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POINTWISE SATURATION THEOREMS FOR THE CEZARO AND ABEL MEANS OF FOURIER SERIES

Abstract

In the paper the problems upword to the known result of G. Sunouchi on (necessary and sufficient) conditions of constancy of conjugate function by given rate of uniform approximation by Fejer means on some interval are considered. The author show that the Fejer summation method is not essential and can be changed to Cesaro and Abel methods, besides the corresponding assertions are true if consider pointwise convergence instead of uniform one.

§1. Approximation rate of periodic functions by Fejer means and properties of conjugate functions.

Let $f(x)$ be a 2π- periodic summable function with Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} (a_k(f)\cos kx + b_k(f)\sin kx) \equiv \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx,$$

$$\sigma_n(x,f) \equiv \sigma_n(x) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)(a_k \cos kx + b_k \sin kx), \quad (n = 0, 1, ...)$$

be its n-th Fejer mean and $\bar{f}(x) = \lim\limits_{r\to 1-0} \bar{f}(r,x) = \lim\limits_{r\to 1-0} \sum\limits_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx) r^k$ be its conjugate function. It's known (se [1], p. 172-174) that function $\bar{f}(x)$ has the same meaning for almost all $x \in [0, 2\pi]$.

Sunouchi [2] proved the following theorem

Theorem A. *If the following relation is fulfilled uniformly on interval* $(a,b) \subset [0, 2\pi]$

$$\sigma_n(x,f) - f(x) = 0\left(\frac{1}{n}\right), \tag{1}$$

then $\bar{f} \equiv const$ *on* (a,b).

On the other hand as Sunouchi showed [2] the following theorem is valid.

Theorem B. *If* $\bar{f}(x) \equiv const$ *on* $(a,b) \subset [0,2\pi]$ *then (1) is fulfilled uniformly on* $[a+\delta, b-\delta]$ *for any fixed* $\delta \in \left(0, \frac{b-a}{2}\right)$ *(after possible correction of* $f(x)$ *on a measure zero set).*

Sunouchi formulated this theorem incorrectly without stipulation placed in brackets. Changing $f(x)$ on a measure zero set we don't change $\bar{f}(x)$ whereas relation (1) can be easily violated.

Theorem A is nothing but some lokal constant criterion of conjugate function. At first note that Sunouchi's method of proving theorem A gives a little bit more than he has formulated. In fact, the following theorem takes place ([3], p.4).

Theorem A'. *If relation (1) is satisfied almost everywhere on* $(a,b) \subset [0,2\pi]$ *and there exists function* $\psi(x) \in L(a,b)$ *such that*

$$n\left|\sigma_n\left(x,f\right)-f\left(x\right)\right|\leq\psi\left(x\right), \qquad (2)$$

a. e. on (a,b) *then* $\bar{f}\equiv const$ *on* (a,b).

It turns out that condition (2) in theorem A' generally speaking, cannot be disregarded, however, one can dispense with it if estimate (1) is satisfied at each point $x\in(a,b)$. In other words the following theorem is valid (see [3], p. 4-6).

Theorem 1. *a) If for* 2π*- periodic summable function* $f(x)$ *relation (1) is satisfied at each point of interval* $(a,b)\subset[0,2\pi]$*, then* $\bar{f}(x)$ *has sense for all* $x\in(a,b)$ *and* $\vec{f}(x)\equiv const$ *on* (a,b).

b) If for 2π*-periodic summable function* $f(x)$ *relation (1) holds everywhere on* $[0,2\pi]$ *or everywhere on* $[0,2\pi]$ *except one interior point or everywhere on* $(0,2\pi)$ *, then* $f(x)\equiv const$ *on* $(-\infty,+\infty)$ *(after possible correction of* $f(x)$ *on measure zero set).*

c) If 2π*-periodic function* $f(x)\in L(0,2\pi)$ *, then in order that* $\bar{f}(x)$ *be constant on some interval* $(a,b)\subset[0,2\pi]$ *it's necessary and sufficient that for function* $f(x)$ *relation (1) be satisfied on* (a,b) *(after possible correction of* $f(x)$ *on a measure zero set).*

d) For any interval $(a,b)\subset[0,2\pi]$ *there exists* 2π*- periodic function* $f_1(x)$ *belonging to all classes* $L^p(0,2\pi)$ *with* $1\leq p<\infty$ *for which relation (1) holds everywhere on* (a,b) *except one point and never the less* $\bar{f}_1(x)\neq const$ *on* (a,b) .

e) There exists 2π*-periodic function* $f_2(x)$ *belonging to all classes* $L^p(0,2\pi)$, $1\leq p<\infty$, *for which (1) is satisfied everywhere on* $[0,2\pi]$*, except of two interior points however,* $\bar{f}_2(x)\neq const$ *on* $[0,2\pi]$.

Remark 1. Under the conditions of point a) of theorem1 function $f(x)$ must be analytical on interval (a,b) since $\bar{f}(x)\equiv const$ on (a,b).

Let's $\langle a,b\rangle$ be a segment, interval or semisegment. A set $E\subset\langle a,b\rangle\subset[0,2\pi]$ will be called exceptional on $\langle a,b\rangle$ for a given class of 2π- periodic functions $f(x)$ if for any function from this class the fulfillment of estimate (1) at each point of set $\langle a,b\rangle\backslash E$ (isn't necessarily uniformly on $\langle a,b\rangle\backslash E$) provides that conjugate function $\bar{f}(x)\equiv const$ on $\langle a,b\rangle$.

In theorem 1 we proved (point a)) that the empty set is exceptional on (a,b) for the class of summable functions and for classes $L^p(0,2\pi)$, $1\leq p<\infty$, even a finite set isn't exceptional one (points *d)* and *e)*). It's natural to expect that constricting the class of considered functions $f(x)$ we simultaneously expand the class of exceptional sets.

Thus, the following theorem is valid (see [3], p.7-10).

Theorem 2. *a) The denumerable set is exceptional on* $(a,b)\subset[0,2\pi]$ *in the class of continuous functions*

b) There exists perfect measure zero M*-set* $P\subset[0,2\pi]$ *which is not exceptional on* $[0,2\pi]$ *in the class of functions belonging to all* $Lip\alpha\,(0<\alpha<1)$.

c) For any $\alpha\in(0,1)$ *there exists perfect* U*-set* $P\subset[0,2\pi]$ *which is not exceptional one in the class of functions* $Lip\alpha$.

d) Every zero measure set is exceptional one on $(a,b)\subset[0,2\pi]$ *in the class of absolutely continuous functions.*

§2. Pointwise saturation theorems for Cesaro and Abel means.

Analyzing proofs of theorems 1 and 2 it's easy to see that three statements underlie them: lemma 1([3], p. 4-5), theorem B and the following Zamansky's theorem C([4], p. 167).

Theorem C. *If $f(x)$ is a 2π-periodic absolutely continuous function, $a_0 = 0$ and $f'(x) = \varphi(x)$, then the relation*

$$\sigma_n(x; f) - f(x) = -\frac{\bar{\varphi}(x)}{n} + \circ(1/n).$$

helds a. e. on $(-\infty, +\infty)$.

Lemma 1. *Let series*

$$\sum_{k=1}^{\infty} u_k \tag{3}$$

be $(C, 1)$-summable to the number c and σ_n be it's $(C, 1)$-means.

If $\sigma_n - c = \circ\left(\frac{1}{n}\right)$, $n \to \infty$, then series

$$-\sum_{k=1}^{\infty} k u_k \tag{4}$$

is $(C, 1)$-summable to zero(and, consequently, summable to zero by the Abel method).

It's found that these three statements with correspondent changes carry over to the case of approximation by Cesaro means of the order $\alpha > 0$

$$\sigma_n^{\alpha}(f; x) = \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k(x) \ \ (n = 0, 1, ...),$$

where $A_n^{\alpha} = \begin{pmatrix} n+\alpha \\ n \end{pmatrix}$, $s_k(x)$- are the partial sums of Fourier series for $f(x)$, and by Abel means

$$f(r, x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) r^k \quad (0 \le r < 1)$$

of Fourier series for $f(x)$. Berens [5] obtained the following generalizations of the author's results [3].

Theorem D. *Let 2π-periodic function $f \in L(0, 2\pi)$ and $\alpha > 0$. If there exists finite limit $\lim_{n\to\infty} \sigma_n^{\alpha}(f; x) = c(x)$, then the sequence*

$$\frac{\sigma_n^{\alpha}(f; x) - c(x)}{a_n^{\alpha}}, \ \textit{where}\ a_n^{\alpha} = \sum_{k=n+1}^{\infty} \frac{\alpha}{k(k+\alpha)} \sim \frac{\alpha}{n} \tag{5}$$

converges iff the series

$$\sum_{k=1}^{\infty} k(a_k \cos kx + b_k \sin kx) \tag{6}$$

is summable by the method (C,α). Moreoves, (C,α)-sum of this series and the limit of sequence (5) coincide.

Remark 2. Indeed theorem D is actually a theorem on summability of numerical series $\sum_{k=0}^{\infty} u_k$ and precisely if there exists a finite limit of (C,α) −means of this series $\lim_{n\to\infty} \sigma_n^\alpha = c$, then sequence $\frac{\sigma_n^\alpha - c}{a_n^\alpha}$ converges iff the mentioned series is (C,α)-summable. Moreover, the limit of this sequence and the (C,α)-sum of the given series coincide.

Theorem E. *Let 2π-periodic function $f \in L(0,2\pi)$, $\alpha > 0$ and $\lim_{n\to\infty} \sigma_n^\alpha (f;x) = f(x)$ for all x from some interval (a,b). If for all $x \in (a,b)$ there exists finite limit*

$$\lim_{n\to\infty} \frac{\sigma_n^\alpha (f;x) - f(x)}{a_n^\alpha} = g(x) \tag{7}$$

with integrable function $g(x)$, then for all $x \in (a,b)$ the conjugate function

$$\bar{f}(x) = C - \int_a^x g(u)\,du, \tag{8}$$

where C- is some constant. In particular, if $f(x)$ is continuous then (8) remains valid even if (7) is violated on a denumerable set.

As a corollary to these theorem we obtain

Theorem F. *a) If for 2π-periodic function $f \in L(0,2\pi)$ at each point of the interval $(a,b) \subset [0,2\pi]$*

$$\sigma_n^\alpha (f;x) - f(x) = \circ(1/n) \tag{9}$$

then $\bar{f}(x)$ makes sense for oll $x \in (a,b)$ and $\bar{f}(x) \equiv const$ on (a,b).

b) If $\lim_{n\to\infty} \sigma_n^\alpha (f,x) = f(x)$, then series (6) $(C,\alpha > 0)$-is summable to zero iff relation (9) is fulfilled.

c) If $f(x)$ is continuous 2π-periodic function, and estimation (9) occurs everywhere on (a,b) except a denumerable set, then function $\bar{f}(x) \equiv const$ on (a,b).

Besides, theorem 3 and 4 hold.

Theorem 3. *If $\bar{f}(x) \equiv const$* on $(a,b) \subset [0,2\pi]$, *then for any fixed δ $\left(0 < \delta < \frac{b-a}{2}\right)$ estimation (9) occurs uniformly on $[a+\delta, b-\delta]$ (after the possible correction of $f(x)$ on measure zero set).*

Theorem 4. *If $f(x)$ is absolutely continuous 2π-periodic function, $a_0 = 0$ and $f'(x) = \varphi(x)$, then equality*

$$\sigma_n^\alpha (f;x) - f(x) = -\frac{\alpha}{n}\bar{\varphi}(x) + \circ(1/n)$$

holds a. e. on $(-\infty, +\infty)$.

Using theorem F, b), theorems 3 and 4, applying method of proving theorems 1 and 2 we'll obtain that the following theorem is valid.

Theorem 5. *The statements of theorem 1 and 2 remain valid if in their formulation we'll change relation (1) by relation (9).*

Note that points a) of both theorems in this case were proved by Berens (theorem F a), b)).

The following statements are also valid.

Lemma 2. *Let for numerical series (3)*

$$f(r)=\sum_{k=0}^{\infty}u_k r^k \to c, \quad (r\to 1-0),$$

hold such that

$$f(r)-c=\circ(1-r), \ (r\to 1-0).$$

Then series (4) is summable to zero by Abel method.

Theorem 6. *If* $\bar{f}(x)\equiv const$ *on some interval* $(a,b)\subset[0,2\pi]$ *, then for any fixed* $\delta\,(0<\delta<(b-a)/2)$ *the relation*

$$f(r,x)-f(x)=\circ(1-r), \quad (r\to 1-0) \tag{10}$$

holds uniformly on $[a+\delta, b-\delta]$ *(after possible correction of* $f(x)$ *on a measure zero set).*

Theorem 7. *If* $f(x)$ *absolutely continuous* 2π*-periodic function,* $a_0=0$ *and* $f'(x)=\varphi(x)$*, then the equality*

$$f(r,x)-f(x)=-(1-r)\,\bar{\varphi}(x)+\circ(1-r)$$

holds a. e. on $(-\infty,+\infty)$.

Now basing on lemma 2 and theorems 6 and 7 and remaining of proving unchanged we shall obtain the following theorem.

Theorem 8. *The statements of theorem 1 and 2 remain valid if in their formulations we'll change relation (1) by relation (10).*

Note that in this case points a) both of theorems were proved by Berens [6] and point a) of the first theorem- also by Hedberg [7] independently and by different methods.

It just remains to verify the fact that lemma 2 and theorems 3, 4, 6 and 7 are valid.

Proof of lemma 2. We shall prove more general statement than lemma 2 and precisely the following theorem 9.

Theorem 9. *If there exists finite limit* $\lim_{r\to 1-0} f(r)=c$ *, then limit*

$$\lim_{r\to 1-0}\frac{f(r)-c}{1-r} \tag{11}$$

exists iff series (4) is summable by Abel method, at that A-sum of this series coincides with limit (11).

First of all we shall show that the following lemma is valid.

Lemma 3. *Let series (3) with real terms be given and* $f(r)=\sum_{k=0}^{\infty}u_k r^k$*, its A- means converge at* $0\leqslant r<1$*. Let for* $0\leqslant r<1$ *the function*

$$t(r)=\frac{1}{1-r}\sum_{k=0}^{\infty}f(\theta_k)(r_{k+1}-r_k).$$

be defined, where

$$r_k=r_k(r), \quad r=r_0<r_1<...<r_k...u \quad \lim_{r\to 1-0} r_k=1, \tag{12}$$

$$1>\theta_k=\theta_k(r)\geqslant r. \tag{13}$$

Then series (3) is A-summable to a number c iff

$$\lim_{r\to 1-0} t(r)=c \tag{14}$$

for arbitrary sequences $\{r_k\}$ and $\{\theta_k\}$ which satisfy conditions (12) and (13).

Proof. 1. Let $f(r)\to c$ as $r\to 1-0$. Then for any $\varepsilon>0$ there exists such r_0 that $|f(r)-c|<\varepsilon$ is valid for all $r\geqslant r_0$. Considering values of $r\geqslant r_0$ we obtain (see (13)) that $|f(\theta_k)-c|<\varepsilon$ holds for all θ_k and, consequently,

$$|t(r)-c|=\frac{1}{1-r}\left|\sum_{k=0}^{\infty}[f(\theta_k)-c](r_{k+1}-r_k)\right|<\frac{\varepsilon}{1-r}\sum_{k=0}^{\infty}(r_{k+1}-r_k)=\varepsilon.$$

2. Let (14) hold for any sequences $\{r_k\}$ and $\{\theta_k\}$ which satisfy conditions (12) and (13). Suppose $\theta_k=r$ $(k=0,1,...)$. Then $t(r)=f(r)$ and lemma 3 is proved.

Proof of theorem 9. It's sufficient to notice that

$$f(r)-c=\sum_{\nu=0}^{\infty}[f(r_\nu)-f(r_{\nu+1})],$$

where $\{r_\nu\}$ satisfies conditions (12). Since

$$f(r_\nu)-f(r_{\nu+1})=\sum_{k=0}^{\infty}u_k\left(r_\nu^k-r_{\nu+1}^k\right)=\sum_{k=1}^{\infty}ku_k\theta_\nu^{k-1}(r_v-r_{v+1})=(r_{\nu+1}-r_\nu)\,g(\theta_\nu),$$

where $r_\nu<\theta_\nu<r_{\nu+1}$ and $g(\theta_\nu)$ are Abel means of series (4), then

$$f(r)-c=\sum_{\nu=0}^{\infty}g(\theta_\nu)(r_{\nu+1}-r_\nu).$$

Applying now lemma 3 to series (4) we obtain the statement of theorem 9.

Proof of theorem 3 and 6. Since $\bar f(x)\equiv const$ on (a,b) then by the theorem E the relation

$$\sigma_n(f;x)-f(x)=\circ(1/n),\ n\to\infty.$$

holds uniformly on $[a+\delta,b-\delta]$.

Further

$$f(r,x)-f(x)=(1-r)\sum_{k=0}^{\infty}[s_k(x)-f(x)]\,r^k=(1-r)^2\sum_{k=0}^{\infty}(k+1)[\sigma_k(f;x)-f(x)]\,r^k.$$

Consequently, for any $\varepsilon>0$ there exists such a number $k_0=k_0(\varepsilon)$ that for all $k\geqslant k_0$ and $x\in[a+\delta,b-\delta]$

$$(k+1)|\sigma_k(f;x)-f(x)|<\varepsilon \tag{15}$$

holds.

Fixing k_0 we obtain that

$$(1-r)^2 \sum_{k=0}^{k_0-1} (k+1) [\sigma_k (f;x) - f(x)] r^k = \circ (1-r), \ r \to 1-0, \tag{16}$$

$$\left| (1-r)^2 \sum_{k=k_0}^{\infty} (k+1) [\sigma_k (f;x) - f(x)] r^k \right| < \varepsilon (1-r) \tag{17}$$

hold uniformly on $[a+\delta, b-\delta]$.

In view of arbitrariness of $\varepsilon > 0$ it follows from (16) and (17) that (10) holds uniformly on $[a+\delta, b-\delta]$.

Thus, theorem 6 is proved.

Theorem 3 can be proved analogously. In fact, let $0 < \alpha \neq 1$. Then

$$\sigma_n^\alpha (f;x) - f(x) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} [s_k (x) - f(x)] =$$

$$= \frac{1}{A_n^\alpha} \sum_{k=0}^{n} (k+1) A_{n-k}^{\alpha-2} [\sigma_k (f;x) - f(x)] .$$

Using inequality (15) for fixed k_0 and $x \in [a+\delta, b-\delta]$ we have:

$$\frac{1}{A_n^\alpha} \left| \sum_{k=0}^{k_0-1} (k+1) A_{n-k}^{\alpha-2} [\sigma_k (f;x) - f(x)] \right| \leqslant \frac{C_1(\alpha)}{n^2} = \circ (1/n) , \tag{18}$$

$$\frac{1}{A_n^\alpha} \left| \sum_{k=k_0}^{n} (k+1) A_{n-k}^{\alpha-2} [\sigma_k (f;x) - f(x)] \right| \leqslant \frac{C_2(\alpha)\varepsilon}{n}. \tag{19}$$

In view of arbitrariness of $\varepsilon > 0$ it follows from (18) and (19) that (9) holds uniformly on $[a+\delta, b-\delta]$.

Proof of theorems 4 and 7. It's known (see, e. g. [1], pp. 170, 174 and 214) that series (6) is $(C, \alpha > 0)$-summable and A-summable to $\bar{\varphi}(x)$ a. e.

On the basis of theorems D and 9 we conclude that

$$\lim_{n\to\infty} \frac{\sigma_n^\alpha (f;x) - f(x)}{a_n^\alpha} = \lim_{r\to 1-0} \frac{f(r;x) - f(x)}{1-r} = -\bar{\varphi}(x) , \tag{20}$$

holds a. e., q. e. d.

Remark 3. Note that relations (20) are generalization for the case of $(C, \alpha > 0)$-means and A-means of the corresponding Zamansky's limit equality for $(C, 1)$-means in theorem C.

Corollary 1. *If limits (20) equal to some function $g(x) \in L(a,b)$ a. e. on $(a,b) \subset [0, 2\pi]$, then for all $x \in (a,b)$ equality (8) occurs.*

Really, since $g(x) = \bar{\varphi}(x)$ a.e. on (a,b), then by theorem P.L. Ulyanov ([8], p.584) $\bar{f}(x)$ absolutly continuous on (a,b) and therefore (8) is true for $x \in (a,b)$.

Corollary 2. *Let arbitrary functional series*

$$\sum_{n=0}^{\infty} c_n \varphi_n (x) \tag{21}$$

with real coefficients be $(C, \alpha > 0)$*-summable (or A-summable) on some set* E *to the function* $f(x)$ *and* $\sigma_n^\alpha (f;x)$ *and* $f(r;x)$ *be* (C,α)*-means and A-means of series (21) respectively. If the series*

$$-\sum_{n=1}^{\infty} n c_n \varphi_n (x) \tag{22}$$

is $(C, \alpha > 0)$*-summability (or A-summable) on* E *to function* $\psi(x)$,*, then for* $x \in E$ *we have*

$$\lim_{n\to\infty} \frac{\sigma_n^\alpha (f;x) - f(x)}{a_n^\alpha} = \psi(x) \quad \left(\text{or } \lim_{r\to 1-0} \frac{f(r;x) - f(x)}{1-r} = \psi(x) \right). \tag{23}$$

And inversely: existence of limit (23) implies $(C, \alpha > 0)$*-summably (or A-summability) of series (22) to the same limit.*

In fact it immediately follows from theorem D (see Remark 2) and from theorem 9.

E. A. Storozhenko's theorem ([9], theorem 3) is a particular case of this statement, in which just necessity is proved for- ONS $\{\varphi_n(x)\}$, which are systems of $(C,1)$-summability for the method $(C,1)$.

In conclusion we perform the theorem which is analogous of statement from [5] for (C,α)-means.

Theorem 10. *If* 2π*-periodic summable function* $f(x)$ *be such that*

$$\|f(r,x) - f(x)\|_1 = 0(1-r), \quad r \to 1-0, \tag{24}$$

then for almost all x

$$\lim_{r\to 1-0} \frac{f(r,x) - f(x)}{1-r} \tag{25}$$

exists.

Proof. By the classical Sunouch-Watari saturation theorem [10] relation (24) is equivalent to the fact that $f(x)$ equivalent to some function of bounded variation. But then (see [1], p.216) series (6) is A-summable a. e. and by theorem 9 limit (25) exists a. e.

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