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WEIGHTED COMPOSITION OPERATORS ON THE SPACES OF VECTOR-VALUED FUNCTIONS

Abstract

In this paper we will investigate compactness of weighted composition operators on uniform spaces of continuous V -valued functions, where V is a Banach algebra.

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Introduction.

Let X be a compact Hausdorff space and V be a complex Banach algebra with norm $\|\cdot\|_V$ and with unit element $\mathbf{1}$ (if there is no unit element we may adjoin $\mathbf{1}$ to the algebra). Let $C(X, V)$ denote the space of all continuous V -valued functions defined on X . It is clear that $C(X, V)$ is a Banach space with respect to the norm $\|f\| = \sup_{x \in X} \|f(x)\|_V$ for any $f \in C(X, V)$, note that $C(X, V)$ is also a Banach algebra with this norm and with pointwise multiplication $(fg)(x) = f(x)g(x)$. By $A(X, V)$ we denote a closed subspace of $C(X, V)$. We will consider a weighted composition operator $T : A(X, V) \rightarrow C(X, V)$ defined on the spaces of vector V -valued functions in the form $Tf(x) = u(x)f(\phi(x))$, where $\phi : X \rightarrow X$ is a continuous selfmap and $u \in C(X, V)$ (we assume the non-trivial case: $\phi \neq \text{const}$ and $u \neq 0$). The importance of this kind of operators is that they are applicable in solving and the existence of the solutions of the functional-differential equations containing both the argument and its shifts and this form operators on the space of complex valued functions are studied in several papers in recent years, see for example [1], [2], [3], [4], [5].

1. Compactness.

For simplicity we assume that the topology of X is metrizable, say, with metric d .

First of all we note that the equicontinuity of the family F of V -valued functions $f \in C(X, V)$ is equivalent to saying, that, if $x_n \rightarrow x_0$ then $f(x_n) \rightarrow f(x_0)$ uniformly on F with respect to $\|\cdot\|_V$ norm. Indeed, this property implies the equicontinuity of the family F , to see this let the property hold but the equicontinuity does not hold, then for some $\delta > 0$ there are sequences $\{x'_n\}$, $\{x''_n\}$ and $\{f_n\} \subset F$ such that $d(x'_n, x''_n) \rightarrow 0$ but $\|f_n(x'_n) - f_n(x''_n)\|_V \geq \delta$. Since X is compact, so we may assume that $x'_n, x''_n \rightarrow x_0 \in X$. Using this property we will have $\|f(x_0) - f(x'_n)\|_V \leq \delta/2$ and $\|f(x_0) - f(x''_n)\|_V \leq \delta/2$ for all $f \in F$ and for all sufficiently large n . So if we take f to be f_n , then we get a contradiction. Therefore the equicontinuity holds. The converse is clear.

Lemma 1.1. *If $\phi : X \rightarrow Y$ is a continuous mapping from (compact) X onto the compact space Y , then the equicontinuity of the family $F \subset C(Y, V)$ is equivalent to the equicontinuity of the family $F \circ \phi = \{f \circ \phi : f \in F\} \subset C(X, V)$.*

Proof. Let F be equicontinuous and $x_n \rightarrow x_0$. Since ϕ is a continuous mapping, then $\phi(x_n) \rightarrow \phi(x_0)$ as $n \rightarrow \infty$. Therefore $f(\phi(x_n)) \rightarrow f(\phi(x_0))$ as $n \rightarrow \infty$ uniformly for $f \in F$, i.e. from above discussion we get that the family $F \circ \phi$ is equicontinuous.

Conversely, let the family $F \circ \phi$ be equicontinuous but F not an equicontinuous family. For this reason there is a sequence $y_n \rightarrow y_0 \in Y$ and a sequence $f_n \in F$ such that $\|f_n(y_n) - f_n(y_0)\|_V \geq \delta > 0$.

Let $y_n = \phi(x_n)$. Now passing to a subsequence we may assume that $x_n \rightarrow x_0$. Since the family $F \circ \phi$ is equicontinuous, so for sufficiently large n we have $\|f \circ \phi(x_n) - f \circ \phi(x_0)\|_V < \delta$ for all $f \in F$. Now if we take $f = f_n$ we get a contradiction, so the lemma is proved.

Lemma 1.2. *The operator $T : A(X, V) \rightarrow C(X, V)$, $f(x) \mapsto u(x)f(\phi(x))$ is compact if and only if for any $\varepsilon > 0$ the restriction of the family $B = \{f \in A(X, V) : \|f\|_V \leq 1\}$ to the compact set $Y_\varepsilon = \phi\{x \in X : \|u(x)\|_V \geq \varepsilon\}$ is equicontinuous.*

Proof. For necessity. Let T be compact and $\varepsilon > 0$. Put $X_\varepsilon = \{x \in X : \|u(x)\|_V \geq \varepsilon\}$ and let the operator $S : C(X, V) \rightarrow C(X_\varepsilon, V)$ be such that $f \mapsto u^{-1}f|_{X_\varepsilon}$, where $f|_{X_\varepsilon}$ denotes the restriction of f to X_ε .

Since $\|u(x)\|_V \geq \varepsilon$ for any $x \in X_\varepsilon$ the operator S is continuous and so the operator $ST : A(X, V) \rightarrow C(X_\varepsilon, V)$ is compact. From this we get that $\phi(B)|_{X_\varepsilon}$ is relatively compact in $C(X_\varepsilon, V)$ and lemma 1.1 implies the equicontinuity of the family $B|_{Y_\varepsilon}$.

Sufficiency. For any $\varepsilon > 0$ the compact set X can be covered by the two following open sets:

$$E_1 = \{x \in X : \|u(x)\|_V > \varepsilon\}, \quad E_2 = \{x \in X : \|u(x)\|_V < 2\varepsilon\}.$$

Let $\delta > 0$ be the Lebesgue number for this covering (i.e. the number such that every ball in X with radius less than δ is contained at least in one of them), then from lemma 1.1 we conclude that the family $\{f \circ \phi : f \in B\}$ is equicontinuous on E_1 . Let $\delta_1 > 0$ be such that if $x_1, x_2 \in E_1$, $d(x_1, x_2) < \delta_1$ then $\|f \circ \phi(x_1) - f \circ \phi(x_2)\|_V < \varepsilon$. Without loss of generality we can assume that $\|u\| \leq 1$ and $\|u(x_1) - u(x_2)\|_V < \varepsilon$ when $d(x_1, x_2) < \delta$.

Hence for $x_1, x_2 \in E_1$ such that $d(x_1, x_2) < \delta$ we have $\|Tf(x_1) - Tf(x_2)\|_V < 2\varepsilon$ (we may assume that $\delta_1 < \delta$). If $d(x_1, x_2) < \delta_1$ and $x_1, x_2 \notin E_1$, then we have $x_1, x_2 \in E_2$ and so again we have: $\|Tf(x_1) - Tf(x_2)\|_V < 4\varepsilon$, since $\|f \circ \phi(x)\|_V \leq 1$ and $\|u(x)\|_V < 2\varepsilon$. So the lemma is proved.

Definition 1.3. *A closed subset $E \subset X$ is called a peak set with respect to $A(X, V)$ if there exists a sequence $\{f_n\}$, $f_n \in A(X, V)$ such that $\|f_n\| = f_n(x) = 1$ for all n and all $x \in E$, moreover, outside any neighborhood of the set E the sequence $\{f_n\}$ tends to 0 uniformly. A peak set consisting of only one point is called peak point.*

Theorem 1.4. *If the operator $T : A(X, V) \rightarrow C(X, V)$ of the form $f \mapsto u(f \circ \phi)$ is compact, then for any connected compact set $K \subset \{x \in X : u(x) \neq 0\}$ and for any peak set E with respect to $A(X, V)$ we have either $\phi(K) \subseteq E$ or $\phi(K) \cap E = \emptyset$.*

Proof. It suffices to prove that if K is a compact and connected set such that $\|u(x)\|_V \geq \varepsilon > 0$ for any $x \in K$ and if E is a peak set such that $\phi(K) \cap E \neq \emptyset$, then $\phi(K) \subset E$.

By lemma 1.2 the restriction $B|_{\phi(K)}$ is an equicontinuous family which B is the unit ball in $A(X, V)$. On the other hand there exists sequence $f_n \in B$ such that $f_n|_E = \mathbf{1}$ and $f_n \rightarrow 0$ uniformly outside any neighborhood of the set E . We may assume that there is a function $g \in C(\phi(K), V)$ such that $\|f_n|_{\phi(K)} - g\| \rightarrow 0$ as $n \rightarrow \infty$.

It is clear that $g|_{E \cap \phi(K)} = \mathbf{1}$ and $g|_{\phi(K) \setminus E} = 0$. But $E \cap \phi(K) \neq \emptyset$, $\phi(K)$ is connected and $g \in C(\phi(K), V)$, thus we deduce that $\phi(K) \setminus E = \emptyset$, i.e. $\phi(K) \subset E$. So the theorem is proved.

Definition 1.5. A function $u : X \rightarrow V$ defined on X is called a multiplier with respect to $A(X, V)$ if for any $f \in A(X, V)$ we have $uf \in A(X, V)$. The set of multiplier functions with respect to $A(X, V)$ will be denoted by $\text{mult}(A(X, V))$.

Definition 1.6. A selfmap $\phi : X \rightarrow X$ is called a compositor with respect to $A(X, V)$ if for every $f \in A(X, V)$ we have $f \circ \phi \in A(X, V)$. The set of compositors with respect to $A(X, V)$ is denoted by $\text{comp}(A(X, V))$.

Remark 1.7. In theorem 1.4 as well as lemmas 1.1 and 1.2 we could assume that the function $\phi : X \rightarrow X$ is continuous only on the set $\{x \in X : u(x) \neq 0\}$.

If $u \in \text{mult}(A(X, V))$ and $\phi \in \text{comp}(A(X, V))$ then it is clear that T is a weighted composition operator on $A(X, V)$, i.e. $T : A(X, V) \rightarrow A(X, V)$. In particular, if $A(X, V)$ is a function algebra, i.e. it is a uniformly closed subalgebra with unit element 1_A and contains constant functions $e_v(x) = v, (\forall x \in X)$, for every $v \in V$ and separates the points of X , then T acting on $A(X, V)$ (of course, if $\phi \in \text{comp}(A(X, V))$). In general, we may consider weighted composition operators on a Banach module M on $A(X, V)$ which contains 1_A , in the case that $\phi \in \text{comp}(M)$ and $u \in A(X, V)$. In such case we can extend the domain of the operator (even using some other topology) to large spaces.

If $A(X, V) = C(X, V)$ we obtain the following theorem which is a generalization of theorem 3 [4].

Theorem 1.7. If V is also a Montel space, then the operator T of the form $f \mapsto u(f \circ \phi)$ from $C(X, V)$ to $C(X, V)$ (where $u \in C(X, V)$ and $\phi : X \rightarrow X$ is continuous on the set $S(u) = \{x \in X : u(x) \neq 0\}$) is compact iff, for each $\varepsilon > 0$, the set $\phi(X_\varepsilon) = \phi(\{x \in X : \|u(x)\|_V \geq \varepsilon\})$ is finite. In particular, if $u(x) \neq 0$ and X is connected set, then T is compact on $C(X, V)$ iff, ϕ is a V -valued constant.

Proof. For sufficiency. Let $\varepsilon > 0$ and $E_1 = \{x \in X : \|u(x)\|_V > \varepsilon\}$, $E_2 = \{x \in X : \|u(x)\|_V < 2\varepsilon\}$, also let l_1, l_2 be the V -valued partition of unity corresponding to this covering of X , i.e. $0 \leq \|l_1\|_V, \|l_2\|_V \leq 1, l_1(x) + l_2(x) = \mathbf{1}$, and $l_k(x) = 0$ outside $E_k, k = 1, 2$. Let $T_k f = (l_k u) \cdot (f \circ \phi), k = 1, 2$.

So we can write $T = T_1 + T_2$. Let $x_1, x_2, \dots, x_p \in X$ be those distinct values of ϕ on E_1 and $F_k = \phi^{-1}(x_k) \subset X, (k = 1, 2, \dots, p)$ then it is clear that $T_1 f(x) = u(x) \sum_{i=1}^p (e_i \chi_i(x)) \cdot f(x_i)$, where χ_i is the characteristic function of F_i , i.e. T_i is a finite rank operator and also $\|T_2\| \leq 2\varepsilon$. Consequently, T is a compact operator.

Necessity. Let $\varepsilon > 0, X_\varepsilon = \{x \in X : \|u(x)\|_V \geq \varepsilon\}$ and $Y_\varepsilon = \phi(X_\varepsilon)$. Since T is compact, then by lemma 1.2, the family $B|_{Y_\varepsilon} = \{f \in C(X, V) : \|f\| \leq 1\}|_{Y_\varepsilon}$ is relatively compact. Since every continuous $g \in C(Y_\varepsilon, V)$ V -valued function may be extended to all X without changing its norm, then the unit ball of $C(Y_\varepsilon, V)$ is relatively compact, consequently, $\dim C(Y_\varepsilon, V)$, and from this we get Y_ε is a finite

set.

Remarkn 1.8. If the compact X is connected and $\varphi : X \rightarrow X$ is continuous then from theorem 1.7 it is clear that there is no compact composition operator on $C(X, V)$ (when $\varphi \neq \text{const}$), but when the subspace $A(X, V)$ has an analytic structure, the by using above mentioned theorem we can get some compactness criteries for the weighted composition operators, e.g. for the cases $A(B^N, V)$, $A(D^N, V)$, etc. where $B^N \subset \mathbb{C}^N$ is the unit ball, $D^N \subset \mathbb{C}^N$ is a polydisk.

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