Abstract

The order of best approximation of multi variable function with the sums of fewer variable functions in the space of mixed norm was established. For this we use exact annihilator of class of fewer variable functions.

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ONE DIRECT THEOREM IN THE SPACE WITH THE MIXED NORM

At the given paper the order of the best approximation of multivariable function with the sums of fewer variable functions in the space by the mixed norm was established. For this the exact annihilator of a class of fewer variable functions is used, previously established in [2]. Let the real function $f = f(x)$, $x = (x_1, ..., x_n)$ be determined on the bounded set Q of n-dimensional Euclidean space $R_n = R_n(x_1, ..., x_n)$. Denote by $x_{\overline{i,k}} = (x_i, x_{i+1}, ..., x_k)$.

Let the set Q be given by the inequalities

$$\begin{cases} \alpha_1\left(x_{\overline{2,n}}\right) \leq x_1 \leq \beta_1\left(x_{\overline{2,n}}\right) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_1\left(x_{\overline{i+1,n}}\right) \leq x_i \leq \beta_i\left(x_{\overline{i+1,n}}\right), \quad i = \overline{1, n-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_n = a \leq x_n \leq b = \beta_n \end{cases}$$

where $\alpha_1\left(x_{\overline{i+1,n}}\right)$, $\beta_i\left(x_{\overline{i+1,n}}\right)$ are fixed real functions.

Consider the norm of multivariable function f relative to the variable x_i in the space L_{p_i}.

$$\|f\|_{p_i} \overset{df}{=} \left(\int_{\alpha_i\left(x_{\overline{i+1,n}}\right)}^{\beta_i\left(x_{\overline{i+1,n}}\right)} |f(x)|^{p_i}\, dx_i\right)^{1/p_i}$$

and denote by $L_p(Q)$,$p = (p_1, ..., p_n)$ the space of the functions $f = f(x)$ for which there exists and finite the integral

$$\|f\|_p \overset{df}{=} \left\|\left(\left\|\cdots\left(\|f\|_{p_1}\right)\cdots\right\|_{p_{n-1}}\right)\right\|_{p_n}.$$

Take the group of the variable $u_v \subset \{x_1, ..., x_n\}$, $u_v \not\subset u_\mu$, $v \neq \mu$; $v, \mu = \overline{1, m}$ and denote by $\sum$ the class of functions of the form

$$\sum \varphi_v \overset{df}{=} \sum_{v=1}^{m} \varphi_v(u_v) \in L_{\overline{p}}Q,$$

where the functions $\varphi_v = \varphi_v(u_v)$ are determined on Q_v projections of the set Q on the space $R(u_v)$. Consider the best approximation of the multivariable function $f \in L_{\overline{p}}(Q)$ by the summ of few variable functions in the space the mixed norm $L_{\overline{p}}(Q)$

$$E_{\overline{p}}\left(f, \sum\right) = \inf_{\Sigma\varphi_v \in \Sigma} \left\|f - \sum_{v=1}^{m} \varphi_v\right\|_{\overline{p}}$$

We'll use the notation of exact annihilator of a class of functions introduced by Babaev M.-B.A. (see for ex. [1]).

Consider the normalized space X and the set $H \subset X$. The family of the continuous operators $\{\nabla_\theta\}_{\theta\in G}$, $\nabla_\theta : X \to X$ depending on some parameter $\theta \in \in G \subset R_{2n}$ we'll call exact annihilator of the set H if

$$f \in H \Longleftrightarrow \nabla_\theta f = 0\,,\ \ \forall\theta \in G$$

Following to [1] let's construct the exact annihilator of the class $\sum$ the sum of fewer variable functions in $L_{\overline{p}}$.

Let $f = f\,(x)\,,\ x \in Q,\ f : Q \to R,$

$$\theta = (\theta_1, ..., \theta_n) \in Q,\ \ \theta^j = (0, ..., \theta_j, ..., 0)\,,$$

$$\Delta_j f \overset{df}{=} \Delta_{\theta_j} f = f\left(x + \theta^j\right) = f\,(x)\,,$$

$$\triangle_{ij} f = \Delta_i\left(\Delta_j f\right), \ldots$$

Let $\mathcal{D}$ be some set of the subsets

$$\overline{n} = \{1, ..., n\} : \mathcal{D} = \{\delta_1, ..., \delta_m\}\,,\ \delta_i \subset \overline{n},\ \ i = \overline{1,m},\ \ \delta_i \not\subset \delta_j,\ \ i \neq j,\ \ \overline{\mathcal{D}} = \underset{\delta\in\mathcal{D}}{\cup}$$

for every $\delta = \left\{\delta^{(1)}, ..., \delta^{|\delta|}\right\}$, where $|\delta|$ is the number of elements of the set δ.

The mixed differences

$$\Delta\theta_{\delta^{(1)}}, ..., \theta_{\delta^{(|\delta|)}} f$$

we'll denote by $\Delta_\delta f = \Delta_{\theta_\delta} f$.

Let $\mathcal{D}$ be a set of all subsets $\overline{n}$ not being the subsets of any

$$\mathcal{D} = \left\{\delta \subset n \,\middle|\, \delta \not\subset \overline{\delta}_{i,i} = \overline{1,m}\right\}$$

from $\delta_1, ..., \delta_m$.

To the set $\mathcal{D}$ let's associate the totality of linear operators for the function f measurable in Q.

$$\mathcal{D}_\theta f = \begin{cases} \sum\limits_{\delta\in} \Delta_\delta f\,, & if \quad x, x+\theta \in Q \\ 0\,, & if \quad x \ \ or \quad x+\theta \notin Q \end{cases}$$

We previosly proved.

Theorem A ([2]). *For the function* $f \in L_{\overline{p}}\,(Q)$

$$f \in \left\{\sum_{v=1}^{m} \varphi_v\,(u_v)\right\} \Longleftrightarrow \mathcal{D}_\theta\ \ f = 0 \quad for \ \ almost \quad all \ \ (x,\theta) \in R^{2n}.$$

In other words the family of the linear operators $\{\nabla_{\mathcal{D}_\theta}\}_{\theta\in G}$ *is an exact anihiator of a class the sums of fewer variable functions*

$$\sum = \left\{\sum_{v=1}^{m} \varphi_v\,(u_v)\right\}$$

in the space $L_p(Q)$.

Allowing for theorem A we'll determine $\mathcal{D}-$modulus of continuity of the function $f \in L_{\overline{p}}(K)$, $K = I^n$, $I = [0,1]$ by the following form

$$\omega_{\mathcal{D}}(f)_{L_{\overline{p}}(K)} = \sup_{\theta \in K} \|\nabla_{\mathcal{D}_\theta} f\|_{L_{\overline{p}}(k)} \qquad (1)$$

Due to the fact that exact annhiator reprsents linear combination of finite mixed diferences it is easy to see that $\omega_{\mathcal{D}}(f)_{L_{\overline{p}}(K)}$ satisfies the continuity modulus properties: it equals to zero at $\theta = (0, ..., 0)$, doesn't decrease as a function from θ and it is continuous function from θ.

Theorem 1. *For every function* $f \in L_{\overline{p}}(K)$ *it holds the estimation*

$$E_{\overline{p}}\left[f, \sum\right] \leq 2^{\sum\limits_{k=1}^{n} \frac{1}{p_i}} \omega_D(f)_{L_{\overline{p}}(K)}.$$

For proving theorem 1 we need

Lemma 1. *For* p_i, $0 < p_i \leq \infty$, $i = \overline{1,n}$ *exist* $\alpha \in K$ *such that*

$$\left\|\nabla_{\mathcal{D}_{x-\alpha}} f(\alpha)\right\|_{\overline{p}} \leq 2^{\frac{1}{p_1}+...+\frac{1}{p_n}} \sup_{h \in K} \|\nabla_{\mathcal{D}_h} f(x)\|_{\overline{p}}.$$

Here in the left-hand side of the inequality the integration is made relatively to $x \in K$.

For proving this lemma we'll use the following results.

Lemma A [1]. *For* $0 < p \leq \infty$, $\exists \alpha \in K$ *is such that*

$$\left\|\nabla_{\mathcal{D}_{x-\alpha}} f(\alpha)\right\|^p_{L_{\overline{p}}(K)} \leq 2^n \sup_{\theta \in K} \|\nabla_{\mathcal{D}_\theta} f(x)\|^p_{L_{\overline{p}}(K)},$$

where on the left part the integration is made relative to $x \in K$.

Proof of lemma 1. Apply lemma A for the case $n = 1$ to the variable x_1

$$\int_0^1 \left|\nabla_{(x_1-\alpha_1)...(x_n-\alpha_n)} f(\alpha_1, ..., \alpha_n)\right|^{p_1} dx_1 \leq$$

$$\leq 2 \sup_{h_1 \in [0,1]} \int_0^1 \left|\nabla_{h_1(x_2-\alpha_2)...(x_n-\alpha_n)} f(x_1, \alpha_2, ..., \alpha_n)\right|^{p_1} dx_1$$

or

$$\left\|\nabla_{(x_1-\alpha_1)...(x_n-\alpha_n)} f(\alpha_1, ..., \alpha_n)\right\|_{p_1} \leq$$

$$\leq \left(2\sup_{h_1} \int_0^1 \left|\nabla_{h_1(x_2-\alpha_2)...(x_n-\alpha_n)} f(x_1, \alpha_2, ..., \alpha_n)\right|^{p_1} dx_1\right)^{\frac{1}{p_1}} =$$

$$= 2^{1/p_1} \sup_{h_1} \left\|\nabla_{h_1(x_2-\alpha_2)...(x_n-\alpha_n)} f(x_1, \alpha_2, ..., \alpha_n)\right\|_{p_1}. \qquad (2)$$

Let's apply now lemma A to the norm of the right-hand of the relation (2) relative to the variable x_2 in L_{p_2} we'll get

$$\left\|\left\|\nabla_{h_1(x_2-\alpha_2)\dots(x_n-\alpha_n)}f\left(x_1,\alpha_2,...,\alpha_n\right)\right\|_{p_1}\right\|_{p_2}^{p_2}\leq$$

$$\leq 2\sup_{h_2}\left\|\left\|\nabla_{h_1h_2(x_3-\alpha_3)\dots(x_n-\alpha_n)}f\left(x_1,x_2,\alpha_3,...,\alpha_1\right)\right\|_{p_1}\right\|_{p_2}^{p_2}\Longrightarrow$$

$$\Longrightarrow\left\|\nabla_{h_1(x_2-\alpha_2)\dots(x_n-\alpha_n)}f\left(x_1,\alpha_2,...,\alpha_n\right)\right\|_{p_1p_2}^{p_2}\leq$$

$$\leq 2\sup_{h_2}\left\|\nabla_{h_1h_2(x_3-\alpha_3)\dots(x_n-\alpha_n)}f\left(x_1,x_2,\alpha_3,...,\alpha_n\right)\right\|_{p_1p_2}^{p_2}$$

$$\Longrightarrow\left\|\nabla_{h_1(x_2-\alpha_2)\dots(x_n-\alpha_n)}f\left(x_1,\alpha_2,...,\alpha_n\right)\right\|_{p_1p_2}\leq$$

$$\leq 2^{\frac{1}{p_2}}\sup_{h_2}\left\|\nabla_{h_1h_2(x_3-\alpha_3)\dots(x_n-\alpha_n)}f\left(x_1,x_2,\alpha_3,...,\alpha_n\right)\right\|_{p_1p_2} \tag{3}$$

Using relation (3) in the inequality (2) we'll get

$$\left\|\nabla_{(x_1-\alpha_1)\dots(x_n-\alpha_n)}f\left(\alpha_1,...,\alpha_n\right)\right\|_{p_1p_2}\leq$$

$$\leq\left\|2^{\frac{1}{p_1}}\sup_{h_1}\left\|\nabla_{h_1(x_2-\alpha_2)\dots(x_n-\alpha_n)}f\left(x_1,\alpha_2,...,\alpha_n\right)\right\|_{p_1}\right\|_{p_2}=$$

$$=2^{\frac{1}{p_1}}\sup_{h_1}\left\|\nabla_{h_1(x_2-\alpha_2)\dots(x_n-\alpha_n)}f\left(x_1,\alpha_2,...,\alpha_n\right)\right\|_{p_1p_2}\leq$$

$$\leq 2^{\frac{1}{p_1}}\sup_{h_1}2^{\frac{1}{p_2}}\sup_{h_2}\left\|\nabla_{h_1h_2(x_3-\alpha_3)\dots(x_n-\alpha_n)}f\left(x_1,x_2,\alpha_3,...,\alpha_n\right)\right\|_{p_1p_2}=$$

$$=2^{\frac{1}{p_1}+\frac{1}{p_2}}\sup_{h_1h_2}\left\|\nabla_{h_1h_2}f\left(x_1,x_2,\alpha_3,...,\alpha_n\right)\right\|_{p_1p_2} \tag{4}$$

Applying this operation consequently to other variables we'll get the relation (4) of the statement of lemma 1.

Proof of theroem 1. In [2] we established that it holds the equality

$$\nabla_{\mathcal{D}_\theta}f\left(x\right)=f\left(x+\theta\right)-\sum_{s=1}^{m}\varphi\left(\overset{\vee\theta}{x}_{\delta_s}\right),$$

where $\varphi\left(\overset{\vee\theta}{x}_{\delta_s}\right)=\varphi\left(x\backslash x_{\delta_s},x_{\delta_s}+\theta_{\delta_s}\right)$

Using this relation for every $\alpha\in K$ we get

$$f\left(x\right)-\nabla_{\mathcal{D}_{x-\alpha}}f\left(\alpha\right)\overset{def}{=}\varphi_0\left(x\right)\in\sum$$

Then by virtue of lemma 1 we have

$$\left\|\nabla_{\mathcal{D}_{x-\alpha}}f\left(\alpha\right)\right\|_{L_{\overline{p}}(K)}=\left\|f\left(x\right)-\varphi_0\left(x\right)\right\|_{L_{\overline{p}}(K)}\leq$$

$$\leq 2^{\sum\limits_{k=1}^{n}\frac{1}{p_k}}\sup_{h}\left\|\nabla_{\mathcal{D}_h}f\left(x\right)\right\|_{L_{\overline{p}}(K)}.$$

It follows that

$$E\left[f,\sum\right]=\inf_{\varphi\in\sum}\|f(x)-\varphi(x)\|_{L_{\overline{p}}(K)}\leq$$

$$\leq 2^{\sum\limits_{k=1}^{n}\frac{1}{p_k}}\sup_{h\in K}\|\nabla_{\mathcal{D}_h}f(x)\|_{L_{\overline{p}}(K)}=2^{\sum\limits_{k=1}^{n}\frac{1}{p_k}}\omega_{\mathcal{D}}(f)_{L_{\overline{p}}(K)}.$$

Theorem 1 is proved.

Theorem 2. *For any function* $f\in L_{\overline{p}}(K)$ *it holds ordinal equality*

$$E_{\overline{p}}\left(f,\sum\right)\overset{\cup}{\underset{\cap}{}}\omega_{\mathcal{D}_\theta}(f)_{L_{\overline{p}}(K)}.$$

Upper estimation is established in theorem 1, and lower estimation follows from lower estimation of a best approximation (theorem 1 of paper [2]) ealier established by us due to definition of a module of the continuity (1).

Actually it is obtained more - in theorem 1 of the given paper is established double-sided estimation with concrete constants.

The quantity of the differnces $\Delta_\delta f$ in $\nabla_{\mathcal{D}_\theta}f$, evidently, depends from $\mathcal{D}$. The greatest number of such differences present in an exact annihilator of set of the sums of form $\sum\limits_{i=1}^{n}\varphi_i(x_i)$ and it equals to $2^n-(n+1)$. This case $\mathcal{D}^0=\{\delta_1,...,\delta_n\}$, $\delta_i=$ $=\{i\}$ $\quad u_i=\{x_i\}$. In $\nabla_{\mathcal{D}^0_\theta}f$ up to cancellation participates $\sum\limits_{i=1}^{n}c_n^i2^i$ values of f . In case when some (more that two) variables take part only in one u_i, number of differences in exact annihilator we can cancel essentially. Take the number $0=k_0<$ $<k_1<...<k_q=n$ and denote

$$t_i=\left\{x_{k_{j-1}+1},...,x_{k_j}\right\},\quad \tau_j=\left\{\theta_{k_{j-1}+1},...,\theta_{k_j}\right\},\ \ j=\overline{1,q}.$$

Let $\eta_i\subset\{1,...,q\}$, $i=\overline{1,l}$. Consider the set

$$V_j=t_{\eta_j}=\left\{t_s|\ \ s\in\eta_j\right\},\ \ V=\{v_1,...,v_l\}.$$

Determine the differences for the function f relative to the group of the variables τ_j

$$\nabla_{\tau_j}f=f\left(x+\tau^j\right)-f(x)\ ,\quad where\quad \tau^j=(0,...,\tau_j,...,0)=$$

$$=\left(0,...,0,\theta_{\kappa_{j-1}+1,...,}\theta_{k_j},0,...,0\right);\ \ \Delta_{\tau_{ij}}f=\Delta_{\tau_i}\left(\Delta_{\tau_j}f\right)$$

For every

$$\eta=\left\{\eta^{(1)},..,\eta^{(|\eta|)}\right\},\quad \Delta_{\tau_\eta}f=\Delta_{\tau_{\eta(1)}},...,\tau_{\eta^{(|\eta|)}}f$$

Finally, let

$$\nabla^Q_{v_\tau}f=\sum_\eta\Delta_{\tau_\eta}f;\quad \nabla^K_{v_\tau}f=\begin{cases}\sum\limits_\eta\Delta_{\tau_\eta}f & if\quad x,x+\tau\in K\\ 0 \quad if & x\ \ or\ \ x+\tau\notin K\end{cases},$$

where summation is taken on all $\eta \subset \{1,...,q\}$, contained in none of $\eta_1,...,\eta_l$. Denote by $\sum_V$ the set of all sums of the form

$$\left\{\sum_{i=1}^{l}\varphi_i\left(V_i\right)\right\}; \qquad \sum_{v}^{c} \stackrel{df}{=} \sum_{v} \cap C\left(K\right) , \quad \sum_{v}^{p} = \sum_{v} \cap L_p\left(K\right)$$

Using the method of proving theorem 1 it is possible to prove that it is true

Theorem 3. *For the function* $f \in L_p\left(T\right)$, $\overline{p} = \left(p_1,...,p_n\right)$, $0 < p_i \leq \infty$ $f \in \sum_v^p \Longleftrightarrow \nabla_{v_\tau}\ f = 0$ *for almost all* $(x,\tau) \in K^2$.

In case $t_i = \{x_i\}$, $i = \overline{1,n}$ from theorem 3 follows theorem 1 from [2].

The exact annihilator of the set of functions of the form $\varphi\left(x_1,x_2,x_3\right)+\psi\left(x_3,x_4\right)$ consisted of 6 differences and 48 values of the function f . If in this example we put $t_1 = \{x_1,x_2\}$, $t_2 = \{x_3\}$, $t_3 = \{x_4\}$ then according to theorem 3 the exact annihilator of this set will take the form $\nabla_{v_\tau}f = \left(\Delta_{\tau_{13}} + \Delta_{\tau_{123}}\right)f$, i.e. will consist of two differences and 12 values of the function f .

Determine v modulus of continuouty of the function $f \in L_{\overline{p}}\left(K\right)$ by the following form

$$\omega_v\left(f\right)_{L_{\overline{p}}(K)} = \sup_{\theta \in K}\left\|\nabla_{v_\tau}^K f\right\|_{L_{\overline{p}}(K)}$$

Theorem 4. *For any function* $f \in L_{\overline{p}}\left(K\right)$ *if holds relation*

$$E_{\overline{p}}\left(f,\sum\right) \mathop{}_{\cap}^{\cup} \omega_v\left(f\right)_{L_{\overline{p}}(K)} .$$

The lower estimation is established similarly to estimation of the best approximation, cited in theorem 1 from [2], where instead of the coefficient $N\left[\nabla_{\mathcal{D}}\right]^{-\gamma}$ will be a number corresponding to quantity of addends of finite differences, participating in a continuity module $\omega_v\left(f\right)$.

The upper estimation is established similarly to theorem 1 of the present paper.

References

[1]. Babaev M.-B.A. *Best approximation by functions of fewer variables.* Dokl. AN SSSR, 1984, v.279, No2, p.273-277.

[2]. Babaev A.M.-B. *On estimation of lower best approximation.* Trudy IMM NAN Azerb. in the press.

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