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ASYMPTOTIC ANALYSIS OF BENDING PROBLEM FOR TRANSVERSAL-ISOTROPIC PLATE OF VARIABLE THICKNESS

Abstract

The homogeneous solutions remaining plate faces stressless were constructed. The classification of homogeneous solutions was made. The constructed homogeneous solutions allow to unload the lateral surface at arbitrary loading. By means of the Lagrange principle a boundary value problem was reduced to solving of infinite linear algebraic equations which are known from the constant thickness plate theory.

Let's consider axisymmetric bending problem of transversal isotopic plate whose thickness is $h = \varepsilon r$ (r is a distance from the origin of the plate, ε is an angular thickness of the plate).

The plate associates with the spherical coordinates r, θ, φ changing in the following limits

$$r_1 \leq r < r_2, \quad \pi/2 - \varepsilon \leq \theta \leq \pi/2 + \varepsilon, \qquad 0 \leq \varphi \leq 2\pi$$

$\theta_0 = \pi/2$ is a middle plane of the plate.

We shall call surfaces $\theta = \pi/2 \pm \varepsilon$ the faces of the plate, spherical coordinates $r = r_s$ $(s = 1, 2)$ the lateral surfaces.

Suppose that the following boundary conditions are given on the faces of the plate

$$\sigma_\theta = (-1)^n \sigma(r), \quad \tau_{r\theta} = \tau(r) \ \ at \theta = \pi/2 + (-1)^n \varepsilon \quad (n = 1, 2) \tag{1.1}$$

The plate is made of transversal isotropic material with spherical anisotropy.

We assume that the origin coincides with the center of the plate which is anisotropy pole.

On the lateral surface the stresses are given

$$\sigma_r = f_{1r}(\theta), \quad \tau_{r\theta} = f_{2r}(\theta) \quad at\ \theta = \pi/2 + (-1)^n \varepsilon \qquad (n = 1, 2) \tag{1.2}$$

Functions $f_{is}(\theta)$ $(i = 1, 2)$ satisfy the equilibrium conditions

$$2\pi r_1^2 \int\limits_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} (f_{11} \sin \varepsilon\eta - f_{21} \cos \varepsilon\eta) \cos \varepsilon\eta d\eta =$$

$$= 2\pi r_2^2 \int\limits_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} (f_{12} \sin \varepsilon\eta - f_{22} \cos \varepsilon\eta) \cos \varepsilon\eta d\eta = P \tag{1.3}$$

$f_{1S}(\theta), f_{2S}(\theta)$ are sufficiently smooth functions. Besides $f_{1S}(\theta)$ are odd functions, $f_{2S}(\theta)$ are even functions with respect to the middle plane of the plate.

Here P is a efforts resultant, acting in an arbitrary cross section $r = const$.

Equilibrium equations in permutations in the spherical coordinate system have the form [1]

$$\frac{b_{11}}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right)+\frac{2}{r^2}\left(b_{12}-b_{22}-b_{23}\right)u_r+$$

$$+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)+\frac{b_{12}+1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\theta}\left(\sin\theta u_\theta\right)+$$

$$+\frac{1}{r^2}\left(b_{12}-b_{22}-b_{23}-1\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta u_\theta\right)=0 \qquad (1.4)$$

$$\frac{b_{12}+1\partial^2 u_r}{r\quad\partial r\partial\theta}+\frac{b_{22}+b_{23}+2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)+$$

$$+\frac{b_{22}}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\ u_\theta\right)\right]+\frac{\left(b_{22}-b_{23}+2\right)u_\theta}{r^2}=0$$

Relations of the generalized Hook's law have the form [2]

$$\sigma_r=G_1\left[b_{11}e_r+b_{12}\left(e_\theta+e_\varphi\right)\right]$$

$$\sigma_\varphi=G_1\left[b_{11}e_r+b_{23}\left(e_\theta+b_{22}e_\varphi\right)\right]$$

$$\sigma_\theta=G_1\left[b_{11}e_r+b_{22}e_\theta+b_{23}e_\varphi\right]$$

$$\tau_{r\theta}=G_1e_{r\theta} \qquad (1.5)$$

where $\sigma_r,\sigma_\varphi,\sigma_\theta,\tau_{r\theta}$ are components of stress tensor, $e_r,e_\varphi,e_\theta,e_{r\theta}$ are components of deformation tensor in the spherical coordinates [2], u_r,u_θ are components of displacement vector

$$mb_{11}=2G_0E_0\left(1-v^2\right),\quad mb_{22}=2G_0\left(1-v_1v_2\right)$$

$$mb_{12}=2G_0v_1\left(1+v\right),\quad mb_{23}=2G_0\left(v+v_1v_2\right)$$

are material constants

$$m=1-v-2v_1v_2,\quad G_0=G\ G_1^{-1},\quad E_0=E_1E^{-1},\ v_1,v_2,E,E_1,G,G_1$$

are technical constants of material.

By using results of paper [1] we shall represent the solutio of the problem in the form

$$u_r=r^\lambda\left[A_1C_{\gamma1}F_{\gamma1}\left(\theta\right)+A_2C_{\gamma2}F_{\gamma2}\left(\theta\right)\right]$$

$$u_\theta=r^\gamma b_0\left[C_{\gamma1}F'_{\gamma1}\left(\theta\right)+C_{\gamma2}F'_{\gamma2}\left(\theta\right)\right] \qquad (1.6)$$

$$F_\gamma\left(\theta\right)=P_\gamma\left(\cos\theta\right)-P_\gamma\left(-\cos\theta\right)$$

$$A_i = -b_{22}\gamma_i(\gamma_i+1) + \lambda(\lambda+1) + 2(G_0-1)$$

$$b_0 = -[(b_{12}+1)\lambda + b_{22} + b_{23} + 2]$$

$P_\gamma(\cos\theta)$ are the first genus Legendre functions, γ are the roots of the biquadratic equation

$$b_{22}\gamma^2(\gamma+1) - \left[\left(b_{11}b_{22} - b_{12}^2 - 2b_{12}\right)\lambda(\lambda+1) + 2b_{22} + \right.$$

$$+2(b_{12}-b_{22}-b_{23})(G_0-1)]\gamma(\gamma+1) + [\lambda(\lambda+1) + 2(G_0-1)]\times$$

$$\times[b_{11}\lambda(\lambda+1) + 2(b_{12}-b_{22}-b_{23})] = 0 \tag{1.7}$$

$C_{\gamma1}, C_{\gamma2}$ are arbitrary constants.

We shall assume that on the faces of the plate homogeneous boundary conditions $\sigma = \tau = 0$ are given

$$\sigma_\theta = 0, \quad \tau_{r\theta} = 0 \tag{1.8}$$

The satisfaction of the boundary conditions on the faces of the plate (1.8) gives the algebraic system of the second order with respect to the constants $C_{\gamma1}, C_{\gamma2}$.

From the conditions of existence of non trivial solutions of this system we obtain the charateristic equation for determination of eigenvalues λ:

$$\Delta(\lambda,\theta_1) = C_{11}d_{12}F_{\gamma1}(\theta_1)F'_{\gamma_2}(\theta_1) - C_{12}d_{11}T_{\gamma2}(\theta_1)T'_{\gamma1}(\theta_1) -$$

$$-C_{13}(d_{12}-d_{11})\,ctg\theta_1 T'_{\gamma1}(\theta_1)T'_{\gamma2}(\theta_1) \tag{1.9}$$

$$C_{1p} = (b_{12}\lambda + b_{22} + b_{23})A_p - b_{22}b_0\gamma_p(\gamma_p+1) \qquad (p=1,2)$$

$$C_{13} = -(b_{22}-b_{23})b_0$$

$$d_{1k} = A_k + (\lambda-1)b_0 \ \ (k=1,2)$$

Transcendental equation (1.9) as integer function of parameter λ defines a denumerable set λ_k with infinite limit point. As in paper [1] in order to study effectively its roots we suppose

$$\theta = \pi/2 + \varepsilon\eta, \qquad -1 \le \eta \le 1 \tag{1.10}$$

η is new variable counted off from the middle plane $\theta\pi/2$.

Substituting (1.10) into (1.9), we obtain

$$D(z,\varepsilon) = \Delta(\lambda,\theta_1) = 0, \qquad z = \lambda + \frac{1}{2} \tag{1.11}$$

Let's prove the following assertion respective to zeros of the functions $D(z,\varepsilon)$: the function $D(z,\varepsilon)$ has two groups of zeros with the following asymptotic properties at $\varepsilon \to 0$.

The first group consists of four zeros and is characterizied by the fact that all of them have a finite limit at $\varepsilon \to 0$ moreover, two of them are independent on the small parameters ε.

The second group of zeros consists on denumerable set of zeros which have the order $\varepsilon \to 0$ at $O\left(\varepsilon^{-1}\right)$.

In order to prove the first assertion by expanding the functions $D(z,\varepsilon)$ in series with respect to $\theta = \pi/2$ in neighbourhood of the plane ε we shall represent equation (1.11) in the form:

$$D(z,\varepsilon) = 3^{-1} A\varepsilon^3 \left(z^2 - \frac{1}{4}\right)\left[D_0(z) + \frac{1}{5(1-v_1v_2)} D_1(z)\varepsilon^2 + ...\right] = 0 \qquad (1.12)$$

where

$$A = 16_0^2 (1+v) E_0 b_0 \left[\gamma_1(\gamma_2+1) - \gamma_1(\gamma_1+1)\right] \sin(\pi/2)\gamma_1 \sin(\pi/2)\gamma_2 \times$$

$$\times\Gamma\left(1+\frac{\gamma_1}{2}\right)\Gamma\left(1+\frac{\gamma_2}{2}\right)\left[\Gamma\left(\frac{1+\gamma_1}{2}\right)\Gamma\left(\frac{1+\gamma_2}{2}\right)\right]^{-1}$$

$$D_0(z) = 4z^2 + 12v_2 - 9 - 4E_0^{-1}$$

$$D_1(z) = -4(1+v)(G_0 - v_2) z^4 + 2\left[2(1-v_1v_2)(3-2v) + \right.$$

$$+ (1+v)(G_0 - v_2)(E_0 + 4E_0G_0 + 2 - 6v_1) -$$

$$\left. -2(1+v)(2E_0G_0 - v_1 - 1)(G_0 - 1)\right] z^2 - (1-v_1v_2)\times$$

$$\times(40E_0G_0 - 60v_1 - 2v + 23) - 1/2(1+v)(G_0 - v_2)\times$$

$$\times(2E_0G_0 + 2 - 6v_1 + E_0/2) + (1+v)(2E_0G_0 - v_1 - 1)(G_0 - 1) +$$

$$+4\left[(1+v)(v_1 - 1)(2E_0C_0 - 3v_1 + 1) + \right.$$

$$\left. +2(1-v_1v_2)(3-2v)E_0\right](G_0 - 1) + 8(1+v)(1-v_1)E_0(G_0-1)^2$$

Here $\Gamma(x)$ is the Euler gamma function.

It's easy to see from (1.12) that $z_{0,1} = \pm 1/2$ are zeros of the function $D(z,\varepsilon)$. Note that existence of these zeros also follows from the equilibrium condition for the plate. In order to define the rest of zeros of the first group we shall seek them in the form

$$z_k = z_{k0} + \varepsilon^2 z_{k2} + ... \qquad (k = 2,3) \qquad (1.13)$$

Substituting (1.13) into (1.12) we obtain

$$z_{k0} = \pm 1/2\left(9 + 4E_0^{-1} - 12v_2\right)^{1/2}$$

$$z_{k2} = -\left(40 z_{k0}\right)^{-1} D_2\left(z_{k0}\right)$$

Let's prove that the rest of zeros of the function $D(z,\varepsilon)$ infinitely increase, when $\varepsilon \to 0$. We proceed from the contrary assuming that $z_k \to z_k^* + \infty \quad (k \geq 4)$ at $\varepsilon \to 0$. Then the limiting relation $D(z,\varepsilon) \to \varepsilon^3 D_0^*(z_k^*)$ is valid at $\varepsilon \to 0$. Thus, limit points of the set $z_k \quad (k \geq 4)$ are defined from equation $D_0^*(z_k^*) = 0$. In the given case

$$D_0^*(z_k^*) = \left(z_k^{*^2} + 1/4\right)\left(4z_k^{*^2} + 12v_2 - 9 - 4E_0^{-1}\right) = \left(z_0^{*^2} - \frac{1}{4}\right) D_0(z_k^*) = 0$$

From the last equality it follows that the other bounded zeros except $Z_{0,1}, Z_{0,3}$ don't exist.

Thus it has been proved that the rest of zeros of the function $D(z,\varepsilon)$ tend to infinity for $\varepsilon \to 0$. They can be divided into three groups depending on their behaviour at $\varepsilon \to 0$.

The following limiting relations are possible:

1) $\varepsilon z_k \to 0$; 2) $\varepsilon z_k \to \infty$; 3) $\varepsilon z_k \to const$ for $\varepsilon \to 0$.

It can be proved that the cases 1) and 2) are impossible here.

In order to construct asymptotics of zeros of the second group (case 3) we shall find $z_n \quad (n = k-, k \geq 4)$ in the form

$$z_n = \varepsilon^{-1}\delta_n + 0(\varepsilon) \quad (n = 1, 2, ...) \tag{1.15}$$

Subsituting (1.15) into (1.7) we have

$$\tau^2 - 2q_1\delta_n^2\tau + q_2\delta_n^4 = 0, \quad \gamma_1 = \sqrt{\tau_i} \quad i = 1.2 \tag{1.16}$$

$$\tau_i^2 = \delta_n^2 S_i \quad S_i = \sqrt{q_1 - (-1)^i\sqrt{q_1^2 - q_2}}$$

$$2q_1 = b_{22}^{-1}\left(b_{11}b_{22} - b_{12}^2 - 2b_{12}\right), \qquad q_2 = b_{11}b_{22}^{-1}$$

As was noted in [1] parameters q_1, q_2 take on different values depending on characteristics of the material v, v_1, v_2, G_0, which implies different representation of solution by means of Legendre functions.

In turn this results in different asymptotic representations of the Legendre function.

Consider the following possible cases:

1. $q_1 > 0, \quad q_1^2 - q_2 > 0, \quad \gamma_{1,2} = \pm S_1\delta_n, \gamma_{3,4} = \pm S_2\delta_n$

$$S_{1,2} = \sqrt{q_1 \pm \sqrt{q_1^2 - q_2}}, \quad q_1^2 > q_2$$

$$S_{1,2} = \alpha \pm 2\beta = \sqrt{q_1 \pm 2\sqrt{q_2 - q_1^2}}, \quad q_1^2 < q_2$$

2. Roots of charaxteristic equation (1.16) are multiple

$$\gamma_{1,2} = \gamma_{3,4} = \pm p\delta_n, \quad q_1^2 - q_2 = 0, \quad p = \sqrt{q_1}$$

3. $q_1 < 0, \quad q_1^2 - q_2 \neq 0, \quad \gamma_{1,2} = \pm iS_1\delta_n, \quad \gamma_{3,4} = \pm iS_2\delta_n$

$$S_{1,2}=\sqrt{|q_1|\pm i\sqrt{q_2-q_1^2}},\quad q_1^2>q_2$$

$$S_{1,2}=\sqrt{|q_1|\pm i\sqrt{q_1^2-q_2}},\quad q_1^2>q_2$$

4. $q_1<0;\quad q_1^2-q_2=0;\qquad \gamma_{1,2}=\gamma_{3,4}=\pm i\delta_n p,\quad p=\sqrt{|q_1|}$

In cases 1 and 2 after substitution of (1.15) into (1.11) and transformation it by means of the asymptotic expansions of $F_\gamma(\theta), F'_\gamma(\theta)$ for δ_n we obtain

$$(S_2-S_1)\sin(S_1+S_2)\delta_n-(S_1+S_2)\sin(S_2-S_1)\delta_n=0 \tag{1.17}$$

$$\alpha\sin 2\beta\delta_n-\beta sh2\alpha\delta_n=0; \tag{1.18}$$

$$\sin 2p\delta_n-2p\delta_n=0; \tag{1.19}$$

As to cases 3 and 4 for them results are obtained from cases 1 and 2 by formal changing of S_1,S_2 by iS_1,iS_2.These equations coincide with equations defining the indices of the Saint-Venant boundary effects in bending problems for plates of constant thickness

As in isotropic case [2,3] it can be proved that the function $D(z,\varepsilon)$ hasn't other zeros except the zeros obtained above.

2. Let's give characteristics of modes of deformation defined by the solutions constructed above.

By assuming that ε is a small parameter we shall perform asymptotic construction of homogeneous solutions correspondent to the different groups of zeros.

We obtain the following expressions for $z_0=1/2$

$$u_r=+C_0\sin\varepsilon\eta,\quad u_\theta=+C_0\cos\varepsilon\eta$$

$$\sigma_r=\sigma_\varphi=\sigma_\theta=\tau_{r\theta}=0 \tag{2.1}$$

It's easy to see that dispalcement of the plate as a solid corresponds to this solution.

The solution corresponding to zero $z_1=-\frac{1}{2}$ has the following asymptotic representation:

$$u_r=-\frac{r_1}{\rho}\eta C_1\langle 4(1-v_1v_2)+\frac{1}{3}\varepsilon^2\{2(1-v_1v_2)\left(\eta^2+3\right)+$$

$$+2(1+v)(G_0-1)\left(\eta^2-3\right)+2\left[G_0E_0(1+v)(G_0-v_2)-\right.$$

$$\left.-2(1+v)(v_1-2)-2(1-v_1v_2)\right](G_0-1)\eta^2\}+...\rangle$$

$$u_\theta=\frac{r_1C_1}{\rho}\varepsilon^{-1}\langle 2(1-v_1v_2)+\frac{1}{2}\{[\frac{1}{2}(1+v)(2-v_1-3v)+2(1-v_1v_2)+$$

$$+(1-2v_1)(1+v)]\eta^2+4(1+v)(v_1-1)(G_0-1)\}\varepsilon^2+...\rangle \tag{2.2}$$

$$\sigma_r=\frac{G_1C_1\eta}{\rho^2}\left[2(3v_1-2)+O\left(\varepsilon^2\right)\right]$$

$$\sigma_\varphi=\frac{G_1C_1\eta}{\rho^2}\left[2(3E_0-2v_1)+O\left(\varepsilon^2\right)\right]$$

$$\sigma_\theta = O\left(\varepsilon^2\right), \quad \tau_{r\theta} = O\left(\varepsilon\right)$$

For the rest of zeros of the first group formulas for calculation of displacements and stresses if we shall represent them by series respective to ε powers have the following form

$$u_r = \frac{1}{\sqrt{\rho}}\eta\sum_{k=2}^{3} C_k \langle (2z_{k0} - 3)(1 - v_1 v_2) + \frac{1}{3!}\{(1+v)\left(z_{k0}^2 + 2G_0 - 9/4\right) \times$$
$$\times (2v_1 z_{k0} + v_1 - 2)\left(\eta^2 - 3\right) + 2(z_{k0} - 3)(1 - v_1 v_2)\left(\eta^2 + 3\right) - (z_{k0} - 3/2) \times$$
$$\times [4G_0 E_0 (1+v)(G_0 - v_2) z_{k0}^2 - G_0 E_0 (1+v)(G_0 - v_2) +$$
$$+4(1 - v_1 v_2) + 4(1+v)(v_1 - 2)] \times$$
$$\times (G_0 - 1)\eta^2 + 2z_{k2}(1 - v_1 v_2)\}\varepsilon^2 + ...\rangle (z_k \ln\rho)$$

$$u_\theta = \frac{1}{\sqrt{\rho}}\varepsilon^{-1}\sum_{k=2}^{3} C_k \langle 2(1 - v_1 v_2) + \frac{1}{2}\{[2(1 - v_1 v_2) + 1/2(1+v)(2 - 3v) +$$
$$+2(2v_1 - 1)(1+v) z_{k0} - 2v_1 (1+v) z_{k0}^2]\eta^2 + 2(1+v)(2E_0 G_0 - v_1) z_{k0}^2 +$$
$$+4(1+v)(v_1 - 1)(G_0 - 1) - (1+v)(E_0 G_0 - v_1/2)\}\varepsilon^2 + ...\rangle \exp(z_k \ln\rho)$$

$$\sigma_r = +\frac{G}{\rho\sqrt{\rho}}\eta\sum_{k=2}^{3} C_k [4E_0 z_{k0}^2 + 4(v_1 - 2E_0) z_{k0} +$$
$$+3E_0 - 2v_1 + O\left(\varepsilon^2\right)] \exp(z_k \ln\rho) \tag{2.3}$$

$$\sigma_\varphi = -\frac{G}{\rho\sqrt{\rho}}\eta\sum_{k=2}^{3} C_k \left[4v_1 z_{k0}^2 + 4(1 - 2v_1) z_{k0} + 3v_1 - 2 + O\left(\varepsilon^2\right)\right] \exp(z_k \ln\rho)$$

$$\sigma_\theta = O\left(\varepsilon^2\right), \quad \tau_{r\theta} = O\left(\varepsilon\right)$$
$$\rho = r_1^{-1} r$$

The second group of zeros describes the mode of deformation rapidly damping far from the eage of the plate. By expanding solutions of this group respective to powers of the small asymptotic expressions:

$$u_r = -\frac{r_1\varepsilon}{\sqrt{\rho}}\sum_{n=1}^{\infty} B_n \left[S_2\left(b_{22}S_2^2 + b_{12}^2 + b_{12} - b_{11}b_{22}\right)\cos S_2\delta_n \sin S_1\delta_n \eta - \right.$$
$$\left. -S_1\left(b_{22}S_1^2 + b_{11}^2 + b_{12} - b_{11}b_{22}\right)\cos S_1\delta_n \sin S_2\delta_n + O(\varepsilon)\right] \exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right)$$

$$u_\theta = +\frac{r_1\varepsilon S_1 S_2}{\sqrt{\rho}}\sum_{n=1}^{\infty} B_n \left[\left(b_{22}S_2^2 + b_{12}\right)\cos S_2\delta_n \cos S_1\delta_n \eta - \right.$$
$$\left. -\left(b_{22}S_1^2 + b_{12}\right)\cos S_1\delta_n \cos S_2\delta_n + O(\varepsilon)\right] \exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right)$$

$$\sigma_r = +\frac{G_1}{\rho\sqrt{\rho}}\left(b_{11}b_{22} - b_{12}^2\right) S_1 S_2 \sum_{n=1}^{\infty} B_n \delta_n \left[S_1 \cos S_2\delta_n \sin S_1\delta_n \eta - \right.$$

$$-S_2\cos S_1\delta_n\sin S_2\delta_n+O(\varepsilon)]\exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right)$$

$$\sigma_\varphi=-\frac{G_1}{\rho\sqrt{\rho}}\sum_{n=1}^{\infty}B_n\delta_n\left[S_2\left(b_{11}b_{22}-b_{12}^2-2G_0-2G_0b_{12}S_1^2\right)\cos S_2\delta_n\sin S_1\delta_n\eta-\right.$$

$$-S_1\left(b_{11}b_{22}-b_{12}^2-2G_0-2G_0b_{12}S_2^2\right)\cos S_1\delta_n\sin S_2\delta_n\eta+$$

$$+O(\varepsilon)\exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right) \qquad (2.4)$$

$$\sigma_\theta=-\frac{G_1}{\rho\sqrt{\rho}}\left(b_{11}b_{22}-b_{12}^2\right)\sum_{n=1}^{\infty}B_n\delta_n\left[S_2\cos S_2\delta_n\sin S_1\delta_n\eta-\right.$$

$$-S_1\cos S_1\delta_n\sin S_2\delta_n\eta+O(\varepsilon)]\exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right)$$

$$\tau_{r\theta}=\frac{G_1}{\rho\sqrt{\rho}}\left(b_{11}b_{22}-b_{12}^2\right)S_1S_2\sum_{n=1}^{\infty}B_n\left[\cos S_2\delta_n\cos S_1\delta_n\eta-\right.$$

$$-\cos S_1\delta_n\cos S_2\delta_n\eta+O(\varepsilon)]\exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right)$$

$$(S_2-S_1)\sin(S_2+S_1)\delta_n-(S_2+S_1)\sin(S_2+S_1)\delta_n=0$$

$$u_r=\frac{r_1}{\sqrt{\rho}}\varepsilon\sum_{n=1}^{\infty}\left[F_{1n}(\eta)+O(\varepsilon)\right]\exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right)$$

$$u_\theta=\frac{r_1}{\sqrt{\rho}}\varepsilon\sum_{n=1}^{\infty}\left[F_{2n}(\eta)+O(\varepsilon)\right]\exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right)$$

$$\sigma_r=\frac{G_1}{\sqrt{\rho}}\sum_{n=1}^{\infty}\left[b_{11}\delta_nF_{1n}(\eta)+b_{12}F'_{2n}(\eta)+O(\varepsilon)\right]\exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right) \qquad (2\ .5)$$

$$\sigma_\varphi=\frac{G_1}{\rho\sqrt{\rho}}\sum_{n=1}^{\infty}\left[b_{12}\delta_nF_{1n}(\eta)+b_{23}F'_{2n}(\eta)+O(\varepsilon)\right]\exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right)$$

$$\sigma_\theta=\frac{G_1}{\rho\sqrt{\rho}}\sum_{n=1}^{\infty}\left[b_{12}\delta_nF_{1n}(\eta)+b_{22}F'_{2n}(\eta)+O(\varepsilon)\right]\exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right)$$

$$\tau_{r\theta}=\frac{G_1}{\rho\sqrt{\rho}}\sum_{n=1}^{\infty}\left[F'_{1n}(\eta)+\delta_nF_{2n}(\eta)+O(\varepsilon)\right]\exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right)$$

where

$$F_{1n}(\eta)=(a_1\Delta_{1n}-a_2\Delta_{2n})\cos\beta\delta_n\eta\ sh\alpha\delta_n\eta-$$

$$-(a_1\Delta_{2n}-a_2\Delta_{1n})\sin\beta\delta_n\eta\ ch\alpha\delta_n\eta$$

$$F_{2n}(\eta)=(b_{12+1})\left[(\beta\Delta_{2n}-\alpha\Delta_{1n})\cos\beta\delta_n\eta\ ch\alpha\delta_n\eta+\right.$$

$$\left.+(\beta\Delta_{2n}-\alpha\Delta_{1n})\sin\beta\delta_n\eta\ ch\alpha\delta_n\eta\right]$$

$$a_1=1-b_{22}\left(\alpha^2-\beta^2\right),\quad a_2=2b_{22}\alpha\beta$$

$$\Delta_{1n} = D_n\{\alpha\left[b_{12} + b_{22}\left(\alpha^2 + \beta^2\right)\right]\sin\beta\delta_n sh\alpha\delta_n +$$
$$+\beta\left[b_{12} + b_{22}\left(\alpha^2 + \beta^2\right)\right]\cos\beta\delta_n ch\alpha\delta_n\}$$
$$\Delta_{2n} = -D_n\{\beta\left[b_{12} + b_{22}\left(\alpha^2 + \beta^2\right)\right]\sin\beta\delta_n sh\alpha\delta_n -$$
$$-\alpha\left[b_{12} + b_{22}\left(\alpha^2 + \beta^2\right)\right]\cos\beta\delta_n ch\alpha\delta_n\}$$
$$\alpha\sin 2\beta\delta_n - \beta sh2\alpha\sigma_n = 0$$
$$u_r = \frac{(b_{12}+1)}{\sqrt{\rho}}\varepsilon\sum_{n=1}^{\infty}E_n\{\{\sin p\delta_n - \left(b_{12}p^2 - b_{11}\right)\times$$
$$\times\left(b_{12}p^2 + b_{11}\right)^{-1}\frac{\cos p\delta_n}{p\delta_n}]\sin p\delta_n\eta +$$
$$+\eta\cos p\delta_n\cos p\delta_n\eta + O(\varepsilon)\}\exp\left(\frac{\delta_n}{\varepsilon}\ln p\right)$$
$$u_\theta = \frac{p^2 - b_{11}}{\sqrt{\rho}}\varepsilon\sum_{n=1}^{\infty}E_n\{[\sin p\delta_n - 2b_{11}(b_{12}+1)\times$$
$$\times p\left(b_{11} - p^2\right)^{-1}\left(b_{12}p^2 + 1\right)^{-1}\frac{\cos p\delta_n}{p\delta_n}]\cos p\delta_n\eta -$$
$$-\eta\cos p\delta_n\sin p\delta_n\eta + O(\varepsilon)\}\exp\left(\frac{\delta_n}{\varepsilon}\ln p\right)$$
$$\sigma_r = \frac{G_1\left(b_{12}p^2 + b_{11}\right)}{\rho\sqrt{\rho}}\sum_{n=1}^{\infty}E_n[(p\delta_n\sin p\delta_n + \cos p\delta_n)\sin p\delta_n\eta +$$
$$+\eta p\delta_n\cos p\delta_n\times\cos p\delta_n\eta + O(\varepsilon)]\exp\left(\frac{\delta_n}{\varepsilon}\ln p\right)$$
$$\sigma_\varphi = \frac{G_1}{\rho\sqrt{\rho}}\sum_{n=1}^{\infty}E_n\langle\{\left(b_{23}p^2 + b_{12}^2 + b_{12} - b_{11}b_{23}\right)p\delta_n\sin p\delta_n -$$
$$-\left(b_{11}b_{23} + b_{12}^2 + b_{12} - b_{11} - b_{23}p^2\right) - 2b_{11}(b_{12}+1)\left(b_{23}p^2 + b_{12}\right)\left(b_{12}p^2 + b_{11}\right)]^{-1}\times$$
$$\times\cos p\delta_n\}\sin p\delta_n\eta + \eta p\delta_n\cos p\sigma_n\cos p\delta_n\eta + O(\varepsilon)\rangle\exp\left(\frac{\delta_n}{\varepsilon}\ln\rho\right)$$
$$\sigma_\theta = \frac{G_1\left(b_{11}b_{22} - b_{12}^2\right)}{\rho\sqrt{\rho}}\sum_{n=1}^{\infty}E_n[(p\delta_n\sin p\delta_n - \cos p\delta_n)\sin p\delta_n\eta +$$
$$+\eta p\delta_n\cos p\delta_n\cos p\delta_n\eta + O(\varepsilon)]\exp\left(\frac{\delta_n}{\varepsilon}\ln p\right)$$
$$\tau_{r\theta} = \frac{G_1\left(b_{12}\ p^2 + 1\right)}{\rho\sqrt{\rho}}\sum_{n=1}^{\infty}E_n\delta_n[(\sin p\delta_n\cos p\delta_n\eta -$$
$$-\eta\cos p\delta_n\sin p\delta_n\eta + O(\varepsilon))]\exp\left(\frac{\delta_n}{\varepsilon}\ln p\right)$$

where

$$\sin 2p\delta_n - 2p\delta_n = 0$$

C_k, B_n, D_n, E_n are arbitrary constants.

From the equation for solutions of the first and second groups one can conclude that the first group of solutions defines the fundamental stress state, the second group - the boundary effect analogous to the Saint-Venant boundary effect in theory of plates of constant thickness.

However for the large G_0 some boundary layer solutions damp very weakly and they have to be included in penetrating solutions.

3. Now let's turn to investigation of stress state situation described by homogeneous solutions (2.1)-(2.6).

Consider relation between the homogeneous solutions and the stress resultant P, acting in the section $\rho = const$

$$P = -2\pi r_1^2 \varepsilon \rho^2 \int_{-1}^{1} (\sigma_r \sin \varepsilon\eta - \tau_{r\theta} \cos \varepsilon\eta) \cos \varepsilon\eta d\eta \qquad (3.1)$$

Supposing $C_0 = 0$ we shall represent displacements and stresses in the form

$$u_r = u_1 + \sum_{k=2}^{\infty} C_k U_k(\eta) \rho^{z_k - 1/2}, \qquad u_\theta = w_1 + \sum_{k=2}^{\infty} C_k W_k(\eta) \rho^{z_k - 1/2}$$

$$\sigma_r = Q_{r1} + \sum_{k=2}^{\infty} C_k Q_{rk} \rho^{z_k - 3/2}, \qquad \sigma_\varphi = Q_{\varphi 1} + \sum_{k=2}^{\infty} C_k Q_{\varphi k}(\eta) \rho^{z_k - 3/2}$$

$$\sigma_\theta = Q_{\theta 1} + \sum_{k=2}^{\infty} C_k Q_{\theta k} \rho^{z_k - 3/2}, \qquad \tau_{r\theta} = T_1 + \sum_{k=2}^{\infty} C_k T_k(\eta) \rho^{z_k - 3/2} \qquad (3.2)$$

In formulas $u_1, ..., T_1$ correspond to the eigenvalues $z_1 = -1/2$. The rest of solutions are included into the second addend.

Substituting (3.2) into (3.1) we obtain

$$P = c_1 \gamma_1 + \rho^{1/2} \sum_{k=2}^{\infty} C_k \rho^{z_k} \gamma_k \qquad (3.3)$$

where

$$\gamma_1 = 16 G_1 \pi (v_1 - E_0) r_1^2 \varepsilon^2 + O(\varepsilon^3)$$

$$\gamma_k = -\pi G_1 \varepsilon \int_{-1}^{1} [Q_{rk}(\eta) \sin \varepsilon\eta + T_k(\eta) \cos \varepsilon\eta] \cos \varepsilon\eta d\eta$$

Let's prove that all $\gamma_k \quad (k = 2, 3, ...)$ are equal to zero. In order to do this let's consider the following boundary problem

$$\sigma_r = \rho_1^{z_k - 3/2} Q_{rs}, \qquad \tau_{r\theta} = \rho_1^{z_k - 3/2} T_s \qquad (\rho = \rho_1)$$

$$\sigma_r = \rho_2^{z_k - 3/2} Q_{rs}, \qquad \tau_{r\theta} = \rho_2^{z_k - 3/2} T_s \qquad (\rho = \rho_1) \qquad (3.4)$$

It's easy to see that solution of problem (3.4) exists and can be obtained from formulas (3.2) if put there $C_k = \delta_{kz}$ where δ_{kz} is the Kronecker symbol.

On the other hand it is well-known that the necessary condition for solvability of first boundary problem of elasticity theory is the vanishing of the resultant and principal moment of all external forces. In the considered case the resultant of external forces in projection onto axis of symmetry $\theta = 0$ has the form:

$$P_s = \left(\rho_2^{z_k - 3/2} - \rho_1^{z_k - 3/2}\right)\gamma_s = 0 \tag{3.5}$$

The last equality is possible only for $\gamma_s = 0$.

For the resultant finally we obtain

$$P = C_1\gamma_1 \tag{3.6}$$

Thus stress state (2.3)-(2.6) is self-balanced in every section $\rho = const$.

Let's make clear the stress state situation correspondent to zeros z_k $(k \geq 2)$. In order to do this let's find bending moment in section $\rho = const$

$$M = 2r_1^2\pi\rho^2\varepsilon\int_{-1}^{1}\left[\sigma_r \sin\varepsilon\eta - \tau_{r\theta}\left(1 - \cos\varepsilon\eta\right)\right]\cos\varepsilon\eta d\eta \approx$$

$$\approx 2\pi r_1^2\rho^2\varepsilon^2\int_{-1}^{1}\eta\sigma_r d\eta + O\left(\varepsilon^4\right) \tag{3.7}$$

Let's find bending moment for stresses (2.3). We have

$$M_1 = 2/3\pi p r_1^2 G\varepsilon^2\rho^{1/2}\sum_{k=2}^{3}C_k[4E_0 z_{k_0}^2 + 4\left(v_1 - 2E_0\right)z_{k_0} +$$

$$+3E_0 - 2v_1 + O\left(\varepsilon^2\right)]\exp\left(z_k \ln\rho\right) \tag{3.8}$$

Let's prove that principle part of bending moment for stresses correspondent to the second group of zeros is equal to zero. Consider the solution defined by formula (2.4). Other cases are considered analogously.

$$M_2 = 2\pi r_1^2\varepsilon^2\rho^2\int_{-1}^{1}\eta\sigma_r d\eta + O\left(\varepsilon^4\right) = 2\pi r_1^2\varepsilon^2 G_1\left(b_{11}b_{22} - b_{12}^2\right)\rho^2\rho^{-3/2}\times$$

$$\times\left(S_2\cos S_2\delta_n \sin S_1\delta_n - S_1\cos S_1\sigma_n \sin S_2\delta_n\right)\exp\left(\frac{z_n}{\varepsilon}\ln\rho\right) - O\left(\varepsilon^4\right) =$$

$$= 2\pi r_1^2\varepsilon^2\rho^2\sigma_\theta\left(\pm 1\right) + O\left(\varepsilon^4\right)$$

Since $\sigma_\theta\left(\pm 1\right) = 0$ finally we obtain

$$M_2 \approx O\left(\varepsilon^4\right)$$

Thus principle we parts of bending moment define solution of the first group.

Expanding the bending moments $M_k^s \quad (k=2,3)$ acting on the surface $\rho=\rho_s$ in series with respect to ε

$$M_k^s = M_{k_0}^z + M_{k_2}^z \varepsilon^2 + \cdots \tag{3.9}$$

and finding at C_k in the form $C_k = C_{k_0} + \varepsilon^2 C_{k_2} + ...$ for defining C_{k_0} we obtain the linear system

$$2/3\pi r_1^2 G_1 \rho_S^{1/2} \sum_{k=2}^{3} C_{k0} [4E_0 z_{k0}^2 + 4\left(v_1 - 2E_0\right) z_{k0} +$$

$$+3E_0 - 2v_1] \exp\left(z_{k_0} \ln_S\right) = M_{k_0}^S \qquad (s=1,2) \tag{3.10}$$

Thus constants C_k are defined by the principle parts of bending moments on the lateral surfacee of the plate.

The first term of expansion (2.2) combined with the first terms of expansions (2.3) can be considered as a solution in applied theory.

It follows from (3.6), (3.9) that first term of asymptotics (2.4), (2.5), (2.6) corresponds the stress state self-balanced in the section $\rho = const$ and the solution itself has a character of boundary effect which is equivalent to the Saint-Venant boundary effect in the theory of plates of constant thickness.

4. Let's examine question on unloading the lateral surface of the plate by means of a class of homogeneous solutions. Let conditions (1.2) be given on the lateral surface. As it was shown above the principle parts of resultant and bending moment are defined by the solution of the first group.

Therefore below we shall suppose that $C_k = 0 \quad (k=1,2,3)$ and consider case (2.4). Other cases are considered analogously.

We shall seek the solution in the form (2.4). In order to define arbitrary constants B_n as in paper [1] we shall use Lagrange variational principle.

Since homogeneous solutions satisfy equilibrium equations and boundary conditions on conic surface the variational principle accepts the form:

$$r_1 \varepsilon \sum_{s=1}^{2} \rho_s^2 \int_{-1}^{1} \left[\left(\sigma_r - f_{1s}\right) \delta u_r + \left(\tau_{r\theta} - f_{2s}\right) u_\theta\right]_{\rho=\rho_s} \cos \varepsilon\eta d\eta = 0 \tag{4.1}$$

Assuming σB_n independent variations from (4.1) we shall obtain the infinite system of linear algebraic equations:

$$\sum_{n=1}^{\infty} M_{kn} B_n = N_k \qquad (k=1,2,...) \tag{4.2}$$

$$M_{kn} = \sum_{s=1}^{2} \exp\left(z_k + z_n\right) \ln \rho_z \int_{-1}^{1} \left(Q_{rn} u_k + T_n w_k\right) \cos \varepsilon\eta d\eta \ ;$$

$$N_k = \sum_{s=1}^{2} \exp\left[\left(z_k + 3/2\right) \ln \rho_z\right] \int_{-1}^{1} \left(f_{1s} u_k + f_{2s} w_k\right) \cos \varepsilon\eta d\eta \ ;$$

The solvability and convergence of the reduction method of system (4.2) follow from work [4].

We shall seek the unknown constant B_n in the from

$$B_n = B_{n0} + \varepsilon B_{n1} + \cdots \tag{4.3}$$

Substituting (4.3) into (4.2) we shall obtain the following system of infinite linear algebraic equations with respect to B_{n_0}

$$\sum_{n=1}^{\infty} m_{kn} B_{n_0} = H_k \tag{4.4}$$

$$m_{kn} = +G_1 \left(b_{11}b_{22} - b_{12}^2\right) S_1 S_2 \sum_{s=1}^{2} \exp\left[\varepsilon^{-1}\left(\delta_n + \delta_k\right)\ln\rho_z\right] \times$$

$$\times \int_{-1}^{1} \{\delta_n \left(S_1 \cos S_2\delta_n \sin S_1\delta_n\eta - S_2 \cos S_1\delta_n \sin S_2\delta_n\eta\right) \times$$

$$\times [S_2 \left(b_{22}S_2^2 + b_{12}^2 + b_{12} - b_{11}b_{22}\right) \cos S_2\delta_k \sin S_1\delta_k\eta -$$

$$- S_1 \left(b_{22}S_1^2 + b_{12}^2 + b_{12} - b_{11}b_{22}\right) \cos S_1\delta_k \sin S_2\delta_k\eta] -$$

$$- \left(\cos S_2\delta_n \cos S_1\delta_n\eta - \cos S_1\delta_n \cos S_2\delta_n\eta\right) \times S_1S_2 \times$$

$$\times \left[\left(b_{22}S_2^2 + b_{12}\right) \cos S_2\delta_k \cos S_1\delta_k\eta - \left(b_{22}S_1^2 + b_{12}\right) \cos S_1\delta_k \cos S_2\delta_k\eta\right]\} d\eta;$$

$$H_k = \sum_{s=1}^{2} \rho_s^{3/2} \exp\left(\frac{\delta_k}{\varepsilon}\ln\rho_s\right) \int_{-1}^{1} \{f_{1s}[S_2 \left(b_{22}S_2^2 + b_{12}^2 + b_{12} - b_{11}b_{22}\right) \times$$

$$\times \cos S_2\delta_k \sin S_1\delta_k\eta - S_1 \left(b_{22}S_1^2 + b_{12}^2 + b_{12} - b_{11}b_{22}\right) \cos S_1\delta_k \sin S_2\delta_k\eta] -$$

$$- f_{2s}S_1S_2[\left(b_{22}S_2^2 + b_{12}\right) \cos S_2\delta_k \cos S_1\delta_k\eta - \left(b_{22}S_2^2 + b_{12}\right) \times$$

$$\times \cos S_2\delta_k \cos S_1\delta_k\eta]\} d\eta;$$

The matrix of system (4.4) has been already met in the theory of transversal-isotropic plate of constant thickness. Defining of B_{ni} $(i = 1, 2, ...)$ is permanently reduced to inversion of t he same matrices which coincide with matrix (4.4).

In conclusion we note that for $G_0 = 1$ we obtain results for the case of bending of isotropic plate of variable thickness.

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