MECHANICS

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ASYMPTOTIC ANALYSIS OF A SPACE PROBLEM OF ELASTICITY THEORY FOR NONHOMOGENEOUS HOLLOW CONE OF SMALL THICKNESS

Abstract

By the method of direct asymptotic integration of equations of elasticity theory the space deflected mode of nonhomogeneous truncated hollow cone of small thickness is investigated. The nonhomogeneous and homogeneous solutions are constructed. On the base of qualitative analysis the nature of deflected mode is clarified.

At the papers [1, 2] the deflected mode of truncated hollow cone of small thickness was investigated. In [3] by the method of asymptotic integratian of equations of elasticity theory the first boundary value problem of elasticity theory for nonhomogeneous truncated hollow cone was investigated. At the given paper by the asymptotic integratian method the second boundary value problem of elasticity theory for nonhomogeneous truncated hollow cone is studied.

1. Consider the axially symmetric problem of elasticity theory for nonhomogeneous truncated hollow cane of small thickness, which represents a body with two conic and two spherical boundaries. Relate the cone to a spherical system of coordinates $r,\ \theta,\ \varphi$:

$$r_1 \le r \le r_2\ ;\quad \theta_1 \le \theta \le \theta_2;\quad 0 \le \varphi \le 2\pi$$

The equilibrium equations in permutation have the form:

$$L\overline{u} \equiv \left(L_0 + \varepsilon \partial_1 L_1 + \varepsilon^2 \partial_1^2 L_2\right) \overline{u} = 0 \tag{1.1}$$

Here $\overline{u} = (u_r, u_\theta)^T\ ;\ u_r = u_r(\rho,\eta)\ ,\ u_\theta = u_\theta(\rho,\eta)$ are the vectors permutation components: L_k are matrix differential operators of the form

$$L_0 = \left\| \begin{matrix} \partial G\partial + \varepsilon \cot(\theta_0+\varepsilon\eta)\,\partial - 2H\varepsilon^2 & -\varepsilon\partial G - \varepsilon H\partial - (G+H)\,\varepsilon^2\cot(\theta_0+\varepsilon\eta) \\ \varepsilon\partial(H+\lambda) + 2\varepsilon G\partial & \partial H\partial + (2\varepsilon G + \varepsilon\partial\lambda)\cot(\theta_0+\varepsilon\eta) - H\varepsilon^2\csc^2(\theta_0+\varepsilon\eta) \end{matrix} \right\|$$

$$L_1 = \left\| \begin{matrix} 2\varepsilon H & \lambda\partial + \partial G + \varepsilon(G+\lambda)\cot(\theta_0+\varepsilon\eta) \\ G\partial + \partial\lambda & 2\varepsilon G \end{matrix} \right\|;\quad L_2 = \left\| \begin{matrix} H & O \\ O & G \end{matrix} \right\|;$$

$$\partial \equiv \frac{\partial}{\partial\eta};\quad \partial_1 \equiv \rho\frac{\partial}{\partial\rho};\quad \partial_1^2 \equiv \rho^2\frac{\partial^2}{\partial\rho^2},$$

$H=2G+\lambda$; $\eta=\frac{\theta-\theta_0}{\varepsilon}$; $\rho=\frac{r}{r_0}$ are new dimensionless variabless; $\theta_0=\frac{\theta_1+\theta_2}{2}$ is an opening angle of mean surface of cone; $\varepsilon=\frac{\theta_2-\theta_1}{2}$ is a small parameter, which characterizes the thickness of cone; $r_0=\sqrt{r_1 r_2}$; $\eta\in[-1;1]$; $\theta_0\in\left(0;\frac{\pi}{2}\right)$.

We'll suppose, that Lame parameter $G=G(\eta)$, $\lambda=\lambda(\eta)$ are arbitrary positive piecewise functions of the variable η.

Suppose, that on lateral surface the following boundary conditions

$$\overline{u}=\overline{q}^{\pm}(\rho) \quad \text{when} \quad \eta=\pm 1 \tag{1.2}$$

are given where $\overline{q}^{\pm}(\rho)=\left(f^{\pm}(\rho);h^{\pm}(\rho)\right)^T$.

2. Consider the construction of partial solution of the equations (1.1), satisfying the boundary conditions (1.2), i.e. nonhomogeneous solutions.

We suppose that the functions given on conic bounds are sufficiently smooth functions and relative to ε have the order unit.

The solution (1.1)-(1.2) we'll look for in the form

$$\overline{u}=\overline{u}_0+\varepsilon\overline{u}_1+\varepsilon^2\overline{u}_2+\cdots$$

$$\overline{u}_i=\left(u_{ri},u_{\theta i}\right); \qquad i=0;1;2;\cdots. \tag{2.1}$$

The construction (2.1) in (1.1)-(1.2) leads to the system, successive integration by η,gives the relation for the expansion coefficients (2.1).

For $\overline{u}_0$ we get

$$u_{r0}=f(\rho)\cdot\left(\int_{-1}^{1}G^{-1}(\eta)\,d\eta\right)^{-1}\cdot\int_{-1}^{\eta}G^{-1}(x)\,dx+f^{-}(\rho);$$

$$u_{\theta 0}=h(\rho)\cdot\left(\int_{-1}^{1}H^{-1}(\eta)\,d\eta\right)^{-1}\cdot\int_{-1}^{\eta}H^{-1}(x)\,dx+h^{-}(\rho), \tag{2.2}$$

where

$$f(\rho)=f^{+}(\rho)-f^{-}(\rho)\ ;\quad h(\rho)=h^{+}(\rho)-h^{-}(\rho).$$

The analysis of stress strate shows, that the stresses relative to ε have the order $O\left(\varepsilon^{-1}\right)$.

3. Consider the question on construction of homogeneous solutions. Suppose in (1.2) $\overline{q}^{\pm}(\rho)=\overline{0}$. Looking for the solutions of homogeneous systems in the form

$$\overline{u}(\rho,\eta)=\rho^{z-\frac{1}{2}}\cdot\overline{\nu}(\eta)\ ; \qquad \overline{\nu}(\eta)=(a;b)^T$$

after the separation of variables we get the following problem:

$$\begin{cases}\left(L_0+\varepsilon\left(z-\frac{1}{2}\right)(L_1-\varepsilon L_2)+\varepsilon^2\left(z-\frac{1}{2}\right)^2L_2\right)\overline{\nu}=\overline{0},\\ \overline{\nu}=\overline{0} \quad \text{when} \quad \eta=\pm 1\ .\end{cases} \tag{3.1}$$

Let's use the asymptotic method [4] for the solution (3.1), founded on three iterative processes.

If in (2.1), (2.2) put the $\overline{q}^{\pm}(\rho) = \overline{0}$ we get that to the first iteration process corresponds the trivial homogeneous solution.

The solution having the character of an edge effect corresponding to the second asymptotic process for nonhomogeneous truncated hollow cone with a fixed lateral surface doesn't exist.

Let's turn into construction of the third iteration process. The solution (3.1) we look in the form

$$\overline{\nu}^{(3)}(\eta) = \varepsilon\left(\overline{W}_0 + \varepsilon\overline{W}_1 + ...\right),$$

$$z = \varepsilon^{-1}(\alpha_0 + \varepsilon\alpha_1 + ...), \tag{3.2}$$

where

$$\overline{W}_i = (a_i, b_i)^T, \quad i = 0, 1, 2, ...$$

After the substitution (3.2) in (3.1) for the first members of expansion we get the spectral problem

$$A(\alpha_0)\overline{W}_0 = \overline{0}, \tag{3.3}$$

where

$$A(\alpha_0)\overline{W}_0 \equiv \left\{\ell(\alpha_0)\overline{W}_0;\ \overline{W}(\pm 1) = \overline{0}\right\},$$

$$\ell(\alpha_0)\overline{W}_0 \equiv \left(L_3\overline{W}_0^{'}\right)' + \alpha_0\left(\left(L_4\overline{W}_0\right)' + L_4^T\overline{W}_0^{'}\right) + \alpha_0^2 L_2\overline{W}_0.$$

$$L_3 = \left\| \begin{matrix} G & 0 \\ 0 & H \end{matrix} \right\|, \quad L_4 = \left\| \begin{matrix} 0 & G \\ \lambda & 0 \end{matrix} \right\|.$$

The spectral problem (3.3) coincides with the problem describing the potential solution of homogeneous on thickness plate, which is studied in [5, 6].

On the next stage of asymptotic integration we receive a boundary value problem for the determination $\overline{W}$ and α_1:

$$\begin{cases} \ell(\alpha_0)\overline{W}_1 = -\left(B_1 + \alpha_0 B_2 + \alpha_1 B_3 + 2\alpha_0\alpha_1 L_2\right)\overline{W}_0, \\ \overline{W}_1(\pm 1) = \overline{0}, \end{cases} \tag{3.4}$$

where

$$B_1 = \left\| \begin{matrix} G\cot\theta_0 \dfrac{d}{d\eta} & -\dfrac{d}{d\eta}\left(\dfrac{3G}{2}\right) - \dfrac{(2H+\lambda)}{2}\dfrac{d}{d\eta} \\ \dfrac{d}{d\eta}\dfrac{(2H+\lambda)}{2} + \dfrac{3G}{2}\dfrac{d}{d\eta} & \left(\dfrac{d}{d\eta}(\lambda) + 2G\dfrac{d}{d\eta}\right)\cot\theta_0 \end{matrix} \right\|,$$

$$B_2 = \left\| \begin{matrix} O & (G+\lambda)\,cot\theta_0 \\ O & O \end{matrix} \right\|,$$

$$B_3 = \left\| \begin{matrix} O & \frac{d}{d\eta}(G) + \lambda\frac{d}{d\eta} \\ \frac{d}{d\eta}(\lambda) + G\frac{d}{d\eta} & O \end{matrix} \right\|.$$

The problem (3.4) is solvable provided that the right-hand side of (3.4) is orthogonal to the solution of the conjugate problem

$$A^*(\alpha_0)\overline{W}_0^* = A(-\overline{\alpha}_0)\overline{W}_0^* = \overline{0}, \tag{3.5}$$

where

$$\overline{W}_0^* = (a_0^*, b_0^*)^T.$$

Satisfying this condition, for α_1 we get

$$\alpha_1 = -\frac{E_1}{E_2},$$

where

$$E_1 = \int_{-1}^{1}\left[G\frac{da_0}{d\eta}\overline{a}_0^*\cot\theta_0 + \frac{3}{2}G\left(\overline{b}_0^*\frac{da_0}{d\eta} + b_0\frac{d\overline{a}_0^*}{d\eta}\right) - \right.$$

$$-\frac{(4G+3\lambda)}{2}\left(\overline{a}_0^*\frac{db_0}{d\eta} + a_0\frac{d\overline{b}_0^*}{d\eta}\right) +$$

$$\left. + \left(2G\frac{db_0}{d\eta}\overline{b}_0^* - \lambda b_0\frac{d\overline{b}_0^*}{d\eta}\right)\cot\theta_0 + \alpha(G+\lambda)\,b_0\overline{a}_0^*\cot\theta_0\right]d\eta;$$

$$E_2 = \int_{-1}^{1}\left[G\left(\frac{da_0}{d\eta}\overline{b}_0^* - b_0\frac{d\overline{a}_0^*}{d\eta}\right) + \lambda\left(\frac{db_0}{d\eta}\overline{a}_0^* - a_0\frac{d\overline{b}_0^*}{d\eta}\right) + \right.$$

$$\left. +\alpha_0\left(2Gb_0\overline{b}_0^* + 2Ha_0\overline{a}_0^*\right)\right]d\eta.$$

So, the solutions corresponding the third iteration process have the form

$$u_r^{(3)} = \rho^{-\frac{1}{2}}\varepsilon\sum_{k=1}^{\infty}B_k U_{rk}^{(3)},$$

$$u_\theta^{(3)} = \rho^{-\frac{1}{2}}\varepsilon\sum_{k=1}^{\infty}B_k U_{\theta k}^{(3)},$$

$$U_{rk}^{(3)} = \left[p_0\alpha_{0k}^{-2}\Psi_k''(\eta) - p_2\Psi_k(\eta) + O(\varepsilon)\right]\exp\left(\varepsilon^{-1}\alpha_{0k}\ln\rho\right),$$

$$U_{\theta k}^{(3)} = \left[\alpha_{0k}^{-3}\left(p_0\Psi_k''\right)' + 2\alpha_{0k}^{-1}p_1\Psi_k' - \alpha_{0k}^{-1}\left(p_2\Psi_k\right)' + O(\varepsilon)\right]\exp\left(\varepsilon^{-1}\alpha_{0k}\ln\rho\right),$$

$$p_0 = \frac{H}{4G\left(G+\lambda\right)};\ p_1 = \frac{1}{2G};\ p_2 = \frac{\lambda}{4G\left(G+\lambda\right)}. \qquad (3.6)$$

Here $\Psi_k(\eta)$ is a solution of Popkovich generalized spectral problem for nonhomogeneous case

$$\begin{cases} \left(p_0\Psi_k''\right)'' + \alpha_{0k}^2\left[2\left(p_1\Psi_k'\right)' - \left(p_2\Psi_k\right)'' - p_2\Psi_k''\right] + \alpha_{0k}^4 p_0\Psi_k = 0, \\ p_0\Psi_k'' - p_2\alpha_{0k}^2\Psi_k = 0 \ \text{ when } \eta = \pm 1, \\ \left(p_0\Psi_k''\right)' + 2\alpha_{0k}^2 p_1\Psi_k' - \alpha_{0k}^2\left(p_2\Psi_k\right)' = 0 \text{ when } \eta = \pm 1. \end{cases}$$

From (3.6) it follows that the stresses $\sigma_{\theta\theta}$, $\sigma_{\theta r}$, σ_{rr}, $\sigma_{\varphi\varphi}$ relative to ε have the order unit.

Note that the third asymptotic process determines the solutions (3.6), which have the character of a boundary layer. The first members (3.6) are completely equivalent to Saint-Venant edge effect of nonhomogeneous plate [5, 6].

4. Consider the question on removal of stresses from end surface of a cone. Suppose, that on spherical part of boundary the stresses

$$\sigma_{rr} = f_{1s}(\eta);\ \sigma_{r\theta} = f_{2s}(\eta) \,\text{when}\rho = \rho_s;\ \ (s = 1, 2). \qquad (4.1)$$

are given.

Here $f_{1s}(\eta)$, $f_{2s}(\eta)$ are smooth functions and satisfy the equilibrium conditions.

Let's represent the permutations in the form $u_r = \sum\limits_{k=1}^{\infty} C_k \rho^{z_k - \frac{1}{2}} \cdot a_k(\eta)$,

$$u_\theta = \sum_{k=1}^{\infty} C_k \rho^{z_k - \frac{1}{2}} \cdot b_k(\eta). \qquad (4.2)$$

For the stresses we'll get

$$\sigma_{rr} = \sum_{k=1}^{\infty} C_k \rho^{z_k - \frac{3}{2}} \cdot E_k(\eta),$$

$$\sigma_{r\theta} = \sum_{k=1}^{\infty} C_k \rho^{z_k - \frac{3}{2}} \cdot T_k(\eta),$$

$$E_k(\eta) = \left(z_k - \frac{1}{2}\right) H a_k + \lambda\left(2a_k + \varepsilon^{-1} b_k' + b_k \cot\left(\theta_0 + \varepsilon\eta\right)\right),$$

$$T_k(\eta) = G\left(\varepsilon^{-1} a_k' + \left(z_k - \frac{3}{2}\right) b_k\right). \qquad (4.3)$$

For the determination of the constants C_k let's use the Lagrange variation principle [7]. In considered case the variational principle has the form

$$\sum_{s=1}^{\infty} \rho_s^2 \int_{-1}^{1} \left[\left(\sigma_{rr} - f_{1s}\right)\delta u_r + \left(\sigma_{r\theta} - f_{2s}\right)\delta u_\theta\right]\Big|_{\rho=\rho_s} \cdot \sin\left(\theta_0 + \varepsilon\eta\right) d\eta = 0. \qquad (4.4)$$

Substituting (4.2)-(4.3) in (4.4) and supposing δC_k as a independent variations we get the infinite system of linear algebraic equations

$$\sum_{k=1}^{\infty} D_{jk} \cdot C_k = t_j; \quad (j = 1, 2, ...) . \tag{4.5}$$

Here

$$D_{jk} = \left(\rho_1^{z_k+z_j} + \rho_2^{z_k+z_j}\right) \cdot \int_{-1}^{1} \left[E_k(\eta)\, a_j(\eta) + T_k(\eta)\, b_j(\eta)\right] \sin(\theta_0 + \varepsilon\eta)\, d\eta,$$

$$t_j = \sum_{s=1}^{2} \rho_s^{z_j+\frac{3}{2}} \int_{-1}^{1} \left[f_{1s}(\eta)\, a_j(\eta) + f_{2s}(\eta)\, b_j(\eta)\right] \sin(\theta_0 + \varepsilon\eta)\, d\eta.$$

In [8] the solvability and convergence of reduction method for the system (4.5) was proved.

We look for the unknown constants B_k in the form

$$B_k = B_{k0} + \varepsilon B_{k1} + \cdots . \tag{4.6}$$

After substituting (4.6) in (4.5) and allowing for (3.6) we get the following system of infinite algebraic equations:

$$\sum_{k=1}^{\infty} B_{k0} \cdot g_{jk} = h_j \qquad (j = 1, 2, ...) . \tag{4.7}$$

Here

$$g_{jk} = \sum_{s=1}^{2} \int_{-1}^{1} \left[\frac{2\rho_1 \Psi_j' \Psi_k'}{\alpha_{0j}} + \frac{\alpha_{0j} - \alpha_{0k}}{\alpha_{0k}\alpha_{0j}^3} \left(p_0 \Psi_j'' - \alpha_{0j}^2 p_2 \Psi_j\right) \Psi_k''\right] d\eta \times$$

$$\times \exp\left(\frac{\alpha_{0j} + \alpha_{0k}}{\varepsilon} \ln \rho_s\right),$$

$$h_j = \sum_{s=1}^{2} \rho_s^{\frac{3}{2}} \int_{-1}^{1} \left[f_{1s}\left(p_0 \alpha_{0j}^{-2} \Psi_j'' - p_2 \Psi_j\right) + f_{2s}\left(\alpha_{0j}^{-1} (p_2 \Psi_j)' - \alpha_{0j}^{-3} \left(p_0 \Psi_j''\right)'\right) - \right.$$

$$\left. -2\alpha_{0j}^{-1} p_1 \Psi_j'\right] d\eta.$$

The determination $B_{k\ell}$ $(\ell = 1, 2, ...)$ is invariably reduced to the reversion of the same matrices, which coincide with the matrices (4.7).

Note that when $G = const$, $\lambda = const$ from the solution of the given problem we get the known results of paper [3].

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