Gambar K. NAMAZOV , Yashar T. MEHRALIYEV

INVESTIGATION OF A CLASSICAL SOLUTION OF A NON- SELF-ADJOINT ONE-DIMENSIONAL INVERSE BOUNDARY VALUE PROBLEM FOR A HIGHER ORDER DIFFERENTIAL EQUATION

Abstract

In the paper the inverse problem for a higher order differential equation with non-self-adjoint boundary conditions is investigated.

Consider the following problem

$$\frac{\partial^2 u}{\partial t^2}+(-1)^p\alpha_2\frac{\partial^{2(1+p)}}{\partial t^2\partial x^{2p}}u(x,t)+(-1)^n\alpha_1\frac{\partial^{1+2n}}{dt\partial x^{2n}}u(x,t)+$$

$$+(-1)^m\alpha_0\frac{\partial^{2m}}{\partial x^{2m}}u(x,t)=a(t)u(x,t)+f(x,t)$$

$$(x,t)\in D_T=\{(x,t):0\le x\le 1\,,\,0\le t\le T\}\,,\left(p=\overline{0,m};n=\overline{0,m};m=1,2,...\right) \tag{1}$$

$$u(x,0)+\delta u(x,T)=\varphi(x)\ ,$$

$$u_t(x,0)+\delta u_t(x,T)=\psi(x)\ ,0\le x\le 1 \tag{2}$$

$$\frac{\partial^{2s}}{\partial x^{2s}}u(0,t)=0,\frac{\partial^{2s+1}}{\partial x^{2s+1}}u(0,t)=\frac{\partial^{2s+1}}{\partial x^{2s+1}}u(1,t)\ \left(s=\overline{0,m-1}\right),0\le t\le T, \tag{3}$$

$$u(x_0,t)=h(t)\ ,\ 0\le t\le T,\ 0<x_0<1, \tag{4}$$

where $u(x,t)$, $a(t)$ are the desired and $f(x,t)$, $\varphi(x)$, $\psi(x)$, $h(t)$ are the given functions, $\alpha_i\ge 0$ $(i=0,1,2)$, δ are the given numbers, where $\alpha_0^2+\alpha_1^2>0$.

Definition. *We call the pair $\{u(x,t),a(t)\}$ of the functions $u(x,t)$ and $a(t)$ satisfying the conditions:*

1) the function $u(x,t)$ is continuous in D_T together with all its derivatives entering in the equation (1);

2) the function $a(t)$ is continuous on $[0,T]$;

3) all the conditions (1)-(4) are satisfied in usual sense, a classical solution of the inverse boundary value problem (1)-(4).

It's known [1] that the sequences of the functions

$$X_0(x)=x,\ X_{2k-1}(x)=x\cos\lambda_k x,\ X_{2k}(x)=\sin\lambda_k x\quad(\lambda_k=2\pi k), \tag{5}$$

and

$$Y_0(x)=2,\ Y_{2k-1}(x)=4\cos\lambda_k x,\ Y_{2k}(x)=4(1-x)\sin\lambda_k x, \tag{6}$$

forms a biorthogonal system in $L_2(0,1)$ and the system (5) forms a basis in $L_2(0,1)$. The arbitrary function $f(x)\in L_2(0,1)$ is decomposed in biorthogonal series

$$\varphi(x)=\sum_{k=0}^{\infty}\varphi_k X_k(x)\,,$$

where

$$\varphi(x)=\int_0^1 \varphi(x)\,Y_k dx \quad (k=0,1,2,...).$$

Besides for any function $\varphi(x)\in L_2(0,1)$ the estimation

$$\frac{3}{4}\|\varphi(x)\|^2_{L_2(0,1)}\le\sum_{k=1}^{\infty}\varphi_k^2\le 16\,\|\varphi(x)\|_{L_2(0,1)}$$

is valid.

We can show that

$$X_0'(x)=1,\quad X_{2k-1}'(x)=\cos\lambda_k x-x\lambda_k\sin\lambda_k x,\quad X_{2k}(x)=\lambda_k\cos\lambda_k x,$$

$$X_0^{(i)}(x)=0\ \ (i\ge 2)\,,\quad X_{2k-1}^{(2i)}(x)=(-1)^i\lambda_k^{2i}X_{2k-1}(x)+2i\,(-1)^i\lambda_k^{2i-1}X_{2k}(x)\,,$$

$$X_{2k}^{(2i)}(x)=(-1)^i\lambda_k^{2i}X_{2k}(x)\,; \tag{7}$$

$$Y_0'(x)=0,\quad Y_{2k-1}'(x)=-4\lambda_k\sin\lambda_k x,\quad Y_{2k}'(x)=-4\lambda_k(1-x)\cos\lambda_k x+4\sin\lambda_k x,$$

$$Y_0^{(2i)}(x)=0,\quad X_{2k-1}^{(2i)}(x)=(-1)^i\lambda_k^{2i}Y_{2k-1}(x)\,,$$

$$Y_{2k}^{(2i)}(x)=(-1)^i\lambda_k^{2i}Y_{2k}(x)+(-1)^{i+1}\,2i\lambda_k^{2i-1}Y_{2k-1}(x)\,. \tag{8}$$

Hence we have

$$X_k^{(2i)}(0)=0,\ X_k^{(2i+1)}(0)=X_k^{(2i+1)}(1)\,, \tag{9}$$

$$Y_k^{(2i)}(0)=\ Y_k^{(2i)}(1)\,,\ Y_k^{(2i+1)}(0)=0. \tag{10}$$

Further, since the system (5) forms a basis in $L_2(0,1)$ and the systems (5) and (6) form a biorthogonal system of functions in $L_2(0,1)$, then it's obvious that for each classical solution $\{u(x,t)\,,a(t)\}$ of the problem (1)-(4) its second component $u(x,t)$ has the form:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)\,X_k(x)\,, \tag{11}$$

where

$$u_k(t)=\int_0^1 u(x,t)\,Y_k(x)\,dx \qquad (k=0,1,2,...).$$

Now multiply the both sides of the equations (1) by $Y_k(x)$ and integrate the obtained equality with respect to x from 0 to 1:

$$\int_0^1\frac{\partial^2}{\partial t^2}u(x,t)\,Y_k(x)\,dx+(-1)^p\,\alpha_2\int_0^1\frac{\partial^{2(1+p)}}{\partial t^2\partial x^{2p}}u(x,t)\,Y_k(x)\,dx+$$

$$+(-1)^n\,\alpha_1\int_0^1\frac{\partial^{1+2n}}{\partial t\partial x^{2n}}u(x,t)\,Y_k(x)\,dx+(-1)^m\,\alpha_0\int_0^1\frac{\partial^{2m}}{\partial x^{2m}}u(x,t)\,Y_k(x)\,dx=$$

$$= \int_0^1 \left(a(t)\,u(x,t) + f(x,t)\right) Y_k(x)\,dx\,.(12) \tag{12}$$

Subject to (3) and (10) integrating by parts we find:

$$\int_0^1 \frac{\partial^{2i}}{\partial x^{2i}} u(x,t) \cdot Y_k(x)\,dx = \int_0^1 u(x,t)\,Y_k^{(2i)}(x)\,dx \quad \left(i = \overline{0,m}\right). \tag{13}$$

Then it's evident that

$$\int_0^1 \frac{\partial^2}{\partial t^2} u(x,t) \cdot Y_k(x)\,dx = \frac{d}{dt^2}\left(\int_0^1 u(x,t)\,Y_k(x)\,dx\right) = u_k''(t)\,,$$

$$\int_0^1 \frac{\partial^{2(1+p)}}{\partial t^2 dx^{2p}} u(x,t) \cdot Y_k(x)\,dx = \frac{d}{dt^2}\left(\int_0^1 u(x,t)\,Y_k^{(2p)}(x)\,dx\right),$$

$$\int_0^1 \frac{\partial^{1+2n}}{\partial t dx^{2n}} u(x,t) \cdot Y_k(x)\,dx = \frac{d}{dt}\left(\int_0^1 u(x,t)\,Y_k^{(2n)}(x)\,dx\right),$$

$$\int_0^1 \frac{\partial^{2m}}{\partial x^{2m}} u(x,t) \cdot Y_k(x)\,dx = \int_0^1 u(x,t)\,Y_k^{(2m)}(x)\,dx. \tag{14}$$

From (12) subject to (8) and (14) we have

$$u_0''(t) = F_0(t)\,, \tag{15}$$

$$\left(1 + \alpha_2 \lambda_k^{2p}\right) u_{2k-1}''(t) + \alpha_1 \lambda_k^{2n} u_{2k-1}'(t) + \alpha_0 \lambda_k^{2m} u_{2k-1}(t) = F_{2k-1}(t)\,, \tag{16}$$

$$\left(1 + \alpha_2 \lambda_k^{2p}\right) u_{2k}(t) + \alpha_1 \lambda_k^{2n} u_{2k}(t) + \alpha_0 \lambda_k^{2m} u_{2k}(t) =$$

$$= F_{2k}(t) - 2p\alpha_2 \lambda_k^{2p-1} u_{2k-1}''(t) - 2n\alpha_1 \lambda_k^{2n-1} u_{2k-1}'(t) - 2\alpha_0 m \lambda_k^{2m-1}, \tag{17}$$

where

$$F_k(t) = a(t)\,u_k(t) + f_k(t)\,, \quad f_k(t) = \int_0^1 f(x,t)\,Y_k(x)\,dx.$$

Further, by using representation (11) from (2) we have

$$u_k(0) + \delta u_k(T) = \varphi_k, \quad u_k'(0) + \delta u_k'(T) = \psi_k, \tag{18}$$

where

$$\varphi_k = \int_0^1 \varphi(x)\,Y_k(x)\,dx, \quad \psi_k = \int_0^1 \psi(x)\,Y_k(x)\,dx \quad (k = 0, 1, 2, \ldots)\,.$$

By solving problem (15), (18) we find

$$u_0(t) = \Big[(1+\delta)^{-2}(1+\delta)\varphi_0 + (t+\delta(t-T))\psi_0 -$$

$$-\delta\int_0^T F_0(\tau)(T+t-\tau)\,d\tau\Big] + \int_0^t F_0(\tau)(t-\tau)\,d\tau. \qquad (19)$$

Hence

$$u_0'(t) = (1+\delta)^{-2}\left[(1+\delta)\psi_0 - \delta\int_0^T F_0(\tau)\,d\tau\right] + \int_0^t F_0(\tau)\,d\tau.$$

1. Assume that

$$n \le m \le 2n-p, \quad \alpha_1 \ge 2\sqrt{\alpha_0(1+\alpha_2)}.$$

Then by solving problem (16), (18) we obtain

$$u_{2k-1}(t) = \frac{1}{\chi_k}\Big[\left(\rho_{1k}(T)\mu_{2k}e^{\mu_{1k}t} - \rho_{2k}(T)\mu_{1k}e^{\mu_{2k}t}\right)\varphi_{2k-1} +$$

$$+\left(\rho_{2k}(T)e^{\mu_{2k}t} - \rho_{1k}(T)e^{\mu_{1k}t}\right)\psi_{2k-1} -$$

$$-\frac{\delta}{1+\alpha_2\lambda_k^{2p}}\int_0^T F_{2k-1}(\tau)\left(\rho_{2k}(T)e^{\mu_{2k}(T+t-\tau)} - \rho_{1k}(T)e^{\mu_{1k}(T+t-\tau)}\right)d\tau +$$

$$+\frac{1}{1+\alpha_2\lambda_k^{2p}}\int_0^t F_{2k-1}(\tau)\left(e^{\mu_{2k}(t-\tau)} - e^{\mu_{1k}(t-\tau)}\right)d\tau +$$

$$+\frac{1}{1+\alpha_2\lambda_k^{2p}}\int_0^t F_{2k-1}(\tau)\left(e^{\mu_{2k}(t-\tau)} - e^{\mu_{1k}(t-\tau)}\right)d\tau, \qquad (20)$$

where

$$\mu_{ik} = -\left(\frac{\alpha_{n,p,k}}{2} - (-1)^i\right)\cdot\sqrt{\alpha_{n,p,k}^2 - 4\alpha_{m,p,k}},$$

$$\chi_k = \mu_{2k} - \mu_{1k} = \sqrt{\alpha_{n,p,k}^2 - 4\alpha_{m,p,k}},$$

$$\rho_{ik}(T) = \left(1+\delta e^{\mu_{ik}T}\right)^{-1} \quad (i=1,2),$$

$$\alpha_{n,p,k} = \frac{\alpha_1\lambda_k^{2n}}{1+\alpha_2\lambda_k^{2p}}, \quad \alpha_{m,p,k} = \frac{\alpha_0\lambda_k^{2m}}{1+\alpha_2\lambda_k^{2p}}.$$

Differentiating (20) two times with respect to t we find:

$$u_{2k-1}'(t) = \frac{1}{\chi_k}\Big[\mu_{1k}\mu_{2k}\left(\rho_{1k}(T)e^{\mu_{1k}t} - \rho_{2k}(T)e^{\mu_{2k}t}\right)\varphi_{2k-1} +$$

$$+\left(\mu_{2k}\rho_{2k}\left(T\right)e^{\mu_{2k}t}-\mu_{1k}\rho_{1k}\left(T\right)e^{\mu_{1k}t}\right)\psi_{2k-1}-\frac{\delta}{1+\alpha_2\lambda_k^{2p}}\int\limits_0^T F_{2k-1}\left(\tau\right)\times$$

$$\times\left(\mu_{2k}\rho_{2k}\left(T\right)e^{\mu_{2k}(T+t-\tau)}-\mu_{1k}\rho_{1k}\left(T\right)e^{\mu_{1k}(T+t-\tau)}\right)d\tau+\frac{1}{1+\alpha_2\lambda_k^{2p}}\times$$

$$\left.\times\int\limits_0^t F_{2k-1}\left(\tau\right)\left(\mu_{2k}e^{\mu_{2k}(t-\tau)}-\mu_{1k}e^{\mu_{1k}(t-\tau)}\right)d\tau\right], \tag{21}$$

$$u''_{2k-1}\left(t\right)=\frac{1}{\chi_k}\left[\mu_{1k}\mu_{2k}\left(\mu_{1k}\rho_{1k}\left(T\right)e^{\mu_{1k}t}-\mu_{2k}\rho_{2k}\left(T\right)e^{\mu_{2k}t}\right)\varphi_{2k-1}+\right.$$

$$+\left(\mu_{2k}^2\rho_{2k}\left(T\right)e^{\mu_{2k}t}-\mu_{1k}^2\rho_{1k}\left(T\right)e^{\mu_{1k}t}\right)\psi_{2k-1}-\frac{\delta}{1+\alpha_2\lambda_k^{2p}}\int\limits_0^T F_{2k-1}\left(\tau\right)\times$$

$$\times\left(\mu_{2k}^2\rho_{2k}\left(T\right)e^{\mu_{2k}(T+t-\tau)}-\mu_{1k}^2\rho_{1k}\left(T\right)e^{\mu_{1k}(T+t-\tau)}\right)d\tau+$$

$$\left.+\frac{1}{1+\alpha_2\lambda_k^{2p}}\int\limits_0^t F_{2k-1}\left(\tau\right)\left(\mu_{2k}^2e^{\mu_{2k}(t-\tau)}-\mu_{1k}^2e^{\mu_{1k}(t-\tau)}\right)d\tau\right]+\frac{1}{1+\alpha_2\lambda_k^{2p}}F_{2k-1}\left(t\right).$$

Finally, by solving problem (17), (18) subject to (20), (21), (22) we have

$$u_{2k}\left(t\right)=\frac{1}{\chi_k}\left[\left(\rho_{1k}\left(T\right)\mu_{2k}e^{\mu_{1k}t}-\rho_{2k}\left(T\right)\mu_{1k}e^{\mu_{2k}t}\right)\varphi_{2k}+\right.$$

$$+\left(\rho_{2k}\left(T\right)e^{\mu_{2k}t}-\rho_{1k}\left(T\right)e^{\mu_{1k}t}\right)\psi_{2k}-\frac{\delta}{1+\alpha_2\lambda_k^{2p}}\times$$

$$\times\int\limits_0^T\left(F_{2k}\left(\tau\right)+b_{0k}F_{2k-1}\left(\tau\right)\right)\left(\rho_{2k}\left(T\right)e^{\mu_{2k}(T+t-\tau)}-\rho_{1k}\left(T\right)e^{\mu_{1k}(T+t-\tau)}\right)d\tau+$$

$$+\frac{1}{1+\alpha_2\lambda_k^{2p}}\int\limits_0^t\left(F_{2k}\left(\tau\right)+b_{0k}F_{2k-1}\left(\tau\right)\right)\left(e^{\mu_{2k}(t-\tau)}-e^{\mu_{1k}(t-\tau)}\right)d\tau+$$

$$\left\{b_{1k}\left(T\right)\left[\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{2k}\left(T\right)e^{\mu_{2k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-e^{\mu_{1k}t}\left(t-\delta\rho_{1k}\left(T\right)\times\right.\right.\right.$$

$$\left.\left.\times Te^{\mu_{1k}T}\right)\right]-b_{2k}\left(T\right)\left[e^{\mu_{2k}t}\left(t-\delta\rho_{2k}\left(T\right)Te^{\mu_{2k}T}\right)-\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{1k}\left(T\right)\times\right.\right.$$

$$\left.\left.\left.\times\ e^{\mu_{1k}t}\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)\right]\right]\right\}\varphi_{2k-1}+\left\{b_{3k}\left(T\right)\left[e^{\mu_{2k}t}\left(t-\delta\rho_{2k}\left(T\right)Te^{\mu_{2k}T}\right)-\frac{1}{\chi_k}\times\right.\right.$$

$$\left.\left.\times\ \left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{1k}\left(T\right)e^{\mu_{1k}t}\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)\right]\right]\right\}-b_{4k}\left(T\right)\left[\frac{1}{\chi_k}\left[e^{\mu_{2k}t}-e^{\mu_{1k}t}-\right.\right.$$

$$\left.\left.\left.-\delta\rho_{2k}\left(T\right)e^{\mu_{2k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-e^{\mu_{1k}t}\left(t-\delta\rho_{1k}\left(T\right)\cdot Te^{\mu_{1k}T}\right)\right]\right\}\psi_{2k-1}-$$

$$-\delta\int\limits_0^T F_{2k-1}(\xi)\times\Big\{b_{5k}(T)\Big[e^{\mu_{2k}(T+t-\xi)}\left(t-\delta\rho_{2k}(T)\cdot Te^{\mu_{2k}T}\right)-$$

$$-\frac{1}{\chi_k}e^{\mu_{2k}(T-\xi)}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\right.$$

$$\left.-\ \delta\rho_{1k}(T)e^{\mu_{1k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]\Big]-b_{6k}(T)\left[\frac{1}{\chi_k}e^{\mu_{1k}(T-\xi)}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\right.\right.$$

$$\left.\left.-\ e^{\mu_{2k}t}\delta\rho_{2k}(T)\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-e^{\mu_{1k}(T+t-\xi)}\left(t-\delta\rho_{1k}(T)Te^{\mu_{1k}T}\right)\right]\Big\}d\xi+$$

$$+\int\limits_0^t\Bigg\{\int\limits_0^\tau F_{2k-1}(\xi)\Big\{b_{7k}\Big[e^{\mu_{2k}(t-\xi)}\left(1-\delta\rho_{2k}(T)e^{\mu_{2k}T}\right)-e^{\mu_{1k}t+(\mu_{2k}-\mu_{1k})\tau-\mu_{2k}\xi}\times$$

$$\times\left(1-\delta\rho_{1k}(T)e^{\mu_{1k}T}\right)\Big]-b_{8k}\Big[e^{\mu_{2k}t+(\mu_{1k}-\mu_{2k})\tau-\mu_{1k}\xi}\left(1-\delta\rho_{2k}(T)e^{\mu_{2k}T}\right)-$$

$$-\ e^{\mu_{1k}(t-\xi)}\left(1-\delta\rho_{1k}(T)e^{\mu_{1k}T}\right)\Big]\Big\}d\xi\Bigg\}d\tau, \tag{22}$$

where

$$b_{0k}=-2\left(1+\alpha_2\lambda_k^{2p}\right)^{-1}p\alpha_2\lambda_k^{2p},$$

$$b_{1k}(T)=-\frac{2}{\chi_k}\mu_{2k}\left(p\alpha_2\lambda_k^{2p-1}\mu_{1k}^2+n\alpha_1\lambda_k^{2n-1}\mu_{1k}+m\alpha_0\lambda_k^{2m-1}\right)\rho_{2k}(T),$$

$$b_{2k}(T)=-\frac{2}{\chi_k}\mu_{1k}\left(p\alpha_2\lambda_k^{2p-1}\mu_{2k}^2+n\alpha_1\lambda_k^{2n-1}\mu_{2k}+m\alpha_0\lambda_k^{2m-1}\right)\rho_{1k}(T),$$

$$b_{3k}(T)=-\frac{2}{\chi_k}\left(p\alpha_2\lambda_k^{2p-1}\mu_{2k}^2+n\alpha_1\lambda_k^{2n-1}\mu_{2k}+m\alpha_0\lambda_k^{2m-1}\right)\rho_{2k}(T),$$

$$b_{4k}(T)=-\frac{2}{\chi_k}\left(p\alpha_2\lambda_k^{2p-1}\mu_{1k}^2+n\alpha_1\lambda_k^{2n-1}\mu_{1k}+m\alpha_0\lambda_k^{2m-1}\right)\rho_{1k}(T),$$

$$b_{5k}(T)=-\left(1+\alpha_2\lambda_k^{2p}\right)^{-1}b_{3k}(T),\ b_{6k}(T)=-\left(1+\alpha_2\lambda_k^{2p}\right)^{-1}b_{4k}(T),$$

$$b_{7k}=-\frac{2}{\chi_k}\cdot\mu_{2k}\left(p\alpha_2\lambda_k^{2p-1}\mu_{1k}^2+n\alpha_1\lambda_k^{2n-1}\mu_{1k}+m\alpha_0\lambda_k^{2m-1}\right)\left(1+\alpha_2\lambda_k^{2p}\right)^{-1},$$

$$b_{8k}=-\frac{2}{\chi_k}\cdot\mu_{1k}\left(p\alpha_2\lambda_k^{2p-1}\mu_{2k}^2+n\alpha_1\lambda_k^{2m-1}\mu_{2k}+m\alpha_0\lambda_k^{2m-1}\right)\left(1+\alpha_2\lambda_k^{2p}\right)^{-1}. \tag{23}$$

Now by differentiating (23) two times with respect to t we obtain

$$u'_{2k}(t)=\frac{1}{\chi_k}\left[\mu_{1k}\mu_{2k}\left(\rho_{1k}(T)e^{\mu_{1k}t}-\rho_{2k}(T)e^{\mu_{2k}t}\right)\varphi_{2k}+\right.$$

$$+\left(\mu_{2k}\rho_{2k}(T)e^{\mu_{2k}t}-\mu_{1k}\rho_{1k}(T)e^{\mu_{1k}t}\right)\psi_{2k}-\frac{\delta}{1+\alpha_2\lambda_k^{2p}}\int\limits_0^T\left(F_{2k}(\tau)+b_{0k}F_{2k-1}(\tau)\right)\times$$

$$\times\left(\mu_{2k}\rho_{2k}(T)e^{\mu_{2k}(T+t-\tau)}-\mu_{1k}\rho_{1k}(T)e^{\mu_{1k}(T+t-\tau)}\right)d\tau+\frac{1}{1+\alpha_2\lambda_k^{2p}}\times$$

$$\times\int_0^t \left(F_{2k}(\tau)+b_{0k}F_{2k-1}(\tau)\right)\left(\mu_{2k}e^{\mu_{2k}(t-\tau)}-\mu_{1k}e^{\mu_{1k}(t-\tau)}\right)d\tau+$$

$$+\left\{b_{1k}(T)\left[\mu_{2k}\cdot\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{2k}(T)\,e^{\mu_{2k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-\right.\right.$$

$$-\mu_{1k}e^{\mu_{1k}t}\left(t-\delta\rho_{1k}(T)\cdot Te^{\mu_{1k}T}\right)\Big]-b_{2k}(T)\left[\mu_{2k}e^{\mu_{2k}t}\left(t-\delta\rho_{2k}(T)\,Te^{\mu_{2k}T}\right)-\right.$$

$$\left.\left.-\ \mu_{1k}\cdot\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{1k}(T)\,e^{\mu_{1k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]\right]\right\}\varphi_{2k-1}+$$

$$+\left\{b_{3k}(T)\left[\mu_{2k}e^{\mu_{2k}t}\left(t-\delta\rho_{2k}(T)\cdot Te^{\mu_{2k}T}\right)-\mu_{1k}\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\right.\right.\right.$$

$$-\ \delta\rho_{1k}(T)\,e^{\mu_{1k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\Big]\Big]-b_{4k}(T)\left[\mu_{2k}\cdot\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)\ -\right.\right.$$

$$\left.\left.-\ \delta\rho_{2k}(T)\,e^{\mu_{2k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-\mu_{1k}e^{\mu_{1k}t}\left(t-\delta\rho_{1k}(T)\,Te^{\mu_{1k}T}\right)\right]\Big\}\,\psi_{2k-1}-$$

$$-\delta\int_0^T F_{2k-1}(\xi)\left\{b_{5k}(T)\left[\mu_{2k}e^{\mu_{2k}(T+t-\xi)}\left(t-\delta\rho_{2k}(T)\,Te^{\mu_{2k}T}\right)\ -\right.\right.$$

$$\left.-\ \mu_{1k}\frac{e^{\mu_{2k}(T-\xi)}}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{1k}(T)\,e^{\mu_{1k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]\right]-$$

$$-b_{6k}(T)\left[\mu_{2k}\frac{e^{\mu_{1k}(T-\xi)}}{\xi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-e^{\mu_{2k}t}\delta\rho_{2k}(T)\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-\right.$$

$$\left.\left.-\ \mu_{1k}e^{\mu_{1k}(T+t-\xi)}\left(t-\delta\rho_{1k}(T)\,Te^{\mu_{1k}T}\right)\right]\right\}d\xi+$$

$$+\int_0^t\left\{\int_0^\tau F_{2k-1}(\xi)\left\{b_{7k}\left[\mu_{2k}e^{\mu_{2k}(t-\xi)}\left(1-\delta\rho_{2k}(T)\,e^{\mu_{2k}T}\right)-\right.\right.\right.$$

$$\left.-\ \mu_{1k}e^{\mu_{1k}t+(\mu_{2k}-\mu_{1k})\tau-\mu_{2k}\xi}\left(1-\delta\rho_{1k}(T)\,e^{\mu_{1k}T}\right)\right]-b_{8k}\left[\mu_{2k}e^{\mu_{2k}t+(\mu_{1k}-\mu_{2k})\tau-\mu_{1k}\xi}\times\right.$$

$$\left.\left.\left.\times\ \left(1-\delta\rho_{2k}(T)\,e^{\mu_{2k}T}\right)-\mu_{1k}e^{\mu_{1k}(t-\xi)}\left(1-\delta\rho_{1k}(T)\,e^{\mu_{1k}T}\right)\right]\right\}d\xi\right\}d\tau, \quad (25)$$

$$u''_{2k}(t)=\frac{1}{\chi_k}\left[\mu_{1k}\mu_{2k}\left(\mu_{1k}\rho_{1k}(T)\,e^{\mu_{1k}t}-\mu_{2k}\rho_{2k}(T)\,e^{\mu_{2k}t}\right)\varphi_{2k}\ +\right.$$

$$+\left(\mu_{2k}^2\rho_{2k}(T)\,e^{\mu_{2k}t}-\mu_{1k}^2\rho_{1k}(T)\,e^{\mu_{1k}t}\right)\psi_{2k}-\frac{\delta}{1+\alpha_2\lambda_k^{2p}}\int_0^T\left(F_{2k}(\tau)+b_{0k}F_{2k-1}(\tau)\right)\times$$

$$\times\left(\mu_{2k}^2\rho_{2k}(T)\,e^{\mu_{2k}(T+t-\tau)}-\mu_{1k}^2\rho_{1k}(T)\,e^{\mu_{1k}(T+t-\tau)}\right)d\tau+\frac{1}{1+\alpha_2\lambda_k^{2p}}\times$$

$$\left.\times\int_0^t\left(F_{2k}(\tau)+b_{0k}F_{2k-1}(\tau)\right)\left(\mu_{2k}^2e^{\mu_{2k}(t-\tau)}-\mu_{1k}^2e^{\mu_{1k}(t-\tau)}\right)d\tau\right]+$$

$$+\left\{b_{1k}(T)\left[\mu_{2k}^2\cdot\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{2k}(T)\,e^{\mu_{2k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-\right.\right.$$

$$-e^{\mu_{1k}t}\left(\left(t-\delta\rho_{1k}(T)\cdot Te^{\mu_{1k}T}\right)\mu_{1k}^2+1\right)\Big]-b_{2k}(T)\left[e^{\mu_{2k}t}\left(1+\mu_{2k}^2\left(t-\rho_{2k}(T)\,Te^{\mu_{2k}T}\right)\right)-\right.$$

$$\left.\left.-\ \mu_{1k}^2\cdot\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{1k}(T)\,e^{\mu_{1k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]\right]\right\}\varphi_{2k-1}+$$

$$+\left\{b_{3k}(T)\left[e^{\mu_{2k}t}\left(1+\mu_{2k}^2\left(t-\rho_{2k}(T)\cdot Te^{\mu_{2k}T}\right)\right)-\mu_{1k}^2\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\right.\right.\right.$$

$$\left.\left.-\ \delta\rho_{1k}(T)\,e^{\mu_{1k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]\right]-b_{4k}(T)\left[\mu_{2k}^2\cdot\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)\ -\right.\right.$$

$$\left.\left.-\ \delta\rho_{2k}(T)\,e^{\mu_{2k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-e^{\mu_{1k}t}\left(\mu_{1k}^2\left(t-\delta\rho_{1k}(T)\,Te^{\mu_{1k}T}\right)+1\right)\right]\psi_{2k-1}-$$

$$-\delta\int_0^T F_{2k-1}(\xi)\left\{b_{5k}(T)\left[e^{\mu_{2k}(T+t-\xi)}\left(1+\mu_{2k}^2\left(t-\delta\rho_{2k}(T)\,Te^{\mu_{2k}T}\right)\right)\ -\right.\right.$$

$$\left.-\ \mu_{1k}^2e^{\mu_{2k}(T-\xi)}\cdot\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{1k}(T)\,e^{\mu_{1k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]\right]-$$

$$-b_{6k}(T)\left[\mu_{2k}^2e^{\mu_{1k}(T-\xi)}\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-e^{\mu_{2k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-\right.$$

$$\left.\left.-\ e^{\mu_{1k}(T+t-\xi)}\left[\mu_{1k}^2\left(t-\delta\rho_{1k}(T)\cdot Te^{\mu_{1k}T}\right)+1\right]\right]\right\}+$$

$$+\int_0^t\left\{\int_0^\tau F_{2k-1}(\xi)\left\{b_{7k}\left[\mu_{2k}^2e^{\mu_{2k}(t-\xi)}\left(1-\delta\rho_{2k}(T)\,e^{\mu_{2k}T}\right)-\right.\right.\right.$$

$$\left.-\ \mu_{1k}^2e^{\mu_{1k}t+(\mu_{2k}-\mu_{1k})\tau-\mu_{2k}\xi}\left(1-\delta\rho_{1k}(T)\,e^{\mu_{1k}T}\right)\right]-b_{8k}\left[\mu_{2k}^2e^{\mu_{2k}t+(\mu_{1k}-\mu_{2k})\tau-\mu_{1k}\xi}\times\right.$$

$$\left.\left.\left.\times\ \left(1-\delta\rho_{2k}(T)\,e^{\mu_{2k}T}\right)-\mu_{1k}^2e^{\mu_{1k}(t-\xi)}\left(1-\delta\rho_{1k}(T)\,e^{\mu_{1k}T}\right)\right]\right\}d\xi\right\}d\tau+$$

$$+\int_0^t F_{2k-1}(\tau)\left(b_{7k}e^{\mu_{2k}(t-\tau)}-b_{8k}e^{\mu_{1k}(t-\tau)}\right)d\tau+$$

$$+\frac{1}{1+\alpha_2\lambda_k^{2p}}F_{2k}(t)-\frac{2p\alpha_2\lambda_k^{2p-1}}{1+\alpha_2\lambda_k^{2p}}\left(1+\frac{\lambda_k}{1+\alpha_2\lambda_k^{2p}}\right)F_{2k-1}(t),\qquad(24)$$

Let's assume the notations

$$\vartheta_{2k-1}(t)=\frac{1}{\chi_k}\left[\mu_{1k}\mu_{2k}\left(\mu_{1k}\rho_{1k}(T)\,e^{\mu_{1k}t}-\mu_{2k}\rho_{2k}(T)\,e^{\mu_{2k}t}\right)\varphi_{2k-1}\ +\right.$$

$$+\left(\mu_{2k}^2\rho_{2k}(T)\,e^{\mu_{2k}t}-\mu_{1k}^2\rho_{1k}(T)\,e^{\mu_{1k}t}\right)\psi_{2k-1}-\frac{\delta}{1+\alpha_2\lambda_k^{2p}}\times$$

$$\times\int_0^T F_{2k}(\tau)\int_0^T F_{2k}(\tau)\left(\mu_{2k}^2\rho_{2k}(T)\,e^{\mu_{2k}(T+t-\tau)}-\mu_{1k}^2\rho_{1k}(T)\,e^{\mu_{1k}(T+t-\tau)}\right)d\tau+$$

$$+\frac{1}{1+\alpha_2\lambda_k^{2p}}\int\limits_0^t F_{2k-1}(\tau)\left(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)}-\mu_{1k}^2 e^{\mu_{1k}(t-\tau)}\right)d\tau\Bigg]+\frac{\alpha_2\lambda_k^{2p}}{1+\alpha_2\lambda_k^{2p}}F_{2k-1}(t), \tag{25}$$

$$\vartheta_{2k}(t)=\frac{1}{\chi_k}\left[\mu_{1k}\mu_{2k}\left(\mu_{1k}\rho_{1k}(T)e^{\mu_{1k}t}-\mu_{2k}\rho_{2k}(T)e^{\mu_{2k}t}\right)\varphi_{2k}+\right.$$

$$+\left(\mu_{2k}^2\rho_{2k}(T)e^{\mu_{2k}t}-\mu_{1k}^2\rho_{1k}(T)e^{\mu_{1k}t}\right)\psi_{2k}-\frac{\delta}{1+\alpha_2\lambda_k^{2p}}\times$$

$$\times\int\limits_0^t F_{2k}(\tau)\left(\mu_{2k}^2\rho_{2k}(T)e^{\mu_{2k}(T+t-\tau)}-\mu_{1k}^2\rho_{1k}(T)e^{\mu_{1k}(T+t-\tau)}\right)d\tau+\frac{1}{1+\alpha_2\lambda_k^{2p}}\times$$

$$\left.\times\int\limits_0^t F_{2k}(\tau)\left(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)}-\mu_{1k}^2 e^{\mu_{1k}(t-\tau)}\right)d\tau\right]+$$

$$+\left\{b_{1k}(T)\left[\mu_{2k}^2\cdot\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{2k}(T)e^{\mu_{2k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-\right.\right.$$

$$\left.-e^{\mu_{1k}t}\left[\mu_{1k}^2\left(t-\delta\rho_{1k}(T)Te^{\mu_{1k}T}\right)+1\right]\right]-b_{2k}(T)\times$$

$$\times\left[e^{\mu_{2k}t}\left[\mu_{2k}^2\left(t-\delta\rho_{2k}(T)Te^{\mu_{2k}T}\right)+1\right]-\mu_{1k}^2\cdot\frac{1}{\chi_k}\times\right.$$

$$\left.\left.\times\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{1k}(T)e^{\mu_{1k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]\right]\right\}\varphi_{2k-1}+$$

$$+\left\{b_{3k}(T)\left[e^{\mu_{2k}t}\left[\mu_{2k}^2\left(t-\delta\rho_{2k}(T)e^{\mu_{2k}T}\right)+1\right]-\mu_{1k}^2\cdot\frac{1}{\chi_k}\right.\right.$$

$$\left.\times\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{1k}(T)e^{\mu_{1k}t}\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)\right]\right]-b_{4k}(T)\times$$

$$\times\left[\mu_{2k}^2\cdot\frac{1}{\chi k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{2k}(T)Te^{\mu_{2k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-\right.$$

$$\left.\left.-e^{\mu_{1k}t}\left[\mu_{1k}^2\left(t-\delta\rho_{1k}(T)Te^{\mu_{1k}T}\right)+1\right]\right]\right\}\psi_{2k-1}-\delta\int\limits_0^t F_{2k-1}(\xi)\times$$

$$\times\left\{b_{5k}(T)\left[e^{\mu_{2k}(T+t-\xi)}\left[\mu_{2k}^2\left(t-\delta\rho_{2k}(T)Te^{\mu_{2k}T}\right)+1\right]-\right.\right.$$

$$\left.-\mu_{1k}^2\frac{1}{\chi_k}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{1k}(T)e^{\mu_{1k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]\right]-$$

$$-b_{6k}(T)\left[\mu_{2k}^2\frac{1}{\chi_k}e^{\mu_{1k}(T-\xi)}\left[\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)-\delta\rho_{2k}(T)e^{\mu_{2k}t}\left(e^{\mu_{2k}T}-e^{\mu_{1k}T}\right)\right]-\right.$$

$$\left.\left.-e^{\mu_{1k}(T+t-\xi)}\left[\mu_{1k}^2\left(t-\delta\rho_{1k}(T)Te^{\mu_{1k}T}\right)+1\right]\right]\right\}d\xi+$$

$$+\int\limits_0^t\left\{\int\limits_0^\tau F_{2k-1}(\xi)\left\{b_{7k}\left[\mu_{1k}^2 e^{\mu_{2k}(t-\xi)}\left(1-\delta\rho_{2k}(T)e^{\mu_{2k}T}\right)-\right.\right.\right.$$

$$\left.-\mu_{1k}^2 e^{\mu_{1k}t+(\mu_{2k}-\mu_{1k})\tau-\mu_{2k}\xi}\left(1-\delta\rho_{1k}(T)e^{\mu_{1k}T}\right)\right]-$$

$$-b_{8k}\left[\mu_{1k}^2 e^{\mu_{2k}t+(\mu_{1k}-\mu_{2k}\tau)-\mu_{1k}\xi}\left(1-\delta\rho_{2k}(T)\,e^{\mu_{2k}T}\right)-\right.$$

$$\left.\left.\left.-\mu_{1k}^2 e^{\mu_{1k}(t-\xi)}\left(1-\delta\rho_{1k}(T)\,e^{\mu_{1k}T}\right)\right]\right\}d\xi\right\}d\tau+\int_0^t F_{2k-1}(\tau)\times$$

$$\times\left(b_{7k}e^{\mu_{2k}(t-\tau)}-b_{8k}e^{\mu_{1k}(t-\tau)}\right)d\tau+\frac{\alpha_2\lambda_k^{2p}}{1+\alpha_2\lambda_k^{2p}}F_{2k}(t)-$$

$$-\frac{2p\alpha_2\lambda_k^{2p-1}}{1+\alpha_2\lambda_k^{2p}}\left(1+\frac{\lambda_k}{1+\alpha_2\lambda_k^{2p}}\right)F_{2k-1}(t). \tag{26}$$

It's obvious that

$$u_k''(t)=\vartheta_k(t)+F_k(t)\quad(k=1,2,...) \tag{27}$$

Now in order to obtain the equation for the second component $a(t)$ of the classical solution $\{u(x,t),a(t)\}$ of problem (1)-(4) we substitute expression (11) subject to the relations (19), (20) and (23) in condition (4)

$$\sum_{k=0}^{\infty}u_k(t)\,X_k(x_0)=h(t). \tag{28}$$

2. Let $h(t)\in c^2[0,T]$ **and** $h(t)\neq 0\ \forall t\in[0,T]$.

Then by differentiating (30) two times with respect to t we have

$$\sum_{k=0}^{\infty}u_k''(t)\,X_k(x_0)=h''(t).$$

Hence subject to (29) we obtain

$$\sum_{k=0}^{\infty}F_k(t)\,X_k(x_0)+\sum_{k=1}^{\infty}\vartheta_k(t)\,X_k(x_0)=h''(t). \tag{29}$$

It's clear that

$$\sum_{k=0}^{\infty}F_k(t)\,X_k(x_0)=a(t)\sum_{k=1}^{\infty}u_k(t)\,X_k(x_0)+\sum_{k=0}^{\infty}f_k(t)\,X_k(x_0).$$

Then from (31), subject to (31) we find

$$a(t)=h^{-1}(t)\left(h''(t)-\sum_{k=0}^{\infty}f_k(t)\,X_k(x_0)-\sum_{k=1}^{\infty}\vartheta_k(t)\,X_k(x_0)\right). \tag{30}$$

Thus, the solution of problem (1)-(4) is reduced to the solution of system (11), (32), where $u_0(t)$, $u_{2k-1}(t)$, $u_{2k}(t)$ are determined by the relations (19), (20), (23) respectively.

The following one is valid.

Theorem. Let conditions 1, 2 and

3. $\varphi(x) \in c^{2(m+n-p)}[0,1]$, $\varphi^{2(m+n-p)+1}(x) \in L_2(0,1)$,
$\varphi^{(2s)}(0) = 0$, $\varphi^{(2s+1)}(0) = \varphi^{(2s+1)}(1)$, $\left(s = \overline{0, m+n-p}\right)$.
4. $\psi(x) \in c^{2(m+n-p)-1}[0,1]$, $\psi^{2(m+n-p)}(x) \in L_2(0,1)$,
$\psi^{(2s)}(0) = 0$, $\psi^{(2s+1)}(0) = \psi^{(2s+1)}(1)$, $\left(s = \overline{0, m+n-p}\right)$.
5. $f(x,t) \in c^{(2(m+n-p)-1,0)}\left(\overline{D}_T\right)$, $\dfrac{\partial^{2(m+n-p)}}{\partial x^{2(m+n-p)}} f(x,t) \in L_2(D_T)$,
$\dfrac{\partial^{2s}}{\partial x^{2s}} f(0,t) = 0$, $\dfrac{\partial^{2s+1}}{\partial x^{2s+1}} f(0,t) = \dfrac{\partial^{2s+1}}{\partial x^{2s+1}} f(1,t)$, $\left(s = \overline{0, m+n-p}\right)$.
6. $h(0) + \delta h(T) = \varphi(x_0)$, $h'(0) + \delta h'(T) = \psi(x_0)$
be satisfied.

Then for the sufficiently small values T and $\left\|h^{-1}(t)\right\|_{C[0,T]}$ problem (1)-(4) has a unique classical solution.

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Gambar K. Namazov
Institute of Mathematics & Mechanics of NAS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.:39-47-20(off.).

Yashar T. Mekhraliyev
Baku State University.
23, Z.I.Khalilov str., 370148, Baku, Azerbaijan.

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