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LIMIT-EQUILIBRIUM STATE OF BUSH OF CONTACT PAIR IN THE PRESENCE OF CRACKS WITH YIELDED END AREAS

Abstract

Limit-equilibrium state of bush of contact pair in the presence of rectilinear cracks with yielded end areas was investigated.

I. The case of one crack with yielded end areas. The higher stress concentration near the apex of a crack as experience shows, leads in some cases to loss of strength of material surrounding the crack, it may appear in zone formation of plastic flow. The analysis of experimental data and also conditions of equilibrium and development of crack subject to interaction of its faces and zone formation of loss of strength leads to the consideration of the model of crack with end area in which the plastic flow holds at constant stress.

First such models for cracks in which it's accepted that in end areas the plastic flow holds at constant stress, are considered in [1,2].

Let the rectilinear crack with the length $2l_1$ be in bush of contact pair near the friction surface. We allocate the origin of local coordinate system $x_1O_1y_1$ in the centre of crack the axis x_1 which coincides with crack line and forms the angle α with the axis x $(\theta = 0)$.

We choose the parts of the crack d_1 and d_2 (end areas) adjoining to its apexes in which the plastic flow holds at constant stress for the given material. The interaction of faces of cracks in end areas is modeled by insertion lines of plastic slip (degenerated placticity strip) between surface of cracks. The physical nature of such interaction and dimensions of end areas depend on the form of material. Since the end areas of plastical flow are small in comparison with the other elastic part of bush, they can be mentally deleted substituting by slits the surfaces of which interact among themselves by some low corresponding to the action of deleted material.

In process of working of contact pair at action of contact pressure (environmental stress) on bush, in end areas connecting the faces of cracks, the normal and tangential forces $q_{y_1}(x_1) = \sigma_T$ and $q_{x_1y_1}(x_1) = \tau_T$ will arise.

Thus the normal and tangential stresses numerically equal to σ_T and τ_T respectively, will be applied to the faces of apex in end areas. The dimensions of end areas are beforehand unknown and subject to definition in process of solving the considered boundary value problem of fracture mechanics.

The boundary conditions of the considered problem have the following form:

$$\sigma_n = -p(\theta); \quad \tau_{nt} = -fp(\theta) \quad \text{when } r = p \text{ on contact area } \left(\rho = R + \varepsilon H/(\theta)\right)$$

$$\sigma_n = 0; \quad \tau_{nt} = 0 \text{ when } r = p \text{ outside of contact area } \left(\rho = R + \varepsilon H/\theta\right) \qquad (1)$$

$v_r = 0;\quad v_\theta = 0$ when $r = R_0$ on surfaces of cracks

$$\sigma_{y_1} = 0;\quad \tau_{x_1y_1} = 0 \text{ when } -l_1 + d_1 < x_1 < l_1 - d_2 \tag{2}$$

$$\sigma_{y_1} = \sigma_T;\quad \tau_{x_1y_1} = \tau_T \text{ when } -l_1 + d_1 < x_1 < -l_1 \text{ and } l_1 - d_2 < x_1 < l_1$$

Using the stated calculation method in [3] we find the boundary conditions at each approximation: for zero approximation of the problem

$$\sigma_r^{(0)} = -p^{(0)}(\theta);\ \tau_{r\theta}^{(0)} = -fp^{(0)}(\theta) \quad \text{when } r = R \text{ on contact area} \tag{3}$$

$\sigma_r^{(0)} = 0;\ \tau_{r\theta}^{(0)} = 0$ when $r = R$ outside of contact area $v_r^{(0)} = 0;\ v_\theta^{(0)} = 0$ when $r = R_0$ on curfaces of crack

$$\sigma_{y_1}^{(0)} = 0;\quad \tau_{x_1y_1}^{(0)} = 0 \text{ when } -l_1 + d_1^0 < x_1 < l_1 - d_2^0 \tag{4}$$

$\sigma_{y_1}^{(0)} = \sigma_T;\quad \tau_{x_1y_1}^{(0)} = \tau_T$ when $-l_1 + d_1^0 > x_1 > -l_1$ and $l_1 - d_2^0 < x_1 < l_1$

For the first approximation of the problem

$$\sigma_r^{(1)} = N - p^{(1)}(\theta);\ \tau_{r\theta}^{(1)} = T - fp^{(1)}(\theta) \quad \text{when } r = R \text{ on contact area} \tag{5}$$

$\sigma_r^{(1)} = N;\quad \tau_{r\theta}^{(1)} = T$ when $r = R$ outside of contact area $v_r^{(1)} = 0;\ v_\theta^{(1)} = 0$ when $r = R_0$ on surfaces of the crack

$$\sigma_{y_1}^{(1)} = 0;\quad \tau_{x_1y_1=0}^{(1)} \text{ when } |x_1| \le l_1 \tag{6}$$

Here the quantities N and T are determined by the relations (12) of [4].

We write the boundary conditions of the considered problem at zero approximation (3)-(4) with help of Kolosov-Muskhelishvili's formulas in the form of boundary problem for finding complex potentials $\Phi(z)$ and $\psi_0(z)$. On the boundary of bush the boundary conditions for complex potentials will have the form (13), (14) in [4] and the boundary conditions on surfaces of crack will be written in the following form:

$$\Phi^{(0)}(t) + \overline{\Phi^{(0)}(t)} + \bar{t}\Phi^{(0)'}(t) + \Psi^{(0)}(t) =$$

$$= \begin{cases} 0 \quad \text{when} \quad -l_1 + d_1^0 < x_1 < l_1 - d_2^0 \\ \sigma_T + i\tau_T \quad \text{when} \quad -l_1 + d_1^0 > x_1 > -l_1 \end{cases} \quad \text{and} \quad l_1 - d_2^0 < x_1 < l_1 \tag{7}$$

where t is affix of points of surfaces of crack.

We choose the complex potentials $\Phi^{(0)}(z)$ and $\Psi^{(0)}(z)$ giving the solution of the boundary value problem (13), (14) in [4], (7) at zero approximation in the form of (16) in [4]. We'll search the functions $\Phi_1^{(0)}(z)$ and $\Psi_1^{(0)}(z)$ in the form of [18] in [4]. Here $g^0(t)$ is a desired function characterizing the displacement discontinuity at passage via crack line. We find the complex potentials $\Phi_2^{(0)}(z)$ and $\Psi_2^{(0)}(z)$ satisfying the condition when $r = R$, in the form (19) of [4].

Satisfying by these functions the boundary condition on surfaces of crack we find the complex singular integral equation with respect to unknown function $g^0(x_l)$

$$\int_{l_1}^{l_1}\left[R(t,x_1)\,g^0(t)+S(t,x_1)\,\overline{g^0(t)}\right]dt=\pi\left[f^0(x_1)+f\right] \tag{8}$$

$|x_1|\leq l_1$.

There

$$f^0(x_1)=-\left[\Phi_0^{(0)}(x_1)+\overline{\Phi_0^{(0)}(x_1)}+x_1\overline{\Phi_0^{(0)'}(x_1)}+\overline{\Psi_0^{(0)}(x_1)}\right]$$

$$f=\begin{cases}0 \quad \textit{when} \quad -l_1+d_1^0<x_1<l_1-d_2^0\\ \sigma_T-i\tau_T \quad \textit{when} \quad -l_1+d_1^0>x_1>-l_1 \quad \textit{and} \quad l_1-d_2^0<x<l_1\end{cases}$$

We write , as usual, an unknown $g^{(0)}(x_1)$, and also loading $f^{(0)}(x_1)$ in the following form: $g^{(0)}(x_1)=v^0(x_1)-iu^0(x_1)$; $f^{(0)}(x_1)=\sigma^0(x_1)-i\tau^0(x_1)$.

Then after separation of real and imaginary parts we obtain two real singular integral equations to dedermine $v^0(x_1)$ and $u^0(x_1)$ respectively.

After algebraization of integral equations, instead of every real integral equation we obtain finite algebraic system consisting of M equations with respect to approximated values of the desired functions $v^0(t_m)$ and $u^0(t_m)$ $(m=1,2,...,M)$ in node points, respectively.

Unknown dimensions of end areas d_1^0 and d_2^0 enter to these algebraic systems. For this relation the algebraic system turned out to be nonlinear.

For construction of missing two equations determining the dimensions of end areas we use the stress finiteness at tip of crack. The stresses finiteness at tip of crack will be proved by mutual action of external load and stress on surface of crack in end areas (postulate on cancellation singularities). The postulate on cancellation the singularities is equivalent to the condition of equality to zero of resulting stress intensity factor defined as difference between stress intensity factor on action external force and stress intensity factor on compressive forces used in end areas of cracks. Thus, the equations used for determination of dimensions of the end areas d_1^0 and d_2^0 in zero approximation on the basis of the said will

$$\sum_{m=1}^{M}(-1)^m p^{(0)}(t_m)\ ctg\frac{2m-1}{4M}\pi=0;$$

$$\sum_{m=1}^{M}(-1)^{M+m} p^{(0)}(t_m)\ tg\frac{2m-1}{4M}\pi=0; \tag{9}$$

$$p^0(t_m)=v^0(t_m)-tu^0(t_m).$$

After solving the algebraic system by the successive approximations method we can pass to the construction of the solution of the problem at the first approximation.

The sequence of constructions of basic solving equations at first approximation is analogous to zero approximation. At first by the found solution we find the functions N and T. We search the complex potentials $\Phi^{(1)}(z)$ and $\Psi^{(1)}(z)$ in the form (16) and choose the potentials $\Phi^{(1)}(z)$, $\Psi^{(1)}(z)$ and $\Phi_2^{(1)}(z)$, $\Psi_2^{(1)}(z)$in the form analogously to (18), (19) with evident variation, respectively.

Then we act as at determination of zero approximation. As a result we find the complex singular integral equation for the desired function $g^1(x_1)$ in the following form:

$$\int_{-l_1}^{l_1}\left[R(t,x_1)\,g^1(t)+S(t,x_1)\,\overline{g^1(t)}\right]dt=\pi f^1(x_1)\,,\ \ |x_1|\le l_1 \tag{10}$$

$$f^1(x_1)=-\left[\Phi_0^{(1)}(x_1)+\overline{\Phi_0^{(1)}(x_1)}+x_1\overline{\Phi_0^{(1)'}(x_1)}+\overline{\Psi_0^{(1)}(x_1)}\right]$$

Representing the unknown function $g^1(x_1)$ and loading $f^1(x_1)$ in the form of

$$g^1(x_1)=v^1(x_1)-iu^1(x_1)\,;\ \ f^1(x_1)=\sigma^1(x_1)-i\tau^1(x_1)\,,$$

and also separating the real and imaginary parts in (10) we obtain two real integral equations to determine $v^1(x_1)$ and $u^1(x_1)$.

Applying the procedure of algebraization to the singular integral equations we reduce every integral equation to the finite algebraic system of M (numbers of Chebushev's nodes of partitions of integration interval) equations relative to approximated values of the desired functions $v^1(t_m)$ and $u^1(t_m)$ $(m=1,2,...,M)$ in node points respectively.

If we require the fulfilment of finiteness conditions of stresses in tip of cracks, we obtain two equations for determination of d_1^1 and d_1^1 dimensions of end areas of plastic flow of crack at the first approximation

$$\sum_{m=1}^{M}(-1)^m p^1(t_m)\ \ ctg\frac{2m-1}{4M}\pi=0;$$

$$\sum_{m=1}^{M}(-1)^{M+m} p^1(t_m)\ \ tg\frac{2m-1}{4M}\pi=0; \tag{11}$$

$$p^1(t_m)=v^1(t_m)-tu^1(t_m)$$

The obtained system of equations substituting every real integral equation jointly with (11) allows to find the values of desired functions $v^1(t_m)$ and $u^1(t_m)$ $(m=1,2,...,M)$ at the node points and the dimensions of plastic areas at first approximation d_1^1 and d_1^1 from numerical solving by the successive approximations method.

For the determination of limiting equilibrium of affix of crack with end domains of plastic flow it is necessary the consideration of critical condition. As such condition we take the condition of limiting (critical) opening displacement on the basis of yielded area. It's assumed that limiting state occur as soon as on the border of end area of plastic flow the following condition we'll be satisfied

$$V(x_0) = \sqrt{u^2(x_0) + v^2(x_0)} = \delta_k,$$

where δ_k is a constant of material characterizing he limiting crack opening displacement under the given conditions.

On the basis of the obtained solution we have

$$-\frac{1+æ}{2G}\int_{-t_1}^{x_0} g(x)\,dx = v(x_0, 0) - iu(x_0, 0),$$

where $g(x) = g^0(x) + \varepsilon g^1(x)$.

Using the substitution of variable and substituting the integral by sum and also separating the real and imaginary parts we find

$$v(x_0, 0) = \frac{1+æ}{2G}\frac{\pi l_1}{M}\sum_{m=1}^{M_1}\left[v^0(t_m) + \varepsilon v^1(t_m)\right],$$

$$u(x_0, 0) = \frac{1+æ}{2G}\frac{\pi l_1}{M}\sum_{m=1}^{M_1}\left[u^0(t_m) + \varepsilon u^1(t_m)\right]. \tag{12}$$

Here M_l is the number of end points continuity in the interval $(-l_1, x_0)$.

Consequently, the advance of appex of crack (break of connection in plasticity strip) occurs, will be the following one

$$\frac{1+æ}{2G}\frac{\pi l_1}{M}\sqrt{A^2 + B^2} = \delta_k; \tag{13}$$

where $A = \sum_{m=1}^{M_1}\left[v^0(t_m) + \varepsilon v^1(t_m)\right]$; $B = \sum_{m=1}^{M_1}\left[u^0(t_m) + \varepsilon u^1(t_m)\right]$

For the dimensions of end areas of plastic flow on appex of crack we have

$$d_1 = d_1^0 + \varepsilon d_1^1; \quad d_2 = d_2^0 + \varepsilon d_2^1.$$

In fig.1 the graphs of dependence of dimensionless length of end area of the plasticity d_1/l_1 for bush of slit borehole-rod pump on pure value of the lengths of cracks for bush of slit sucker-rod pump on pure value of the contact pressure p_0/σ_T for $V = 0,4\ m/sek$ and various values of lengths of cracks for rough interior contour by law v) of roughness distribution [3] are reduced.

The joint solution of the joined nonlinear system of the equations consisting of the equations substituting every singular integral equation, (11) and (13) allows to define the critical dependence of contact pressure (exterior loading) on length of crack, the dimensions of end yielded areas and values of the desired functions $u(t_m) = u^0(t_m) + \varepsilon u^1(t_m)$; $v(t_m) = v^0(t_m) + \varepsilon v^1(t_m)$ $(m = 1, 2, ..., M)$.

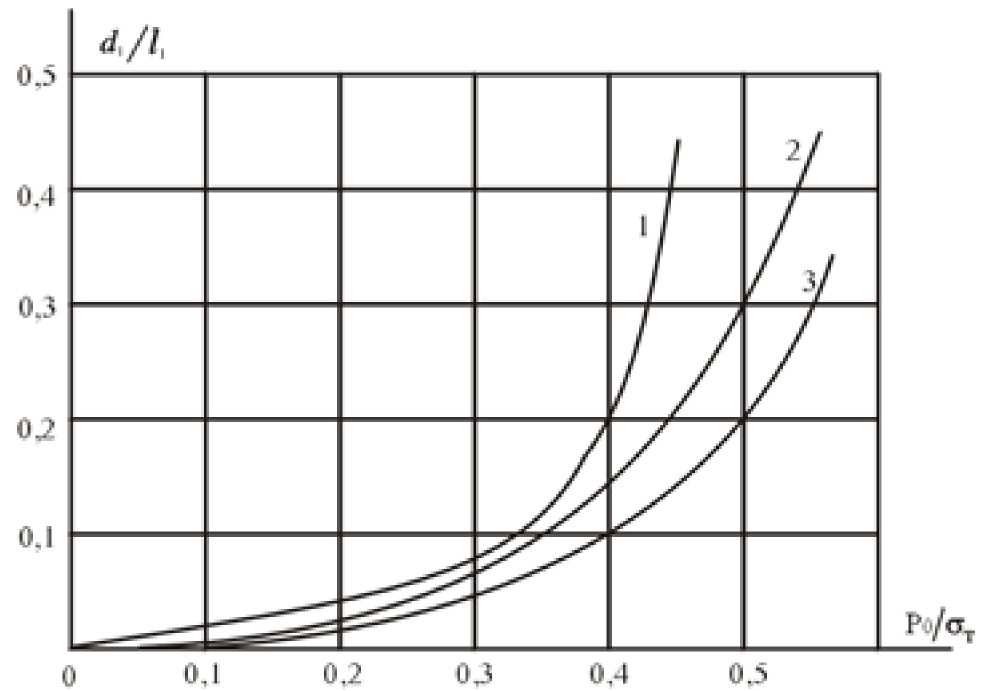
In fig.2 graphs of dependence of dimensionless limit loading $p_* = p/\sigma_1$ on dimensionless length of crack $\lambda = l/(R_0 - R)$ of bush of slit sucker-rod pump on the velocity of movement of plunger $V = 0,4\ m/sek$ are given. In this figure curve 1 belongs to the smooth interior contour, and curve 2 to rough contour by law v) of roughness distribution [3].

The experience shows that before the onset of limiting equilibrium state the stage of stable subcritical development of crack is almost everywhere observed. The stage of subcritical growth of crack is of great importance. We describe subcritical development of crack in brush with the help of generalized δ_c-theory [6].

According to the generalized δ_c-theory the equation of subcritical diagram of failure in the considered case is found from the equation

$$\frac{1+æ}{2G}\,\frac{\pi l_1}{M}\sqrt{A^2+B^2}=\delta_c\left[1+\left(\frac{\xi_k-\xi}{\xi_k-\xi_0}\right)^2\right] \tag{14}$$

Here $\xi_k = \xi_0\left(2-\varepsilon^{-\xi_0}\right)$; $\xi_0 = C$; $\xi = l/C$; $C = \dfrac{E\delta_c}{4\sigma_T}$.



$1 - \lambda = 0.2$
$2 - \lambda = 0.3$
$3 - \lambda = 0.4$
Fig.1

Solving the obtained condition (14) jointly with the mentioned above joined non-linear system we find the dependence of contact pressure (exterior loading) on the length of crack l (diagram of failure).

The obtained condition (14) of subcritical growth of crack may be used also at solving the problem on growth of cracks up to failure at any preassigned way of loading including repeated loading. Crack development here occurs at every loading

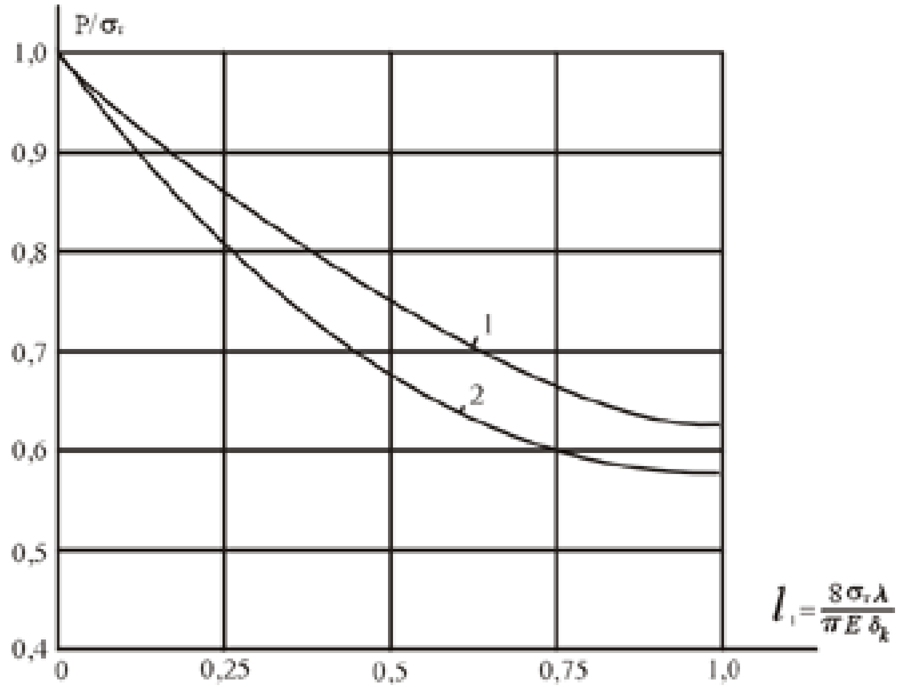


Fig.2

cycle and at unloading it's considered that the crack length remains constant.

2. The case of arbitrary number of cracks with yielded end areas. We generalize the previous solution of the problem to the case of arbitrary number of cracks. Let in bush near the friction surface, there be N_0 rectilinear arbitrarily disposed cracks of length $2l_k$ $(k = 1, 2, .., N_0)$. On centers of cracks as earlier we locate origin of local coordinate system $x_k O_k y_k$ of the axes x_k which coincide with lines of cracks and from the angle α_k with the axis x $(\theta = 0)$. We'll assume the availability of end areas in which the plastical flow holds at constant stress for the given material.

Interaction of faces of crack in end domains is modeled by introducing lines of plastical slip (degenerate strip of plasticity) between surfaces of cracks. Subject to the fact that the end domains of plastical flow are small in comparison with the other elastic parts of bush we can conceptually delete then by replacing sections the surface of which interact among themselves by certain law corresponding to the action of deleted material.

In process of working of contacting pair under the action of contact pressure, the friction and temperature forces on bush, in end areas, joining the surfaces of cracks, will arise the normal and tangential forces $q_{y_1}(x_k) = \sigma_T$ and $q_{x_1 y_1}(x_k) = \tau_T$ respectively. Consequently, the contracting normal and tangential stresses numerically, equaled to σ_T and τ_T, respectively, will be applied to the surfaces of cracks in end domains. The dimensions of end domains are unknown beforehand and are subject to determination in process of solving boundary value problem of fracture mechanics.

Denote by $L=\sum\limits_{k=1}^{N_0}L_k$ a set of free banks of cracks and by $L"=\sum\limits_{k=1}^{N_0}L"_k$ a set of end areas of cracks in which the plastical flow holds.

The boundary conditions of the considered problem on the contours $r=\rho$ and $r=R_0$ have the form (1) and on surfaces of cracks will have form

$$\sigma_{y_k}-i\tau_{x_ky_k}=0 \qquad on \quad L \tag{15}$$

$$\sigma_{y_k}-i\tau_{x_ky_k}=\sigma_T-i\tau_T \qquad on \quad L"$$

Not stopping on details not losing the clearness of statement we lead the basic resolving equations of problem at zero and first approximations

$$\frac{1}{M}\sum_{m=1}^{M}\sum_{k=1}^{N_0}l_k\left[p_k^{(0)}(t_m)R_{nk}(l_kt_m,l_nx_r)+\right.$$

$$\left.+\overline{p_k^{(0)}(t_m)}S_{nk}(l_kt_m,l_nx_r)\right]=f_n^{(0)}(x_r) \tag{16}$$

$$\sum_{m=1}^{M}p_n^{(0)}(t_m)=0 \qquad (n=1,2,...,N_0;\quad r=1,2,...,M-1))$$

$$\sum_{m=1}^{M}(-1)^m p_n^{(0)}(t_m)\ \ ctg\frac{2m-1}{4M}\pi=0$$

$$\sum_{m=1}^{M}(-1)^{M+m}p_n^{(0)}(t_m)\ \ tg\frac{2m-1}{4M}\pi=0 \qquad (n=1,2,...,N_0) \tag{17}$$

where $t_m=\cos\dfrac{2m-1}{4M}\pi \quad (m=1,2,...,M)$

$$x_r=\cos\frac{\pi r}{M} \qquad (r=1,2,...,M-1)$$

$$f_n(x_n)=-\left[\Phi_0^{(0)}(x_n)+\overline{\Phi_0^{(0)}(x_n)}+x_n\overline{\Phi_0^{(0)'}(x_n)}+\overline{\Psi_0^{(0)}(x_n)}\right] \tag{18}$$

$$f_n^0(x_n)=\begin{cases} f_n(x_n) & on \quad L' \\ f_n(x_n)+\sigma_T-i\tau_T & on \quad L" \end{cases}$$

$$p_n^0(t_m)=v_n^0(t_m)-iu_n^0(t_m)$$

$$\frac{1}{M}\sum_{m=1}^{M}\sum_{k=1}^{N_0}l_k\left[p_k^{(1)}(t_m)R_{nk}(l_kt_m,l_nx_r)+\right.$$

$$\left.+\overline{p_k^{(1)}(t_m)}S_{nk}(l_kt_m,l_nx_r)\right]=f_n^{(1)}(x_r) \tag{19}$$

$$\sum_{m=1}^{M} p_n^{(1)}(t_m) = 0 \qquad (n = 1, 2, ..., N_0; \ \ r = 1, 2, ..., M-1)$$

$$\sum_{m=1}^{M} (-1)^m \, p_n^{(1)}(t_m) \ \ ctg \frac{2m-1}{4M}\pi = 0 \tag{20}$$

$$\sum_{m=1}^{M} (-1)^{M+m} \, p_n^{(1)}(t_m) \ \ tg \frac{2m-1}{4M}\pi = 0 \qquad (n = 1, 2, ..., N_0)$$

$$f_n^1(x_n) = -\left[\Phi_0^{(1)}(x_n) + \overline{\Phi_0^{(1)}(x_n)} + x_n \overline{\Phi_0^{(1)'}(x_n)} + \overline{\Psi_0^{(1)}(x_n)}\right] \tag{21}$$

$$p_n^1(t_m) = v_n^1(t_m) - i u_n^1(t_m) \, .$$

Separating the real and imaginary parts in (16), (17), (18) and also in (19), (20), (21) we double the number of equations at zero and first approximations respectively. From unknown dimensions of yielded areas, the systems of equations are non-linear. For solving the nonlinear systems we can use the method of successive approximations. For the considered class of problems it's appropriate to use the inverse approach. We give beforehand the dimensions of yielded areas and in process of solving we find loading corresponding to such end areas of plastical flow adjoining the apexes of cracks.

For investigation of limiting equilibrium of each apex of crack we use the analogue of the relation (13) for the corresponding apex of crack.

The joint solution of non-linear systems (16), (18), (19), (21) for each apex of crack allows to determine the critical dependence of loading (contact pressure) of dimensions of cracks, the dimensions of end yielded areas and the values of desired functions

$$v_n(t_m) = v_n^0(t_m) + \varepsilon v_n^1(t_m) \, ;$$

$$u_n(t_m) = u_n^0(t_m) + \varepsilon u_n^1(t_m) \quad (m = 1, 2, ..., M; \ n = 1, 2, ..., N_0) \, .$$

References

[1]. Leonov M.Ya., Panasyuk V.V. *Development of the smallest cracks in solid.* // Prik. mechanika, 1959, v.5, No4, p.391-401 (Russian)

[2]. Dugdale D.S. *Yielding of steel sheets contacting slits* // J. Mech. and Phys. Solids, 1960, v.8, No2, p.100-108.

[3]. Hajiyev V.J. *Modelling of abrasive damage of details of contact pair "bush-shaft"* // Dynamics and durability of mechanical systems. Baku, elm, 1999, p.3-10 (Russian)

[4]. Hajiyev V.J., Mirsalimov V.M. *Minimization of failure parameters of details of type bush of contact pair* // Mechanics-Machine building, Baku, 2001, No1, p.8-12 (Russian)

[5]. Muskhelishvili N.I. *Some basic problems of mathematical theory of elasticity.* M.: Nauka, 1966, 708 p. (Russian)

[6]. Morozov E.M. *On one generalization of σ_c-theory of cracks.* // Prikl. Mekhanika, 1970, v.6, iss.4, p.128-131 (Russian)

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