[1]**Azer Kh. KHANMAMEDOV**

EXISTENCE OF A GLOBAL ATTRACTOR FOR THE PLATE EQUATION IN AN UNBOUNDED DOMAIN

Abstract

In the paper the asymptotical behaviour of solutions for the plate equation in R^n is studied.The existence of global attractor in $W_2^2(R^n) \times L_2(R^n)$ is proved.

The subject of investigation of the present paper is the question on the existence of global attractor for the following Cauchy problem:

$$\vartheta_{tt} + \alpha\vartheta_t + \Delta^2\vartheta + \lambda\vartheta + f(\vartheta) = g(x), \quad (t,x) \in R_+ \times R^N, \tag{1}$$

$$\vartheta(0,x) = \vartheta_0(x) \ , \qquad \vartheta_t(0,x) = \vartheta_1(x), \quad x \in R^N, \tag{2}$$

where $\alpha > 0,\ \lambda > 0,\ g \in L_2\left(R^N\right)$ are given and non-linear function $f(\cdot)$ satisfies the following conditions:

$$f \in C^1(R), \ \left|f'(u)\right| \le C \cdot (1 + |u|^p), \quad p > 0, \quad (N-4)p < 4, \tag{3}$$

$$f(u) \cdot u \ge \delta \int_0^u f(s)\,ds \ge 0, \quad for \quad some \quad \delta > 0. \tag{4}$$

The existence of global attractor of initial-boundary value problem for the equation (1) in a bounded domain is studied in [1], where the asymptotical compactness of solutions follows directly from the compactness of imbedding of the Sobolev spaces. Since the imbedding of the Sobolev spaces in unbounded domains, in general, isn't compact, then we can't apply the method used in [1] in our case.

The basic aim of the present paper is the proof of asymptotical compactness of solutions which implies the existence of global attractor.

Denote the spaces $W_2^s\left(R^N\right)$ and $L_2\left(R^N\right)$ by $H^s\,(s \ne 0)$ and H respectively, and the norms in H^s and H by $\|\cdot\|_s$ and $\|\cdot\|$ respectively.

By the transformation $\theta = \begin{pmatrix} \vartheta \\ \vartheta_t \end{pmatrix}$ the problem (1)-(2) in the space $H^2 \times H$ is reduced to the following problem

$$\theta'(t) = A\theta(t) + F(\theta(t)) + G, \quad t \in R_+, \tag{5}$$

$$\theta(0) = \theta_0, \tag{6}$$

where

$$A = \begin{pmatrix} 0 & I \\ -\Delta^2 - \lambda I & -\alpha I \end{pmatrix}, \ D(A) = H^4 \times H^2, \ F(\theta(t)) = \begin{pmatrix} 0 \\ -f(\vartheta(t)) \end{pmatrix},$$

[1]Author was supported by TUBITAK NATO-PC B Programm

$$G = \begin{pmatrix} 0 \\ g \end{pmatrix}, \ \theta_0 = \begin{pmatrix} \vartheta_0 \\ \vartheta_1 \end{pmatrix}.$$

The problem (5)-(6) for any $\theta_0 \in H^2 \times H$ under the conditions (3)-(4) has the unique solution $\theta(\cdot) \in C\left([0,\infty); H^2 \times H\right)$ (see, for example [2]), which satisfies the following integral equation

$$\theta(t) = e^{tA}\theta_0 + \int\limits_0^t e^{(t-s)A}\left(F(\theta(s)) + G\right) ds. \tag{7}$$

Hence the problem (1)-(2) in the space $H^2 \times H$ generates strongly continuous non-linear semi-group $V(t) \ \ (t \geq 0)$ such that $\begin{pmatrix} \vartheta(t) \\ \vartheta_t(t) \end{pmatrix} = V(t)\begin{pmatrix} \vartheta_0 \\ \vartheta_1 \end{pmatrix}$, where $\vartheta(t)$ is a solution of the problem (1)-(2).

Multiplying (1) by $\vartheta_t + \varepsilon_0 \cdot \vartheta$ and integrating over R^N we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\vartheta_t\|^2 + \|\Delta\vartheta\|^2 + 2\varepsilon_0 \int\limits_{R^N} \vartheta_t \cdot \vartheta dx + \right. \\ &\left. +\alpha \cdot \varepsilon_0 \|\vartheta\|^2 + \lambda \|\vartheta\|^2 + 2\int\limits_{R^N} \Phi(\vartheta)\, dx\right) + \\ &+(\alpha - \varepsilon_0)\|\vartheta_t\|^2 + \varepsilon_0\left(\|\Delta\vartheta\|^2 + \lambda\|\vartheta\|^2\right) + \\ &+\varepsilon_0 \int\limits_{R^N} f(\vartheta) \cdot \vartheta dx = \int\limits_{R^N} g \cdot (\vartheta_t + \varepsilon_0\vartheta)\, dx, \end{aligned} \tag{8}$$

where $\Phi(t) = \int\limits_0^t f(s)\, ds$.

By virtue of (4), (8) we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\vartheta_t\|^2 + \|\Delta\vartheta\|^2 + 2\varepsilon_0 \int\limits_{R^N} \vartheta_t \cdot \vartheta dx + \right. \\ &\left. +\alpha \cdot \varepsilon_0 \|\vartheta\|^2 + \lambda \|\vartheta\|^2 + 2\int\limits_{R^N} \Phi(\vartheta)\, dx\right) + \\ &+\left(\frac{3\alpha}{4} - \varepsilon_0\right)\|\vartheta_t\|^2 + \varepsilon_0\|\Delta\vartheta\|^2 + \varepsilon_0\left(\lambda - \frac{\varepsilon_0}{4}\right)\|\vartheta\|^2 + \\ &+\varepsilon_0 \cdot \delta \int\limits_{R^N} \Phi(\vartheta)\, dx \leq \left(1 + \frac{1}{\alpha}\right) \cdot \|g\|^2. \end{aligned}$$

Choosing $\varepsilon_0 \in \left(0, \min\left\{4\lambda, \frac{\sqrt{\lambda}}{2}, \frac{3\alpha}{4}\right\}\right)$ and denoting

$$\omega = \frac{1}{4}\min\left\{\frac{3\alpha}{4} - \varepsilon_0, \frac{\varepsilon_0\left(4\lambda - \varepsilon_0\right)}{4\left(\lambda + \alpha\varepsilon_0\right)}, 2\varepsilon_0, \varepsilon_0\delta\right\}$$

we obtain from the last inequality

$$\frac{d}{dt}\left(\|\vartheta_t\|^2 + \|\Delta\vartheta\|^2 + 2\varepsilon_0\int\limits_{R^N}\vartheta_t\cdot\vartheta dx + \alpha\cdot\varepsilon_0\|\vartheta\|^2 + \lambda\|\vartheta\|^2 + 2\int\limits_{R^n}\Phi(\vartheta)\,dx\right) +$$

$$+2\omega\left(\|\vartheta_t\|^2 + 2\varepsilon_0\int\limits_{R^N}\vartheta_t\cdot\vartheta dx + \|\Delta\vartheta\|^2 + \lambda\|\vartheta\|^2 + \right.$$

$$\left. + \alpha\cdot\varepsilon_0\|\vartheta\|^2 + 2\int\limits_{R^n}\Phi(\vartheta)\,dx\right) \le \frac{2(\alpha+1)}{\alpha}\cdot\|g\|^2. \qquad (9)$$

Let $B \subset H^2 \times H$ be a bounded set. Then from (9) according to (3)-(4) for $\forall\theta_0 \in B$ we have

$$\|V(t)\theta_0\|_{H^2\times H} \le C\left(\|B\|_{H^2\times H}\right)\cdot e^{-\omega t} + L, \quad t > 0 \qquad (10)$$

where

$$\|B\|_{H^2\times H} = \sup_{u\in B}\|u\|_{H^2\times H}, \qquad L = \sqrt{\frac{\alpha+1}{\omega\alpha}}\cdot\|g\|.$$

In particular from (9) the inequality

$$\left\|e^{tA}\theta_0\right\|_{H^2\times H} \le M\cdot e^{-\omega t}\cdot\|\theta_0\|_{H^2\times H}, \quad t > 0, \qquad (11)$$

also follows and consequently

$$\left\|e^{tA}\theta_0\right\|_{H^s\times H^{s-2}} \le M\cdot e^{-\omega t}\cdot\|\theta_0\|_{H^s\times H^{s-2}}, \quad t > 0, \qquad (12)$$

Now consider the following sequence of the problems:

$$\vartheta_{tt}^{(m)} + \alpha\vartheta_t^{(m)} + \Delta^2\vartheta^{(m)} + \lambda\vartheta^{(m)} + f\left(\vartheta^{(m-1)}\right) = g(x), \ (t,x) \in R_+\times R^N, \quad (13)_m$$

$$\vartheta^{(m)}(0,x) = 0, \qquad \vartheta_t^{(m)}(0,x) = 0, \qquad x \in R^N, \qquad (14)_m$$

where $\vartheta^0(t,x) \equiv \vartheta(t,x)$ is a solution of the problem (1)-(2).

To prove the asymptotical compactness of solutions of the problem (1)-(2) we use the following lemmas.

Lemma 1. *Let the conditions (3)-(4) are satisfied. Then there exists* $m_0 \ge 1$ *such that for* $\forall m \ge m_0$ *and* $\forall\begin{pmatrix}\vartheta_0\\ \vartheta_1\end{pmatrix} \in B$

$$\left\|\vartheta^{(m)}\right\|_4 + \left\|\vartheta_t^{(m)}\right\|_2 + \left\|\vartheta_{tt}^{(m)}\right\| \le C_m\left(\|B\|_{H^2\times H}, \|g\|\right), \quad t > 0 \qquad (15)$$

holds.

Proof. Differentiating $(13)_m$ with respect to t, taking into account $(14)_m$ we have

$$\begin{cases} \vartheta_{ttt}^{(m)}+\alpha\vartheta_{tt}^{(m)}+\Delta^2\vartheta_t^{(m)}+\lambda\vartheta_t^{(m)}+f'\left(\vartheta^{(m-1)}\right)\vartheta_t^{(m-1)}=0, \\ \vartheta_t^{(m)}(0,x)=0, \;\; \vartheta_{tt}^{(m)}(0,x)=g(x). \end{cases}$$

Using the representation of solutions we obtain

$$\begin{pmatrix} \vartheta_t^{(m)} \\ \vartheta_{tt}^{(m)} \end{pmatrix}=e^{tA}\begin{pmatrix} 0 \\ g \end{pmatrix}+\int_0^t e^{(t-s)A}\begin{pmatrix} 0 \\ f'\left(\vartheta^{(m-1)}\right)\vartheta_t^{(m-1)} \end{pmatrix}ds,\; t>0. \qquad (16)_m$$

Using the relations (3) and (10) it's easy to check that for $\forall t\geq 0$ the inequalities

$$\left\|f'(\vartheta)\vartheta_t\right\|\leq C\left(\|B\|_{H^2\times H},\|g\|\right), \quad if \quad N\leq 3, \qquad (17)$$

and

$$\left\|f'(\vartheta)\vartheta_t\right\|_{-2+\delta}\leq C\left(\|B\|_{H^2\times H},\|g\|\right), \quad if \quad N\geq 4, \qquad (18)$$

hold, where

$$\begin{cases} 0<\delta<2, & \text{if} \quad n=4, \\ \delta=\frac{4-p(n-4)}{4}, & \text{if} \quad n>4. \end{cases}$$

If $N\leq 3$, then estimates (11) and (17) allow to derive the estimate (15) from $(16)_m$ for $\forall m\geq 1$. If $N\geq 4$, then according to (11), (12) and (18) from $(16)_1$ we have

$$\left\|\vartheta_t^{(1)}\right\|_\delta+\left\|\vartheta_{tt}^{(1)}\right\|_{-2+\delta}\leq \tilde{c}\left(\|B\|_{H^2\times H},\|g\|\right), \quad t\geq 0.$$

Allowing for the last inequality in $(13)_1$, we obtain

$$\left\|\vartheta^{(1)}\right\|_{2+\delta}+\left\|\vartheta_t^{(1)}\right\|_\delta+\left\|\vartheta_{tt}^{(1)}\right\|_{-2+\delta}\leq \tilde{c}_1\left(\|B\|_{H^2\times H},\|g\|\right), \quad t\geq 0.$$

Using this estimate in $(16)_2$ we obtain more smooth estimate for $\vartheta^{(2)}(t,x)$. Thus, the application of this proceedure after finite number of steps gives us the estimate (15).

Lemma 2. *Assume that (3)-(4) hold. Then the estimate*

$$\left\|\vartheta^{(m)}-\vartheta\right\|_2+\left\|\vartheta_t^{(m)}-\vartheta_t\right\|\leq$$

$$\leq C_m^1\left(\|B\|_{H^2\times H},\|g\|\right)\left(1+t+\cdots t^{m-1}\right)e^{-\omega t},\; t\geq 0 \qquad (19)$$

is valid for $\forall m\geq 1$ *and* $\forall\begin{pmatrix} \vartheta_0 \\ \vartheta_1 \end{pmatrix}\in B$.

Proof. According to $(13)_1$-$(14)_1$ we conclude that

$$\begin{pmatrix} \vartheta \\ \vartheta_t \end{pmatrix}-\begin{pmatrix} \vartheta^{(1)} \\ \vartheta_t^{(1)} \end{pmatrix}=e^{tA}\begin{pmatrix} \vartheta_0 \\ \vartheta_1 \end{pmatrix}.$$

Then by virtue of (11) we have:

$$\left\|\vartheta^{(1)}-\vartheta\right\|_2+\left\|\vartheta_t^{(1)}-\vartheta_t\right\|\le M\left(\|\vartheta_0\|_2+\|\vartheta_1\|\right)\cdot e^{-\omega t},\quad t>0.$$

Using the equation (7) and the representation of solutions of the problem $(13)_m$-$(14)_m$ taking into account (3), (10) and (11) we obtain

$$\left\|\vartheta^{(m)}-\vartheta\right\|_2+\left\|\vartheta_t^{(1)}-\vartheta_t\right\|\le M\left(\|\vartheta_0\|_2+\|\vartheta_1\|\right)\cdot e^{-\omega t}+$$

$$+M\cdot\int_0^t e^{-\omega(t-s)}\cdot\left\|f\left(\vartheta^{(m-1)}(s)\right)-\right.$$

$$\left.-f\left(\vartheta(s)\right)\right\|ds\le M\left(\|\vartheta_0\|_2+\|\vartheta_1\|\right)\cdot e^{-\omega t}+\approx{c}_{m-1}\left(\|B\|_{H^2\times H},\|g\|\right)\times$$

$$\times\int_0^t e^{-\omega(t-s)}\left\|\vartheta^{(m-1)}(s)-\vartheta(s)\right\|_2 ds.$$

Further we obtain the estimate (19) by using the method of mathematical induction.

Now we prove the asymptotical compactness of the semigroup $V(t)\quad(t\ge 0)$ in the space $H^2\times H$.

Theorem 3. *Assume (3)-(4) hold. Then for any bounded set $B\subset H^2\times H$ the set $\{V(t_n)\theta_n\}_{n=1}^{\infty}$ is $H^2\times H$ precompact in $H^2\times H$ where $\{\theta_n\}_{n=1}^{\infty}\subset B,\ t_n\to+\infty$.*

Proof. Denote the operator generated by the problem $(14)_m$ by $V^m(t)$, i.e.

$$V^m(t)\begin{pmatrix}\vartheta_0\\ \vartheta_1\end{pmatrix}\equiv\begin{pmatrix}\vartheta^{(m)}\\ \vartheta_t^{(m)}\end{pmatrix}=\int_0^t e^{(t-s)A}\begin{pmatrix}0\\ -f\left(\vartheta^{(m-1)}(s)\right)+g\end{pmatrix}ds.$$

According to lemma 2 for the proving of the theorem it's sufficient to show the precompactness $\{V^{m_0}(t_n)\theta_n\}_{n=1}^{\infty}$.

Let $\varphi(\cdot)\in C^{\infty}\left(R^N\right)$ such that $0\le\varphi(x)\le 1$ and $\varphi(x)=\begin{cases}1, & |x|>2,\\ 0, & |x|<1.\end{cases}$

Multiplying $(13)_{m_0}$ by $\varphi\left(\frac{x}{k}\right)\left(\vartheta_t^{(m_0)}+\varepsilon_0\vartheta^{(m_0)}\right)$ and integrating over R^N we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\int_{R^N}\varphi\left(\frac{x}{k}\right)\cdot\left(\vartheta_t^{(m_0)}\right)^2dx+\int_{R^N}\varphi\left(\frac{x}{k}\right)\cdot\left(\Delta\vartheta^{(m_0)}\right)^2dx+\right.$$

$$+2\varepsilon_0\int_{R^N}\varphi\left(\frac{x}{k}\right)\cdot\vartheta_t^{(m_0)}\cdot\vartheta^{(m_0)}dx+\lambda\int_{R^N}\varphi\left(\frac{x}{k}\right)\cdot\left(\vartheta^{(m_0)}\right)^2+$$

$$\left.+\alpha\varepsilon_0\int_{R^N}\varphi\left(\frac{x}{k}\right)\cdot\left(\vartheta^{(m_0)}\right)^2dx+2\int_{R^N}\varphi\left(\frac{x}{k}\right)\cdot\Phi\left(\vartheta^{(m_0)}\right)dx\right)+$$

$$+(\alpha-\varepsilon_0)\cdot\int_{R^N}\varphi\left(\frac{x}{k}\right)\cdot\left(\vartheta_t^{(m_0)}\right)^2dx+\varepsilon_0\int_{R^N}\varphi\left(\frac{x}{k}\right)\cdot\left(\Delta\vartheta^{(m_0)}\right)^2dx+$$

$$+\varepsilon_0\cdot\lambda\cdot\int\limits_{R^N}\varphi\left(\frac{x}{k}\right)\cdot\left(\vartheta^{(m_0)}\right)^2dx+\varepsilon_0\int\limits_{R^N}\varphi\left(\frac{x}{k}\right)\cdot f\left(\vartheta^{(m_0)}\right)\vartheta^{(m_0)}dx=$$

$$=\int\limits_{R^N}\varphi\left(\frac{x}{k}\right)\cdot g\left(x\right)\cdot\left(\vartheta_t^{(m_0)}+\varepsilon\vartheta^{(m_0)}\right)dx-$$

$$-\frac{1}{k^2}\int\limits_{R^N}\left(\Delta\varphi\right)\left(\frac{x}{k}\right)\cdot\vartheta_t^{(m_0)}\cdot\Delta\vartheta^{(m_0)}dx-$$

$$-\frac{2}{k}\sum_{i=1}^{N}\int\limits_{R^N}\varphi_{x_i}\left(\frac{x}{k}\right)\cdot\vartheta_{tx_i}^{(m_0)}\cdot\Delta\vartheta^{(m_0)}dx-$$

$$-\frac{\varepsilon_0}{k^2}\int\limits_{R^N}\left(\Delta\varphi\right)\left(\frac{x}{k}\right)\cdot\vartheta^{(m_0)}\cdot\Delta\vartheta^{(m_0)}dx-$$

$$-\frac{2\varepsilon_0}{k}\cdot\sum_{i=1}^{N}\int\limits_{R^N}\varphi_{x_i}\left(\frac{x}{k}\right)\cdot\vartheta_{x_i}^{(m_0)}\cdot\Delta\vartheta^{(m_0)}dx+$$

$$+\int\limits_{R^N}\left(f\left(\vartheta^{(m_0)}\right)-f\left(\vartheta^{(m_0-1)}\right)\right)\varphi\left(\frac{x}{k}\right)\left(\vartheta_t^{(m_0)}+\varepsilon_0\vartheta^{(m_0)}\right)dx.$$

Using the definitions of the numbers ε_0, ω and the functions $\varphi(x)$ and allowing for (3), (4), (10), (15) and (19) from the last equality we have

$$\left\|\vartheta_t^{(m_0)}\right\|_{L_2(R^N\backslash B_{2k})}+\left\|\vartheta^{(m_0)}\right\|_{W_2^2(R^N\backslash B_{2k})}\le\left(m\left(t\right)+\frac{1}{k}\right)\times$$

$$\times C\left(\|B\|_{H^2\times H},\|g\|,\|\varphi\|_{W_\infty^2(R^N)}\right)+L\cdot\int\limits_{R^N\backslash B_k}g\left(x\right)dx,$$

where $B_r=\left\{x/x\in R^N,\ |x|\le r\right\}$, $m\left(t\right)\underset{t\to+\infty}{\to}0$.

The last estimate means that for $\forall\varepsilon>0$ there exist n_ε and k_ε, such that for $\forall n\ge n_\varepsilon$ and $\forall k\ge k_\varepsilon$

$$\left\|V^{(m_0)}\left(t_n\right)\theta_n\right\|_{W_2^2(R^N\backslash B_{k_\varepsilon})\times L_2(R^N\backslash B_{k_\varepsilon})}\le\frac{\varepsilon}{3}. \tag{20}$$

On the other hand according to lemma 1 the restriction of $\left\{V^{(m_0)}\left(t_n\right)\theta_n\right\}_{n=n_\varepsilon}^\infty$ is precompact in $W_2^2\left(B_{k_\varepsilon}\right)\times L_2\left(B_{k_\varepsilon}\right)$. So in $\left\{V^{(m_0)}\left(t_n\right)\theta_n\right\}_{n=n_\varepsilon}^\infty$ there is a finite number of elements being $\frac{\varepsilon}{3}$-net in the space $W_2^2\left(B_{k_\varepsilon}\right)\times L_2\left(B_{k_\varepsilon}\right)$. And by virtue of (20) these elements will be ε-net in $H^2\times H$ for the set $\left\{V^{(m_0)}\left(t_n\right)\theta_n\right\}_{n=n_\varepsilon}^\infty$. It completes the proof of theorem 1.

Now we can formulate the basic result by using the results of [1] and [3].

Theorem 4. *The problem (1)-(2) under the conditions (3)-(4) in the space $H^2\times H$ has global attractor which is invariant and compact.*

References

[1]. Ghidaglia J.M., Temam R. *Attractors for damped nonlinear hyperbolic equations.* J.Math.pures et appl., 1987, v.66, No3, pp.273-319.

[2]. Aliev A.B., Solvability "in the large" of the Cauchy problem for quasilinear equations of hyperbolic type. DAN SSSR, 1978, v.240, No2, pp.249-252. (Russian)

[3]. Ladyzhenskaya O.A. *On the determination of minimal global attractors for Navier-Stokes equations and other partial differential equations.* UMN, 1987, v.42, issue6(258), pp.25-60. (Russian)

Azer Kh. Khanmamedov
Institute of Mathematics & Mechanics of NAS of Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20(off.).

Received September 2, 2002; Revised November 21, 2002.
Translated by Mirzoyeva K.S.