\theta,\gamma}(R_+^n,\varphi)}$$

$$\Big(\|f\|_{W^{l_1,\ldots,l_n}\Gamma^*_{p,\theta,\gamma}\left(R^n_+,\varphi\right)}=\|f\|_{\Gamma^*_{p\theta,\gamma}\left(R^n_+,\varphi\right)}+$$
$$+\sum_{i=1}^{n-1}\left\|D_i^{l_i}f\right\|_{\Gamma^*_{p\theta,\gamma}\left(R^n_+,\varphi\right)}+\left\|B_n^{l_n}f\right\|_{\Gamma^*_{p\theta,\gamma}\left(R^n_+,\varphi\right)}\Big).$$

are finite.

Such functional spaces adapted to work with generalized shift of the form (B_n- is a shift) (see ex. [1], [6]).

$$T^y f(x)=C_\gamma\int_0^\pi f\left(x'-y',\ \sqrt{x_n^2+y_n^2-2x_ny_n\cos\alpha}\right)\sin^{\gamma-1}\alpha d\alpha,$$

where $x=(x',x_n)$, $y=(y',y_n)$, $C_\gamma=\pi^{-\frac{1}{2}}\frac{\Gamma\left(\gamma+\frac{1}{2}\right)}{\Gamma(\gamma)}$.

By means of B_n-shift anisotropic B_n-Riesz potential

$$R^\alpha_{B_n}f(x)=\int_{R^n_+}T^y\rho(x)^{\alpha-|a|_\gamma}f(y)\,y_n^\gamma dy,\quad 0<\alpha<|a|_\gamma$$

and isotropic B_n-Riesz potential

$$I^\alpha_{B_n}f(x)=\int_{R^n_+}T^y|x|^{\alpha-n-\gamma}f(y)\,y_n^\gamma dy,\ 0<\alpha<n+\gamma.$$

are determined.

Let $\triangle_{B_n}=\sum\limits_{i=1}^{n-1}\partial^2/\partial x_i^2+B_n$. The following theorems are true.

Theorem 1. *[7] If α is an even non-negative integer, $f(x)$-is a finite, even by the variable x_n function having $\alpha/2$ continuous derivatives by the variables $x_1,...,x_{n-1}$ and α are continuous derivatives by x_n, then the potential $I^\alpha_{B_n}f(x)$ is a solution of the equation*

$$\triangle^{\alpha/2}_{B_n}u(x)=f(x)$$

Note, that in [8] the boundedness of isotropic B_n-Riesz potential $I^\alpha_{B_n}$ from $L^\gamma_p\left(R^n_+\right)$ in $L^\gamma_q\left(R^n_+\right)$, $1<p<q<\infty$, $1/p-1/q=\alpha/(n+\gamma)$ and in [9] the boundedness anisotropic B_n-Riesz potential from $L^\gamma_p\left(R^n_+\right)$ in $L^\gamma_q\left(R^n_+\right)$, $1<p<q<\infty$, $1/p-1/q=\alpha/|a|_\gamma$ was proved.

Theorem 2. *[3] [9] Let $1<p<q<\infty$, $0<\alpha<|a|_\gamma$, $1<\theta<\theta_1<\infty$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{|a|_\gamma}$.*

If

$$\sup_{t>0}\left(\int_t^\infty\psi(\tau)\,\tau^{-\frac{|a|_\gamma\theta_1}{p'}}d\tau\right)^{\frac{\theta}{\theta_1}}\left(\int_0^t\varphi(\tau)^{1-\theta'}\,\tau^{\frac{\theta'}{p'}(\theta-p')}d\tau\right)^{\theta-1}<\infty,$$

then

$$\left\|R^\alpha_{B_n}f\right\|_{\Gamma_{q\theta_1,\gamma}\left(R^n_+,\psi\right)}\leqslant C\,\|f\|_{\Gamma_{p\theta,\gamma}\left(R^n_+,\varphi\right)}$$

with the constant C independent of the function f.

Theorem 3. *[4] [9] Let* $1 < p < q < \infty$, $0 < \alpha < |a|_\gamma$, $1 < \theta < \theta_1 < < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|a|_\gamma}$.

If

$$\sup_{t>0} \left(\int_0^t \psi(\tau)\, \tau^{\frac{|a|_\gamma \theta_1}{q}} d\tau \right)^{\frac{\theta}{\theta_1}} \left(\int_t^\infty \varphi(\tau)^{1-\theta'} \tau^{-\frac{|a|_\gamma \theta'}{q} - \theta'} d\tau \right)^{\theta - 1} < \infty,$$

then

$$\left\| R^\alpha_{B_n} f \right\|_{\Gamma^*_{q\theta_1,\gamma}(R^n_+,\psi)} \leqslant C_2 \|f\|_{\Gamma^*_{p\theta,\gamma}(R^n_+,\varphi)}$$

with the constant C_2 *independent of the function* f.

Note, that in case $\theta = \theta_1 = \infty$ the analogy of theorem 1,2 is also true (see [3], [4]).

Theorem 4. *Let the function* $f \in W^{l_1,\ldots,l_n}\Gamma_{p,\theta,\gamma}\left(R^n_+, \varphi\right)$ *the weight pairs* (φ, ψ) *satisfy the conditions of theorem 1. Let* $l_i > 0$, $v_i \geqslant 0$ *be integers, such, that*

$$\sigma_1 = 1 - \sum_{i=1}^n \frac{v_i}{l_i} - \left(\frac{1}{p} - \frac{1}{q} \right) \left(\sum_{i=1}^{n-1} \frac{1}{l_i} + \frac{\gamma + 1}{2l_n} \right) > 0.$$

Then, the operator $D^v_{B_n} f$ *as the operator from* $W^{l_1,\ldots,l_n}\Gamma_{p,\theta,\gamma}\left(R^n_+, \varphi\right)$ *in* $\Gamma_{q\theta_1,\gamma}\left(R^n_+, \psi\right)$ $(1 < p < q < \infty,\ 1 < \theta < \theta_1 < \infty,\ \gamma \neq 1, 3, \ldots, 2l_{n-1})$ *is bounded, moreover*

$$\left\| D^v_{B_n} f \right\|_{\Gamma_{q\theta_1,\gamma}(R^n_+,\psi)} \leqslant C \|f\|_{W^{l_1,\ldots,l_n}\Gamma_{p,\theta,\gamma}(R^n_+,\varphi)}$$

with the constant C, *not depending on* f.

Proof. According to (1) we have at $\gamma \neq 1, 3, \ldots, 2l_{n-1}$

$$\left\| D^v_{B_n} f \right\|_{\Gamma_{q\theta_1,\gamma}(R^n_+,\psi)} \leqslant c \left(\|J_0\|_{\Gamma_{q\theta_1,\gamma}(R^n_+,\psi)} + \left\| \sum_{i=1}^n J_i \right\|_{\Gamma_{q\theta_1,\gamma}(R^n_+,\psi)} \right).$$

Accept in (1) $a_i = \frac{1}{l_i}$ $(i = 1, 2, \ldots, n-1)$, $a_n = \frac{1}{2l_n}$. By virtue of theorem 1 we have

$$\|J_i\|_{\Gamma_{q\theta_1,\gamma}(R^n_+,\psi)} \leqslant C \|g_i\|_{\Gamma_{q\theta_1,\gamma}(R^n_+,\psi)}, \quad i = 1, 2, \ldots, n,$$

and also

$$\|J_0\|_{\Gamma_{q\theta_1,\gamma}(R^n_+,\psi)} \leqslant C \|f\|_{\Gamma_{p\theta,\gamma}(R^n_+,\varphi)}$$

Theorem is proved.

The proof of following theorem is analogous to lastone.

Theorem 5. *Let the function* $f \in W^{l_1,\ldots,l_n}\Gamma_{p,\theta,\gamma}\left(R^n_+, \varphi\right)$ *the weight pairs* (φ, ψ) *satisfy the conditions of theorem 2. Let* $l_i > 0$, $v_i \geqslant 0$ *be integers such that*

$$\sigma_1 = 1 - \sum_{i=1}^n \frac{v_i}{l_i} - \left(\frac{1}{p} - \frac{1}{q} \right) \left(\sum_{i=1}^{n-1} \frac{1}{l_i} + \frac{\gamma + 1}{2l_n} \right) > 0.$$

Then the operator $D^v_{B_n} f$ *as operator from* $W^{l_1,\ldots,l_n}\Gamma^*_{p,\theta,\gamma}\left(R^n_+, \varphi\right)$ in $\Gamma^*_{q\theta_1,\gamma}\left(R^n_+, \psi\right)$ $(1 < p < q < \infty,\ 1 < \theta < \theta_1 < \infty,\ \gamma \neq 1, 3, \ldots, 2l_{n-1})$*is bounded, moreover*

$$\left\| D^v_{B_n} f \right\|_{\Gamma^*_{q\theta_1,\gamma}(R^n_+,\psi)} \leqslant C \|f\|_{W^{l_1,\ldots,l_n}\Gamma^*_{p,\theta,\gamma}(R^n_+,\varphi)}$$

with the constant C independent of f.

Note, that the analogy of theorem 4,5 is also true in case $\theta = \theta_1 = \infty$.

From theorem 1 and theorem 2 we have.

Theorem 6. *Let* $1 < p < q < \infty$, $1 < \theta < \theta_1 < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{2}{n+\gamma}$ *and the positive functions* φ, ψ *be summable on every interval* $(0,\tau) \subset (0,\infty)$.

If

$$\sup_{t>0} \left(\int_t^\infty \psi(\tau)\, \tau^{-\frac{(n+\gamma)\theta_1}{p'}} d\tau \right)^{\frac{\theta}{\theta_1}} \left(\int_0^t \varphi(\tau)^{1-\theta'} \tau^{\frac{\theta'}{p'}(\theta - p')} d\tau \right)^{\theta-1} < \infty,$$

Then, the following a priori estimations:

$$\|u\|_{\Gamma_{q\theta_1,\gamma}(R^n_+,\psi)} \le C \, \|\Delta_{B_n} u\|_{\Gamma_{p,\theta,\gamma}(R^n_+,\varphi)} \, .$$

are true.

And also from theorem 1 and theorem 3 we have.

Theorem 7. *Let* $1 < p < q < \infty$, $1 < \theta < \theta_1 < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{2}{n+\gamma}$ *and the positive functions* φ, ψ *be summable on every interval* $(0,\tau) \subset (0,\infty)$.

If

$$\sup_{t>0} \left(\int_0^t \psi(\tau)\, \tau^{\frac{(n+\gamma)\theta_1}{q}} d\tau \right)^{\frac{\theta}{\theta_1}} \left(\int_t^\infty \varphi(\tau)^{1-\theta'} \tau^{-\frac{(n+\gamma)\theta'}{q} - \theta'} d\tau \right)^{\theta-1} < \infty,$$

Then, the following a priori estimations:

$$\|u\|_{\Gamma^*_{q\theta_1,\gamma}(R^n_+,\psi)} \le C \, \|\Delta_{B_n} u\|_{\Gamma^*_{p,\theta,\gamma}(R^n_+,\varphi)} \, .$$

are true.

The author expresses his gratitude to the academician A.D. Gadjiev and prof. V.S. Guliyev for discussion of the obtained results.

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Received December 10, 200; Received September 23, 2002.
Translated by Mamedova V.I.