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THE UNIQUENESS OF SOLUTION OF INVERSE SCATTERING PROBLEM FOR THE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS ON THE WHOLE AXIS

Abstract

Nonstationary approach to scattering problem for a system of ordinary differential equations on the whole axis is considered. The theorem on unique solution of the inverse scattering problem is proved.

Consider the system of equations

$$-i\frac{dy}{dx}+Q\left(x\right)y=\lambda\tau y\qquad\left(-\infty<x<+\infty\right)\tag{1}$$

with nondegenerate diagonal matrix

$$\tau=diag\left(\xi_1 I_{n_1},...,\xi_r I_{n_r}\right)\in R^{n\times n},$$

where $\xi_1>...>\xi_r$ and with summable $n\times n$ matrix Q which has block representation $Q=\left(q_{ij}\right)_{i,j=1}^{r}$.

Matrix function $Q\left(x\right)$ is assumed to be complexvalued, in this connection, an operator defined in the Hilbert space $L^2\left(R;C^n\right)$ by the differential expression $l\left(y\right)=-i\tau^{-1}y'+\tau^{-1}Q\left(x\right)y$ is generally nonself-adjoint.

For $r=2,\ n_1=n_2=n,\ \xi_1=-\xi_2=1$ from system (1) system of Dirac equations is obtained and in the case of self-adjoint canonic system of Dirac equations the inverse scattering problem on the semi-axis $(x\geq 0)$ was studied in papers [1,2,3,etc.], and on the whole axis in [4,5, etc.].

At $n_1=...=n_r=1$ and $q_{ii}=0,\ i=1,...,r$ is the inverse scattering problem for the system (1) on semi-axis was studied in [6,7,8, etc.], and on the whole axis in [9,10, etc.].

In the paper the inverse scattering problem on the whole axis is investigated for the system of differential equations obtained from (1) at $r=2,\ q_{11}=q_{22}=0$, i.e. for the system of differential equations of the form

$$-i\frac{dy}{dx}+\left(\begin{array}{cc}0&q_{12}\left(x\right)\\q_{21}\left(x\right)&0\end{array}\right)y=\lambda\left(\begin{array}{cc}\xi_1 I_{n_1}&0\\0&\xi_2 I_{n_2}\end{array}\right)y,\qquad y=\left(\varphi_1,\varphi_2,..,\varphi_n\right),\tag{2}$$

where $I_{n_i}\left(i=1,2\right)-n_i\times n_i$ are unit matrices, $n_1+n_2=n,\quad\xi_1>0>\xi_2;\quad q_{ij}\left(x\right)$ $\left(i,j=1,2\right)-n_i\times n_j$ are matrix functions with measurable complex-valued elements, whose Euclidean norms satisfy the conditions

$$\int\limits_{-\infty}^{\infty}\left\|q_{ij}\left(x\right)\right\|dx<+\infty,\quad i,j=1,2,\quad\left(i\neq j\right)\tag{3}$$

Let's consider the problem generated by system (2) with one of the boundary conditions on infinity:

$$\lim_{x\to-\infty} y_1(x,\lambda)\, e^{i\lambda\xi_1 x} = A_1, \qquad \lim_{x\to-\infty} y_2(x,\lambda)\, e^{i\lambda\xi_2 x} = A_2 \tag{4}$$

and

$$\lim_{x\to+\infty} y_1(x,\lambda)\, e^{i\lambda\xi_1 x} = B_1, \qquad \lim_{x\to+\infty} y_2(x,\lambda)\, e^{i\lambda\xi_2 x} = B_2 \tag{5}$$

Here

$$y_1(x,\lambda) = \left(\varphi_1(x,\lambda),\ldots,\varphi_{n_1}(x,\lambda)\right), \quad y_2(x,\lambda) = \left(\varphi_{n_1+1}(x,\lambda),\ldots,\varphi_n(x,\lambda)\right).$$

At first let's prove the following theorem.

Theorem 1. *Let λ be real number and the coefficients of system (2) satisfy condition (3). Then the following statements are valid:*

1) there exists a unique solution in the class of bounded functions of problem (2)-(4) and (2)-(5),

2) for any bounded solution $y(x,\lambda)$ of system of equation (2) there exist (4) and (5).

Proof. The first statement of the theorem is equivalent to the unique solvability in $L^\infty(R,C^n)$ of the system of integral equations

$$\begin{aligned} y_1(x,\lambda) &= A_1 e^{i\lambda\xi_1 x} - i\int_{-\infty}^{x} q_{12}(s)\, y_2(s)\, e^{i\lambda\xi_1(x-s)}ds \\ y_2(x,\lambda) &= A_2 e^{i\lambda\xi_2 x} - i\int_{-\infty}^{x} q_{21}(s)\, y_1(s)\, e^{i\lambda\xi_2(x-s)}ds \end{aligned} \tag{6}$$

and

$$\begin{aligned} y_1(x,\lambda) &= B_1 e^{i\lambda\xi_1 x} + i\int_{x}^{+\infty} q_{12}(s)\, y_2(s)\, e^{i\lambda\xi_1(x-s)}ds \\ y_2(x,\lambda) &= B_2 e^{i\lambda\xi_2 x} + i\int_{x}^{+\infty} q_{21}(s)\, y_1(s)\, e^{i\lambda\xi_2(x-s)}ds \end{aligned} \tag{7}$$

which are equivalent to problems (2)-(4) and (2)-(3) respectively.

By virtue of (3), integral operators here are contractive in $L^\infty(R,C^n)$ and therefore system of integral equations (6) and (7) is uniquely solvable in the space $L^\infty(R,C^n)$.

The second statement of the theorem is reduced to the direct checking of the fact that if $y(x,\lambda) = (y_1(x,\lambda), y_2(x,\lambda))$ is bounded solution of system (2), then vector-function

$$z_1(x,\lambda) = y_1(x,\lambda) + i\int_{-\infty}^{x} q_{12}(s)\, y_2(s)\, e^{i\lambda\xi_1(x-s)}ds,$$

$$z_2(x,\lambda) = y_2(x,\lambda) + i\int_{-\infty}^{x} q_{21}(s)\, y_1(s)\, e^{i\lambda\xi_2(x-s)}ds,$$

belongs to the space $L^{\infty}(R, C^n)$ and is the solution of nonperturbed system (system (2) when $q_{12} = q_{21} = 0$). Therefore there exist vectors $A_1 = A_1(\lambda)$ and $A_2 = A_2(\lambda)$ such that

$$z_1(x,\lambda) = A_1 e^{i\lambda\xi_1 x}, \qquad z_2(x,\lambda) = A_2 e^{i\lambda\xi_2 x}$$

i.e.

$$y_1(x,\lambda) = A_1 e^{i\lambda\xi_1 x} - i\int\limits_{-\infty}^{x} q_{12}(s)\, y_2(s)\, e^{i\lambda\xi_1(x-s)}ds,$$

$$y_2(x,\lambda) = A_2 e^{i\lambda\xi_1 x} - i\int\limits_{-\infty}^{x} q_{21}(s)\, y_1(s)\, e^{i\lambda\xi_2(x-s)}ds,$$

From here taking into account (3), we obtain (4). The proof of the second statement for (5) is analogous.

By virtue of theorem 1 Cauchy problem with the data on infinity is uniquely solvable in the class of bounded functions. This allows to define scattering operator for system of equations (2). According to theorem 1 to each vector $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ corresponds the unique solution of system (2) with boundary condition (4). For the obtained solution $y(x,\lambda)$ vector $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ exists by formula (6). We shall define the scattering operator S (according to the definition of the scattering operator for the nonstationary case [4]) as operator

$$S : \begin{pmatrix} A_1 \\ B_2 \end{pmatrix} \to \begin{pmatrix} B_1 \\ A_2 \end{pmatrix} \tag{8}$$

The operator $S = S(\lambda)$ is linear in the space R^n and is defined by matrix $S(\lambda) = (S_{ij}(\lambda))_{i,j=1}^{n}$.

Later on the following strict condition on coefficients of system (1) becomes convenient. Let the elements of matrix function $q_{12}(x)$ and $q_{21}(x)$ be complex-valued and measurable, besides, Euclidean norms satisfy the inequality

$$||q_{ij}(x)|| \leq \frac{C}{1+|x|^{1+\varepsilon}}, \quad i,j = 1,2 \quad (i \neq j), \quad \varepsilon, C > 0. \tag{9}$$

Let's establish the following theorem.

Theorem 2. *Let λ be a real number and the coefficients of system (2) satisfy condition (9).*

Then solution of problems (2)-(4) can be represented in the form

$$\begin{aligned} y_1(x,\lambda) &= A_1 e^{i\lambda\xi_1 x} + \int\limits_{-\infty}^{x} A_{11}(x,t)\, A_1 e^{i\lambda\xi_1 t}dt + \int\limits_{-\infty}^{x} A_{12}(x,t)\, A_2 e^{i\lambda\xi_2 t}dt \\ y_2(x,\lambda) &= A_2 e^{i\lambda\xi_2 x} + \int\limits_{-\infty}^{x} A_{21}(x,t)\, A_1 e^{i\lambda\xi_1 t}dt + \int\limits_{-\infty}^{x} A_{22}(x,t)\, A_2 e^{i\lambda\xi_2 t}dt \end{aligned} \tag{10}$$

and solution of problems (2)-(5) can be represented in the form

$$\begin{aligned} y_1(x,\lambda) &= B_1 e^{i\lambda\xi_1 x} + \int\limits_{x}^{+\infty} B_{11}(x,t)\, B_1 e^{i\lambda\xi_1 t}dt + \int\limits_{x}^{\infty} B_{12}(x,t)\, B_2 e^{i\lambda\xi_2 t}dt \\ y_2(x,\lambda) &= B_2 e^{i\lambda\xi_2 x} + \int\limits_{x}^{+\infty} A_{21}(x,t)\, B_1 e^{i\lambda\xi_1 t}dt + \int\limits_{x}^{+\infty} B_{22}(x,t)\, B_2 e^{i\lambda\xi_2 t}dt \end{aligned} \tag{11}$$

Here for fixed x matrix functions $A_{ij}(x,t)$, $B_{ij}(x,t)$, $i,j=1,2$ are summable by t and satisfy the conditions

$$\begin{array}{ll} A_{12}(x,\lambda) = -i\frac{\xi_2}{\xi_2-\xi_1}q_{12}(x)\ , & B_{12}(x,\lambda) = i\frac{\xi_2}{\xi_2-\xi_1}q_{12}(x)\ , \\ A_{21}(x,\lambda) = -i\frac{\xi_1}{\xi_1-\xi_2}q_{21}(x)\ , & B_{21}(x,\lambda) = i\frac{\xi_1}{\xi_1-\xi_2}q_{12}(x)\,, \end{array} \tag{12}$$

Proof. We shall lead proof for the first representation. For representation (11) proof is led analogously. Let's prove that solution of system of equations (2) with boundary condition (4) is representable in form (10). In fact, proceeding from (11) and (6), we obtain the systems of integral equations with respect to kernels $A_{ij}(x,t)$, $i,j=1,2$:

$$\begin{cases} A_{11}(x,t) = -i\int\limits_{-\infty}^{x} q_{12}(s)\,A_{21}\quad(s,t+s-x)\,ds\ , \\ A_{21}(x,t) = -i\frac{\xi_1}{\xi_1-\xi_2}q_{21}\left(\frac{\xi_1}{\xi_1-\xi_2}t-\frac{\xi_2}{\xi_1-\xi_2}x\right) - \qquad\qquad t\le x \\ -i\int\limits_{\frac{\xi_1}{\xi_1-\xi_2}t-\frac{\xi_2}{\xi_1-\xi_2}x}^{x} q_{21}(s)\,A_{11}\left(s,t-\frac{\xi_2}{\xi_1}x+\frac{\xi_2}{\xi_1}s\right)ds, \end{cases} \tag{13}$$

$$\begin{cases} A_{12}(x,t) = -i\frac{\xi_2}{\xi_2-\xi_1}q_{12}\left(\frac{\xi_2}{\xi_2-\xi_1}t-\frac{\xi_1}{\xi_2-\xi_1}x\right) - \\ -i\int\limits_{\frac{\xi_2}{\xi_2-\xi_1}t-\frac{\xi_1}{\xi_2-\xi_1}x}^{x} q_{12}(s)\,A_{22}\left(s,t+\frac{\xi_1}{\xi_2}s-\frac{\xi_1}{\xi_2}x\right)ds, \\ A_{22}(x,t) = -i\int\limits_{-\infty}^{x} q_{21}(s)\,A_{12}\quad(s,t+s-x)\,ds\ , \qquad t\le x \end{cases} \tag{14}$$

Integral operators here are Volterra ones in the direction $\beta=(1,0)$ with integrable majorant $\alpha(\tau)\equiv\max\{||q_{12}(\tau)||,\ \ ||q_{21}(\tau)||\}$. Then by theorem 4.11 [11] system of integral equations (13) and (14) there exists a unique solution in the space of matrix-valued bounded functions of two variables. At that matrix functions $A_{ij}(x,t)$, $i,j=1,2$ satisfy condition (12). The theorem is proved.

Now from (10) and (11) we have

$$y_1(x,\lambda) = \left(I_{n_1} + \int\limits_{-\infty}^{x} A_{11}(x,t)\,e^{i\lambda\xi_1(t-x)}dt\right)A_1e^{i\lambda\xi_1 x} +$$

$$+\left(\int\limits_{-\infty}^{x} A_{12}(x,t)\,e^{i\lambda\xi_2(t-x)}dt\right)A_2e^{i\lambda\xi_2 x} =$$

$$= \left(I_{n_1} + \int\limits_{x}^{+\infty} B_{11}(x,t)\,e^{i\lambda\xi_1(t-x)}dt\right)B_1e^{i\lambda\xi_1 x} +$$

$$+\left(\int\limits_{x}^{+\infty} B_{12}(x,t)\,e^{i\lambda\xi_2(t-x)}dt\right)B_2e^{i\lambda\xi_2 x}$$

and

$$y_2(x,\lambda)=\left(\int\limits_{-\infty}^{x}A_{21}(x,t)\,e^{i\lambda\xi_1(t-x)}dt\right)A_1e^{i\lambda\xi_1x}+$$

$$+\left(I_{n_2}+\int\limits_{-\infty}^{x}A_{22}(x,t)\,e^{i\lambda\xi_2(t-x)}dt\right)A_2e^{i\lambda\xi_2x}=$$

$$=\left(\int\limits_{x}^{+\infty}B_{21}(x,t)\,e^{i\lambda\xi_1(t-x)}dt\right)B_1e^{i\lambda\xi_1x}+$$

$$+\left(I_{n_2}+\int\limits_{x}^{+\infty}B_{22}(x,t)\,e^{i\lambda\xi_2(t-x)}dt\right)B_2e^{i\lambda\xi_2x}\ .$$

Hence,

$$\left(I+A_-(x,\lambda)\right)e^{i\lambda\tau x}\begin{pmatrix}A_1\\B_2\end{pmatrix}=\left(I+B_+(x,\lambda)\right)e^{i\lambda\tau x}\begin{pmatrix}B_1\\A_2\end{pmatrix}\tag{15}$$

Here

$$A_-(x,\lambda)=$$

$$=\begin{pmatrix}\frac{1}{\xi_1}\int\limits_{-\infty}^{0}A_{11}\left(x,\frac{t}{\xi_1}+x\right)e^{i\lambda t}dt & \frac{1}{\xi_2}\int\limits_{-\infty}^{0}B_{12}\left(x,x+\frac{t}{\xi_2}\right)e^{i\lambda t}dt\\ -\frac{1}{\xi_1}\int\limits_{-\infty}^{0}A_{21}\left(x,x+\frac{t}{\xi_1}\right)e^{i\lambda t}dt & -\frac{1}{\xi_2}\int\limits_{-\infty}^{0}B_{22}\left(x,x+\frac{t}{\xi_2}\right)e^{i\lambda t}dt\end{pmatrix}$$

$$B_+(x,\lambda)=$$

$$=\begin{pmatrix}\frac{1}{\xi_1}\int\limits_{0}^{+\infty}B_{11}\left(x,x+\frac{t}{\xi_1}\right)e^{i\lambda t}dt & \frac{1}{\xi_2}\int\limits_{0}^{+\infty}A_{12}\left(x,x+\frac{t}{\xi_2}\right)e^{i\lambda t}dt\\ -\frac{1}{\xi_1}\int\limits_{0}^{+\infty}B_{21}\left(x,x+\frac{t}{\xi_1}\right)e^{i\lambda t}dt & -\frac{1}{\xi_2}\int\limits_{0}^{+\infty}A_{22}\left(x,x+\frac{t}{\xi_2}\right)e^{i\lambda t}dt\end{pmatrix}$$

$$e^{i\lambda\tau}=\begin{pmatrix}e^{i\lambda\xi_1x}I_{n_1} & 0\\ 0 & e^{i\lambda\xi_2x}I_{n_2}\end{pmatrix}$$

are denoted.

Matrix functions $A_-(x,\lambda)$ and $B_+(x,\lambda)$ admit analytical extension to lower $(\operatorname{Im}\xi\le 0)$ and upper $(\operatorname{Im}\lambda\ge 0)$half plane respectively. We suppose that matrix functions $I+A_-(x,\lambda)$ and $I+B_+(x,\lambda)$ in their domains of their analyticity nowhere degenerate, i.e.

$$\det\left(I+A_-(x,\lambda)\right)\neq 0,\ \ \operatorname{Im}\lambda\le 0\ \text{ and }\ \det\left(I+B_+(x,\lambda)\right)\neq 0,\ \operatorname{Im}\lambda\ge 0.\tag{16}$$

From (15) by definition (8) we obtain

$$S(\lambda)=e^{-i\lambda\tau x}\left(I+B_+(x,\lambda)\right)^{-1}\left(I+A_-(x,\lambda)\right)e^{i\lambda\tau x}\tag{17}$$

Thus, we obtain regular Riemann problems with normalization to unit matrix in the point at infinity:

$$e^{i\lambda\tau x}S(\lambda)\,e^{i\lambda\tau x}=(I+B_{+}(x,\lambda))^{-1}(I+A_{-}(x,\lambda)) \tag{18}$$

For inverse problem the following uniqueness theorem is valid.

Theorem 3. *Let $S(\lambda)$ be the scattering matrix for system of equations (2) with the coefficients $q_{12}(x)$ and $q_{21}(x)$ satisfying condition (9).*

Then at condition (16), the coefficients $q_{12}(x)$ and $q_{21}(x)$ are uniquely defined by $S(\lambda)$.

Proof. Let $S(\lambda)$ be the scattering matrix for system of equations (2) with two different coefficients $q_{12}^1(x)$, $q_{21}^1(x)$ and $q_{12}^2(x)$, $q_{21}^2(x)$. For the coefficients $q_{12}^1(x)$, $q_{21}^1(x)$ and $q_{12}^2(x)$, $q_{21}^2(x)$ by theorems 1 and 2 there exist matrix functions $A_{-}^1(x,\lambda)$, $B_{+}^1(x,\lambda)$ and $A_{-}^2(x,\lambda)$, $B_{+}^2(x,\lambda)$ respectively which satisfy the equality according to (18)

$$\left(I+B_{+}^1(x,\lambda)\right)^{-1}\left(I+A_{-}^1(x,\lambda)\right)=$$
$$=e^{i\lambda\tau x}S(\lambda)\,e^{i\lambda\tau x}=\left(I+\ B_{+}^2(x,\lambda)\right)^{-1}\left(I+A_{-}^2(x,\lambda)\right)$$

Suppose

$$\Psi(\lambda)=\left(I+A_{-}^2(x,\lambda)\right)\left(I+A_{-}^1(x,\lambda)\right)^{-1}=$$
$$=\left(I+\ B_{+}^2(x,\lambda)\right)\left(I+\ B_{+}^1(x,\lambda)\right)^{-1}$$

Under condition (16) function $\Psi(\lambda)$ is analytical on the whole plane. It is bounded by module in it, then by Liouville theorem, we have

$$\Psi(\lambda)=I$$

Then

$$A_{-}^1(x,\lambda)=A_{-}^2(x,\lambda)$$

and

$$B_{+}^1(x,\lambda)=\ B_{+}^2(x,\lambda)$$

Hence $q_{12}^1(x)=q_{12}^2(x)$ and $q_{21}^1(x)=q_{21}^2(x)$.

The theorem is proved.

Remark 1. One can show that only for system (2) solutions satisfying condition (4) have form (10). For general systems (1) for $q_{ii}=0\quad(i=1,...,2)$ solution of form (10) can be constructed by the method suggested in [11] for the nonstationary analogue of system (1).

Remark 2. The uniqueness of the inverse scattering problem is proved by satisfying condition (16). This condition can be replaced by the absence of discrete spectrum of the operator defined in $L_2(R,C^n)$ generated by differential expression $l(y)$.

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