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ON BEHAVIOUR OF SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS IN CONIC DOMAINS

Abstract

The exact estimation of behaviour of solutions and their derivatives near the conic point have been obtained, it has been proved that $u(x)$ has squaresummable exactly wheighted second generalized derivatives.

Let's consider a mixed boundary value problem in bounded domain $\Omega \subset R^n$, $n \geq 2$ for the equation

$$\frac{d}{dx_i} a_i(x, u, u_x) + a(x, u, u_x) = 0, \qquad x \in \Omega \tag{1}$$

Denote by $\partial\Omega$ the boundary of domain Ω and $\partial\Omega = \Gamma_1 \cup \Gamma_2$. The Dirichlet conditions are given on Γ_1; Neumann conditions on Γ_2. Relative to domain Ω we shall require the fulfilment of isoperimetric inequalities [1].

In the given paper our aim is to obtain exact estimates of behaviour of solution and its derivative near the conic points and, unlike paper [2], obtain estimates for $|u(x)|$ and $|\nabla u(x)|$ with $\varepsilon = 0$. In paper [2] the review of results on these themes is given.

Let's make some denotations $B_d(0)$ is ball of radius d with the center at the point 0. $\Omega_0^d = \Omega \cap B_d(0)$ is come in R^n, i.e. for sufficiently small d

$$\Omega_0^d = \{(r, \omega)/0 < r < d; \ \omega = (\omega_1, \omega_2, ..., \omega_{n-1}) \in G\},$$

(r, ω) are spherical coordinates. G is a domain on a unit sphere S^{n-1} with infinitely differentiable boundary ∂G, $\Gamma_0^d = \{(r, \omega)/0 < r < d; \omega \in \partial G\} = \Gamma_{0,1}^d \cup$ $\cup \Gamma_{0,2}^d \subset \partial\Omega$ is lateral surface, of the cone Ω_0^d, $G_\rho = \Omega_0^d \cap \{|x| = \rho\}$, $0 < \rho < d$. $dx = r^{n-1} dr d\omega$, $d\Omega_\rho = \rho^{n-1} d\omega$, $d\omega$ is an element of area of the unit sphere, $|\nabla u|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2} |\nabla_\omega u|^2$, where $|\nabla_\omega u|$ is projection of vector ∇u on tangent plane to the sphere S^{n-1} at the point ω

$$\nabla u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{n} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Delta_\omega u.$$

Here $\Delta_\omega u$ is the Laplace-Beltrami operator on a unit sphere.

Denote by $W_{\alpha,0}^m(\Omega)$ the space of functions having generalized derivatives till the order m in Ω with norm

$$\|u\|_{W_{\alpha,0}^m(\Omega)}^2 = \sum_{|k|=0}^{m} \int_\Omega r^{\alpha - 2(m-k)} \left| \frac{\partial^{|k|} u}{\partial x_1^{k_1} ... \partial x_n^{k_n}} \right|^2 dx$$

in which all continuously differentiable functions in $\overline{\Omega}$ vanishing on Γ_1, in particular

$$\|u\|_{W_{\alpha,0}^2(\Omega)}^2 = \int_\Omega \left(r^\alpha u_{xx}^2 + r^{\alpha-2} |\nabla u|^2 + r^{\alpha-4} u^2 \right) dx$$

are dense set.

Denote by $W^1_{2,0}(\Omega)$ the Sobolev space of functions $W^1_2(\Omega)$ in which all continuously differentiable functions in $\overline{\Omega}$ vanishing on Γ_1 are dense set.

Later on we shall need Hardy inequalities and different consequences of this inequality.

For any function $u\in W^1_{2,0}\left(\Omega^d_0\right)$ the following inequality is valid

$$\int\limits_{\Omega^d_0} r^{\alpha-4}u^2dx\le\frac{4}{(4-n-\alpha)^2}\int\limits_{\Omega^d_0} r^{\alpha-2}u^2_r dx\,,\quad \alpha<4-n, \tag{2}$$

which is obtained by integration with respect to $\omega\in G$ of the correspondent Hardy inequality (see [3]) provided that integral in the right-hand side is finite.

Allowing for isoperimetricity condition of domain Ω consider the eigenvalues problem

$$\Delta_\omega u+\lambda(\lambda+n-2)\,u=0\,,\ \ \omega\in G,\ \ u|_{\gamma_0}=0,\ \ \frac{\partial u}{\partial u}|_{\gamma_1}=0, \tag{3}$$

where $\partial G\in\gamma_0\cup\gamma_1$. It follows from paper [4] that there exists the least positive eigenvalue $\lambda=\lambda(G)$ of this problem. Then by means of the variational principle $\forall u\in W^1_{2,0}(G)$ we obtain that

$$\int\limits_G u^2d\omega\le\frac{1}{\lambda^2+\lambda(n-2)}\int\limits_G|\nabla_\omega u|^2d\omega. \tag{4}$$

Note that constants in inequalitites (2), (4) are the best ones.

If we'll multiply inequality (4) by $1/r$, integrate with respect to $r\in(0,d)$, then for any function

$$u\in V=\left\{v\in W^1_2(\Omega)/v(x)=0,\ x\in\Gamma^d_{0,1}\,,\ \ \frac{\partial v}{\partial n}=0,\ \ x\in\Gamma^d_{0,2}\right\}$$

$$\int\limits_{\Omega^d_0} r^{-n}u^2dx\le\frac{1}{\lambda^2+\lambda(n-2)}\int\limits_{\Omega^d_0} r^{2-n}|\nabla u|^2dx, \tag{5}$$

if integral in the right-hand side is finite.

For any function $u\in V$ the following inequality is valid

$$\int\limits_{\Omega^d_0} r^{\alpha-4}u^2dx\le\left[\left(2-\frac{n+\alpha}{2}\right)^2+\lambda(\lambda+n-2)\right]^{-1}\int\limits_{\Omega^d_0} r^{\alpha-2}|\nabla u|^2dx, \tag{6}$$

at the finiteness of the integral in the right-hand side. Here $\alpha\le 4-n$. In order to obtain this inequality we shall muliply inequality (4) by $1/r$ and integrarte with respect to $r\in(0,d)$, then

$$\int\limits_{\Omega^d_0} r^{\alpha-4}u^2dx\le\frac{1}{\lambda^2+\lambda(n-2)}\int\limits_{\Omega^d_0} r^{\alpha-4}|\nabla_\omega u|^2dx. \tag{7}$$

If $\alpha < 4-n$ inequality (6) is obtained by summation of inequalitites (2) and (7). If $\alpha = 4-n$ inequality (6) coincides with (5).

We call function $u(x) \in W^1_{2,0}(\Omega)$ satisfying the integral identity

$$\int_\Omega \left[a_i(x,u,u_x)\eta_{x_i} + a(x,u,u_x)\eta(x)\right]dx = 0, \tag{8}$$

for any function $\eta(x) \in W^1_{2,0}(\Omega)$ the generalized solution of the mixed boundary-value problem for equation (1).

Relative to the coefficient we'll require the fulfilment of the following conditions. Functions $a_i(x,u,p)$ are measurable at $x \in \Omega$ and any $u \in R$, $p \in R$, differentiable with respect to p_j, $j = 1,...,n$ and satisfy the inequalities

$$v(|u|)\xi^2 \le \frac{\partial a_i(x,u,p)}{\partial p_j}\xi_i\xi_j \le \mu(|u|)\xi^2, \quad \forall \xi \in R^n, \tag{9}$$

$$\frac{\partial a_i(0,0,p)}{\partial p_j} = \delta_i^j, \; i,j = \overline{1,n}, \tag{10}$$

$$\left[\sum_{i=1}^n a_i^2(x,u,p)\right]^{1/2} \le \mu_1(|u|)(|p| + g(x)), \quad 0 \le g(x) \in L_q(\Omega), \tag{11}$$

where δ_i^j is Kronecker symbol, $q > n$, $g(0) < \infty$.

Function $a(x,u,p)$ measurable at $x \in \Omega$, $u \in R$, $p \in R^n$ satisfying the inequality

$$|a(x,u,p)| \le \mu_2(|u|)\left(p^2 + f(x)\right), \tag{12}$$

where $0 \le f(x) \in L_{q/2}(\Omega)$, $q > n$, $v(t)$ $(\mu(t), \mu_1(t), \mu_2(t))$ is positive nondecreasing function at $t \ge 0$, μ, $v > 0$, $\mu_1, \mu_2 \ge 0$.

In paper [5] the boundedness and Hōlder continuity of generalized solution of (8) have been proved under the conditions (9)-(12). Assuming the value $M = vrai\max_\Omega |u(x)|$ to be known there exists $\gamma > 0$, $C_0 > 0$ dependent only on $M, n, q, \mu, \mu_1, \mu_2, v, \Omega$ that

$$|u(x)| = |u(x) - u(0)| \le C_0|x|^\gamma, \quad |x| < d.$$

Theorem 1. *Let $u(x)$ be a generalized solution of (8) and conditions (9)-(12) and the conditions that for any $k > 0$ there exists $d_0 > 0$ such that for any $p \in R^n$*

$$\left(\sum_{i=1}^n \left[a_i(x,u,p) - a_i(0,0,p)\right]^2\right)^{1/2} \le K|p| + h(x), \tag{13}$$

as soon as $|x| + |u| < d_0$, $0 \le h(x) \in L_q$, $q > n$ be fulfilled.

Besides if $g(x) \in \overset{\bullet}{W}{}^0_{\alpha-2}(\Omega)$, $h(x) \in W^0_{\alpha-2,0}(\Omega)$, $f(x) \in W^0_{\alpha,0}(\Omega)$, $\alpha \le 4-n$, moreover,

$$\lambda > 2 - (n+\alpha)/2 \tag{14}$$

then the following estimate is valid

$$\int\limits_{\Omega} r^{\alpha-2} |\nabla u|^2 dx \le C(1 + \|g\|_{W^0_{\alpha-2}(\Omega)} +$$

$$+ \|f\|_{q/2,\Omega} + \|h\|_{W^0_{\alpha-2,0}(\Omega)} + \|f\|^2_{W^0_{\alpha,0}(\Omega)}), \tag{15}$$

where constant C depends only on quantities $M, v, \mu_1, \mu_2, \mu, \alpha, n, \lambda, q, mes\Omega, mesG$.

Proof. For any $\delta \in (0, d)$ if r is radius vector of the point $x \in \overline{\Omega}$ then quantities $r_\delta = |r - \delta l| \ne 0$, $\forall x \in \overline{\Omega}$, where for the fixed point $z \in S^{n-1}\backslash\overline{G}$ and unit radius vector $l = \overrightarrow{0z} = (l_1, ..., l_n)$ vector δl does't belong to Ω^d_0. Therefore the function $\eta(x) = r_\delta^{\alpha-2} u(x)$ is admissible in identity (8). We obtain

$$\int\limits_{\Omega} r_\delta^{\alpha-2} a_i(x, u, u_x) u_{x_i} dx + \int\limits_{\Omega} r_\delta^{\alpha-2} u(x) a(x, u, u_x) dx +$$

$$+ \int\limits_{\Omega} (\alpha - 2) u(x) r_\delta^{\alpha-4} a_i(x, u, u_x)(x_i - \delta l_i) dx = 0. \tag{16}$$

By means of condition (10) we have

$$a_i(0, 0, p) = p_i + a_i^0, \quad a_i^0 \equiv a_i(0, 0, 0), \ i = \overline{1, n}$$

$$a_i(x, u, p) p_i = |p|^2 + a_i^0 p_i + [a_i(x, u, p) - a_i(0, 0, p)] p_i. \tag{17}$$

Taking this into account, choosing some small number d and dividing domain Ω into two subdomains Ω^d_0 and $\Omega\backslash\Omega^d_0$ we estimate the obtained integrals in each of subdomains separately. At that we apply inequality (6), use one estimate from [6] and the fact that $u(x)$ is Hōlder continuous. Finally, using conditions of the theorem passing to the limit as $\delta \to +0$ we obtain the required estimate.

Remark. If $n = 2$ $0 \in \partial\Omega$ is a corner point $G = (0, \omega_0)$, ω_0 is size of the angle in the neighbourhood of 0, $\Omega^d_0 = (0, d) \times (0, \omega_0)$. In this case eigenvalues problem (3) has the following form

$$u" + \lambda^2 u = 0, \quad u = u(\omega), \ \omega \in G$$

$$u(\omega)|_{\omega=0} = 0, \quad \left.\frac{\partial u}{\partial n}\right|_{\omega=\omega_0} = 0 \ . \tag{18}$$

The least positive eigenvalue of this problem is $\lambda = \frac{\pi}{2\omega_0}$. Condition (14) will take on the form

$$\frac{\pi}{\omega_0} > 2 - \alpha \ , \qquad \alpha \le 2.$$

Let's go over to the estimation of $|u(x)|$. Previously we'll prove a lemma

Lemma. *Let $u(x)$ be a generalized solution of problem (1) and conditions (9)-(12) be satisfied. Then for any function*

$$v(x) \in V \left\{ v \in W^1_2(\Omega^\rho_0) / v(x) = 0, \quad x \in \Gamma^\rho_{0,1}; \quad \frac{\partial v}{\partial n} = 0, \ x \in \Gamma^\rho_{0,2} \right\}$$

and almost all $\rho \in (0, d)$ the following equality is fulfilled

$$\int\limits_{\Omega_0^\rho} \left[a_i(x, u, u_x)\, v_{x_i} + a(x, u, u_x)\, v(x) \right] dx = \int\limits_{G_\rho} a_i(x, u, u_x)\, v(x) \cos(r, x_i)\, dG_\rho \quad (19)$$

In order to prove it we sabstitute $\eta(x) = v(x) \left(\chi_\rho\right)_h(x), \quad \forall v \in W^1_{2,0}(\Omega)$ into the integral identity (8), where $\chi_\rho(x)$ is characteristic function of the set Ω_0^ρ and $\left(\chi_\rho\right)_h$ is its Sobolev averaging. Such η is admissible by virtue of theorem 1. In the obtained equation passing to the limit as $h \to 0$ we obtain (37). Passage to the limit is justified by usage of properties of mean functions [7] and theorem 1.

Theorem 2. *Let $u(x)$ be a generalized solution of problem (1), conditions (9)-(12) and the condition*

$$\left(\sum_{i=1}^{n} \left[a_i(x, u, p) - a_i(0, 0, p) \right]^2 \right)^{1/2} \le \delta(|x|)\, |p| + h(x), \quad (20)$$

for any $x \in \Omega_0^d$, $u \in R$, $p \in R^n$ be fulfilled, where $\delta(r)$ is nondecreasing positive function satisfying the Diny condition $\int\limits_0^d \frac{\delta(r)}{r} dr < \infty$. In addition we assume that the following conditions are satisfied

$$a_i(x, u, p)\, p_i \ge v_0 |p|^2 - \mu_3 |u|^\beta - u^2 \varphi(x);$$

$$a(x, u, p)\, u \le \mu_0 |p|^2 + \mu_3 |u|^\beta + u^2 \varphi(x), \quad (21)$$

where $2n/(n-2) > \beta > 2; \quad 0 \le \varphi(x) \in L_{q/2}(\Omega), \quad q > n, \ v_0 > 0, \ \mu_0, \ \mu_3 \ge 0;$

$$g(x) \in W^0_{2-n}(\Omega), \ h(x) \in W^0_{2-n,0}(\Omega), \quad f(x) \in W^0_{4-n,0}(\Omega), \quad (22)$$

$$\rho^2 \int\limits_G g^2(\rho, \omega)\, d\omega + \rho^2 \int\limits_G h^2(\rho, \omega)\, d\omega + \int\limits_{\Omega_0^\rho} r^{4-n} f^2(x)\, dx \le k \rho^s,$$

$s > 2\lambda(G), \quad 0 < \rho < d.$

Then the estimation

$$|u(x)| \le C\, |x|^{\lambda(G)} \quad (23)$$

is valid, where $\lambda(G)$ is the least positive eigenvalue of problem (3) and constant C depends only on the known quantities of the problem.

Proof. Let's substitute $v(x) = r^{2-n} u(x)$ into identity (19). Such a function is admissible by virtue of inequality (5) and theorem 1. Taking into account (17) and estimating integrals with multipliers a_i^0 and expression $u\ u_{x_0}$ we obtain

$$\int\limits_{\Omega_0^\rho} r^{2-n} |\nabla u|^2 dx \le \frac{n-2}{2} \int\limits_G u^2 d\omega + \int\limits_{\Omega_0^\rho} \left| \left[a_i(x, u, u_x) - a_i(0, 0, u_x) \right] \right| \times$$

$$\times \left[r^{2-n} |u_{x_i}| + (2-n)\, r^{-n} |x_i| \ |u(x)| \right] dx + \int\limits_{\Omega_0^\rho} r^{2-n} |u(x)| \ |a(x, u, u_x)|\, dx +$$

$$+\rho\int\limits_G |u(x)|\,[a_i(x,u,u_x)-a_i(0,0,u_x)]\,|\cos(r,x_i)|_{r=\rho}\,d\omega+$$

$$+C_9\rho^{-\varepsilon}\,||g||_{W^0_{2-n}(\Omega)}+\rho^{2-\varepsilon}\int\limits_G g^2(\rho,\omega)\,d\omega+\rho\int\limits_G u\;u_\rho d\omega \tag{24}$$

Using denotation $v(\rho)=\int\limits_0^\rho dr\int\limits_G\left(ru_r^2+\frac{1}{r}|\nabla_\omega u|^2\right)d\omega$ and estimating integrals in the right-hand side of (24) by means of inequalitites (4), (5) Cauchy inequality with $\varepsilon>0$ and Hōlder property of $u(x)$ we obtain

$$v(\rho)\le c\rho^{\,2\lambda}\,,\quad 0<\rho<d \tag{25}$$

where constant C depends on $M,d,v,\mu_1,\mu_2,\mu,n,\lambda,q,mesG,mes\Omega,||g||_{q,\Omega}\,,$

$$||h||_{W^0_{2-n,0}(\Omega)}\,,||g||_{W^0_{2-n}(\Omega)}\,,||f||_{W^0_{4-n,0}(\Omega)}\,,||f||_{q/2,\Omega}\,,\int\limits_0^d\frac{\delta(r)}{r}dr,k,s$$

Let's consider function

$$z(x')=\rho^{-\lambda(G)}u(\rho x')\,,\quad 0<\rho<d \tag{26}$$

in layer $Q'=\{x'/\ 1/2<|x'|<1\}\,,\ \ u\equiv 0$ out of Ω, and use one of inequalities from [6]. Taking into account estimate (25) we obtain

$$\int\limits_{\rho/2<|x|<\rho}|u(x)|^q\,dx\le C\rho^{n+q\lambda},\quad 2\le q\le 2n/(n-2)\,,\quad n>2. \tag{27}$$

Then taking into consideration one of results from [6] by virtue of assumption of our theorem we obtain

$$|u(x)|\le M\rho^{\lambda(G)} \tag{28}$$

where $x\in\Omega_0^d\cap\{\rho/2<|x|<\rho<d\}$ and M is a constant dependent on the known quantities. Supposing $|x|=\frac{2}{3}\rho$ we'll obtain the required estimate (23).

The theorem is proved.

Theorem 3. *Let $u(x)$ be a generalized solution of problem (1) and assumptions of theorem 1 be satisfied. Besides, let's for $x\in\overline{\Omega}$ and for any $u,p\in R^n$ functions $a_i(x,u,p)\,,\ \ i=\overline{1,n}$ and $a(x,u,p)$ be differentiable with respect to their arguments and the inequalities*

$$a_i(x,u,p)\,p_i\ge v_0\,|p|^2-\varphi_0(x)$$

$$\left[\sum_{i=1}^n\left(\left|\frac{\partial a_i}{\partial u}\right|^2+\left|\frac{\partial a}{\partial x_i}\right|^2\right)\right]^{1/2}+\sum_{i,j=1}^n\left(\left|\frac{\partial a_i}{\partial x_j}\right|^2\right)^{1/2}\le\mu_4(|u|)\,(|p|+\varphi_1(x)) \tag{29}$$

$$\left(\left|\frac{\partial a}{\partial u}\right|^2+\sum_{i=1}^n\left|\frac{\partial a}{\partial x_i}\right|^2\right)^{1/2}\le\mu_5(|u|)\left(|p|^2+\varphi_2(x)\right),$$

be fulfilled, where $\varphi_i(x)$, $i = 0,1,2$ *are nonnegative functions, moreover,* $\varphi_0(x)$, $\varphi_2(x) \in L_{q/2}(\Omega)$, $\varphi_1(x) \in L_q(\Omega)$, $q > n$. *Then* $u(x) \in W^2_{\alpha,0}(\Omega)$ *and the following estimate is valid*

$$||u||^2_{W^2_{\alpha,0}(\Omega)} \leq C_1(1 + ||f||_{q,\Omega} + ||f||_{q/2,\Omega} + ||\varphi_0||_{q/2,\Omega} + ||\varphi_2||_{q/2,\Omega} + ||\varphi_1||_{q,\Omega} +$$

$$+ ||h||^2_{W^2_{\alpha-2,0}(\Omega)} + ||g||^2_{W^0_{\alpha-2}(\Omega)} + ||f||^2_{W^0_{\alpha,0}(\Omega)} + C_2\{\int\limits_{\Omega} r^{(\alpha+h)q/4-n}[\varphi_0^{q/2}(x) + \varphi_1^q(x) +$$

$$+\varphi_2^{q/2}(x) + f^{q/2}(x) + g^q(x)]\}^{4/q} \tag{30}$$

where $\alpha \leq 4-n$ *and provided that the last integral is finite,* $c_1, c_2 > 0$ *depend on the known parameters.*

Proof. In order to prove the theorem equation is considered in the sequence of domains $\Omega_{k,\rho}$, which are intersections of Ω_0^d and some layers. Making some transformations and using one estimate from [6] and summing all the obtained inequalities over $k = 1,2,...$ using theorem 1 we obtain the required corollary.

Corollary. *Let all be the conditions of theorem 3 except equation (14) be fulfilled. Then generalized solution of problem (1)* $u(x) \in W^2(\Omega)$, *if*

1) $n \geq 4$;

2) $n = 2$ *and* $0 < \omega_0 < \frac{\pi}{2}$;

3) $n = 3$ $G \subset G_0 = \{\omega = (\theta;\varphi) \,/\, 0 < |\theta| < \omega_0 < \pi,\ \ 0 < \varphi < 2\pi\}$, *where* ω_0 *is solution of equation* $p_{1/2}(\cos\omega_0) = 0$ *for Legendre function.*

Proof. 1) According to theorem 3 $u(x) \in W^2_{4-n,0}(\Omega)$. Condition (14) turns on to trivial one if $\alpha = 4-n$ because $\lambda = \lambda(G) > 0$. Now the statement follows from inequality

$$\int\limits_{\Omega_0^d} u^2_{xx}dx \leq d^{n-4}\int\limits_{\Omega_0^d} r^{4-n}u^2_{xx}dx \leq const\ .$$

2) If we suppose $\alpha = 0$ in theorem 3 then condition (13) will be trivial one. If $n = 2$ the statement follows from the remark.

3) Equation (14) turns on to $\lambda(G) > 1/2$. Let $\Omega_0 \subset S^2$ be domain in which the eigenvalue problem (3) is solvable for $\lambda(G) = 1/2$ and $\partial\Omega_0 = \partial^1\Omega_0 \cup \partial^2\Omega_0$:

$$\Delta_\omega u + (1/2)(1+1/2)\,u = 0,\ \ \omega \in \Omega_0 \tag{31}$$

$$u|_{\partial^1\Omega_0} = 0, \qquad \frac{\partial u}{\partial u}\bigg|\,\partial^2\Omega_0 = 0$$

Condition $\lambda > 1/2$ means that $\Omega \subset \Omega_0$ [8]. We solve problem (31) in the form $u \equiv v(\theta)$. Then for $v(\theta)$ we obtain

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\frac{dv}{d\theta}\right) + \frac{1}{2}\left(1+\frac{1}{2}\right)v = 0,\ \ 0 < |\theta| < \omega_0, \tag{32}$$

$$v(-\omega_0) = 0\ , \qquad \frac{\partial v}{\partial n}(\omega_0) = 0.$$

Solution of this equation is Legendre function of the first genus $v(\theta) = p_{1/2}(\cos\theta)$, which has exactly one zero in the interval $0 < \theta < \pi$ which we denote by ω_0 (see [9]).

The corollary is proved.

Theorem 4. *Let $u(x)$ be a generalized solution of (1). Let funcions $a_i(x,u,p)$, $a(x,u,p)$ be differentiable with respect to their arguments and conditions (9)-(12), (29) with $q=\infty$ be satisfied. Besides, let conditions of theorem 2 be satisfied. Then*

$$|\nabla u(x)| \le c|x|^{\lambda(G)-1} \tag{33}$$

where $\lambda(G)$ is the least positive eigenvalue of problem (3), and constant c depends only on the known quantities.

Proof. As in the proof of theorem 2 consider function $z(x') = \rho^{-\lambda(G)}u(\rho x')$, $0<\rho<d$ in layer $Q' = \{x'/\ \ 1/2 < |x'| < 1\}$ assuming that $u \equiv 0$ outside of Ω. Under our conditions it is valid the theorem from [6] on boundedness of modulus of gradient of solution inside of domain and smooth pieces of boundary

$$vrai\max_{Q'} \left|\nabla' z\right| \le M_1 \tag{34}$$

where $M_1 > 0$. Then for the function $u(x)$ we obtain

$$|\nabla u(x)| \le M_1\rho^{\lambda(G)-1}, \quad x \in \Omega_0^d \cap \{\rho/2 < |x| < \rho < d\}. \tag{35}$$

If we suppose $|x| = \frac{2}{3}\rho$, then we obtain the required estimate.

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