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THE SCATTERING PROBLEM FOR A SYSTEM OF THE FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS ON A SEMIAXIS

Abstract

In the paper the direct scattering problem is considered on a semiaxis in a general case when there are k incident and $n-k$ scattered waves.

Let's consider on the semiaxis $x \geq 0$ a system of the first order ordinary differential equations of the form

$$-i\frac{dy_i(x)}{dx} + \sum_{s=1}^{n} c_{js}(x)\, y_s(x) = \lambda \xi_j y_j(x), \qquad n \geq 3 \tag{1}$$

in the case when $\xi_1 > \xi_2 > ... > \xi_k > 0 > \xi_{k+1} > ... > \xi_n$.

Here the coefficients $c_{js}(x)\,(j,s = 1,...,n)$ are complex-valued measurable functions $c_{jj}(x) \equiv 0\ \ (j = 1,...,n)$ and satisfy the following conditions

$$\int_0^{+\infty} |c_{js}(x)|\, dx < +\infty, \ \ j,s = 1,2,...,n. \tag{2}$$

When $k = n-1$ the inverse scattering problem was considered in [1], for k=1in [2]. The inverse scattering problem on a whole axis was investigated in [3-6].

Let $\lambda \in R$ be fixed and the coefficients satisfy conditions (2) then equation (1) has such solutions $y_j(x)\ \ (j = 1,...,n)$ that

$$\lim_{x\to+\infty} y_j(x)\, e^{-i\lambda\xi_j x} = A_j, \quad j = 1,2,...,k, \tag{3}$$

$$\lim_{x\to+\infty} y_j(x)\, e^{-i\lambda\xi_j x} = B_j, j = k+1,...,n. \tag{4}$$

This statement follows e.g. from [7]. Let's consider system (1) on a semiaxis under the different boundary conditions.

$$y_n^m(0,\lambda) = c_n^m y_{p_1^m}^m(0,\lambda),$$

$$y_{n-1}^m(0,\lambda) = c_{n-1}^m y_{p_2^m}^m(0,\lambda),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_{k+1}^m(0,\lambda) = c_{k+1}^m y_{p_k^m}^m(0,\lambda), \quad (2 \leqslant m \leqslant n-1). \tag{5}$$

where $c_j^m\ \ (j = k+1,...,n)$ take on values 0 or 1; for the each of problems the sets $\{p_1^m, p_2^m, ..., p_k^m\}$ are assumed to be different and $\{p_1^m, p_2^m, ..., p_k^m\} \in \{1,2,...,k\}$. Here by m$\in N$ the number of problem is noted.

For example, if $k = n-1; c_n^m = 1\ \ \ (m = 1,...,n-1),\ \ p_1^m = m.$

$$y_n^m(0,\lambda) = y_m^m(0,\lambda).$$

if $k=1$; $c_n^m=1\ \ (m=1,...,n-1)\,,\ c_{n-1}^m=0,...,c_{k+1}^m=0,\ p_1^m=m$

$$y_n^m(0,\lambda)=y_m^m(0,\lambda)$$

$$y_k^m(0,\lambda)=0,\quad k\neq n,m.$$

we obtain problems [1,2].

Note that for $n=5$ if $k=2$ it can be considered

$$y_5^1(0,\lambda)=y_1^1(0,\lambda)\,,\ \ c_5^1=1,\ \ p_1^1=1,\ \ m=1,$$

$$y_4^1(0,\lambda)=y_2^1(0,\lambda)\,,\ \ c_4^1=1,\ \ p_2^1=2,$$

$$y_3^1(0,\lambda)=0,\ \ c_3^1=0.$$

$$y_5^2(0,\lambda)=y_2^2(0,\lambda)\,,\ \ c_5^2=1,\ \ p_1^2=2,\ \ m=2,$$

$$y_3^2(0,\lambda)=y_1^2(0,\lambda)\,,\ \ c_3^2=1,\ \ p_3^2=1,$$

$$y_4^2(0,\lambda)=0,\ \ c_4^2=0.$$

$$y_4^3(0,\lambda)=y_1^3(0,\lambda)\,,\ \ c_4^3=1,\ \ p_2^3=1,\ \ m=3,$$

$$y_3^3(0,\lambda)=y_2^3(0,\lambda)\,,\ \ c_3^3=1,\ \ p_3^3=2,$$

$$y_5^3(0,\lambda)=0,\ \ c_5^3=0.$$

and for $k=3$

$$y_5^1(0,\lambda)=y_1^1(0,\lambda)\,,\ \ c_5^1=1,\ \ p_1^1=1,\ \ m=1,$$

$$y_4^1(0,\lambda)=y_2^1(0,\lambda)\,,\ \ c_4^1=1,\ \ p_1^2=2,$$

$$y_5^2(0,\lambda)=y_2^2(0,\lambda)\,,\ \ c_5^2=1,\ \ p_2^1=2,\ \ m=2,$$

$$y_4^2(0,\lambda)=y_1^2(0,\lambda)\,,\ \ c_4^2=1,\ \ p_2^2=1,$$

$$y_4^3(0,\lambda)=y_3^3(0,\lambda)\,,\ \ c_4^3=1,\ \ p_2^3=1,\ \ m=3,$$

$$y_5^3(0,\lambda)=y_1^3(0,\lambda)\,,\ \ c_5^3=1,\ \ p_1^3=2.$$

For system (1) on a semiaxis $r\ \ (2\leq r\leq n-1)$ problems are considered:

The m–th $(m=1,2,...,r)$ problem is in finding solution of system of equations (1) with boundary condition (5) at the given conditions (3).

We shall call the combined consideration of these r problems the scattering problem for system (1) on a semiaxis.

Theorem. *Let the coefficients of system (1) satisfy conditions (2). Then for any $A_1,...,A_k$ there exists a unique bounded solution of the scattering problem on a semiaxis for system (1).*

Proof. The scattering problem for the m-th problem is equivalent to the following system of integral equations

$$y_j^m(x,\lambda)=A_je^{i\lambda\xi_jx}+i\int\limits_x^{+\infty}\sum_{p=1}^{n}c_{jp}\left(x'\right)y_p^m\left(x',\lambda\right)e^{i\lambda\xi(x-x')}dx',\quad (j=1,2,...,k)$$

$$y_j^m(x,\lambda) = B_j^m e^{i\lambda\xi_j x} + i\int\limits_x^{+\infty}\sum_{p=1}^{n} c_{jp}(x')\, y_p^m(x',\lambda)\, e^{i\lambda\xi_j(x-x')}dx',\ (j=k+1,...,n) \tag{6}$$

where

$$B_{k+1}^m = c_{k+1}^m A_{p_k^m} +$$
$$+ i\sum_{p=1}^{n}\int\limits_0^{+\infty}\left[c_{k+1}^m c_{p_k^m,p}(x')\, e^{-i\lambda\xi_{p_k^m}x'} - c_{k+1,p}(x')\, e^{-i\lambda\xi_{k+1}x'}\right] y_p^m(x',\lambda)\, dx',$$

....

$$B_{n-1}^m = c_{n-1}^m A_{p_2^m} +$$
$$+ i\sum_{p=1}^{n}\int\limits_0^{+\infty}\left[c_{n-1}^m c_{p_2^m,p}(x')\, e^{-i\lambda\xi_{p_2^m}x'} - c_{n-1,p}(x')\, e^{-i\lambda\xi_{n-1}x'}\right] y_p^m(x',\lambda)\, dx',$$

$$B_n^m = c_n^m A_{p_1^m} + i\sum_{p=1}^{n}\int\limits_0^{+\infty}\left[c_n^m c_{p_1^m,p}(x')\, e^{-i\lambda\xi_{p_1^m}x'} - c_{np}(x')\, e^{-i\lambda\xi_n x'}\right] y_p^m(x',\lambda)\, dx',$$

Existence and uniqueness of solutions of system (6) in the class of bounded functions follow from Volterra property of these systems of integral equations.

By means of conditions (2) from (6) we obtain

$$\lim_{x\to+\infty} y_j^m(x,\lambda)\, e^{-i\lambda\xi_j x} = B_j^m, \tag{7}$$

$$j = k+1,...,n;\ m = 1,2,...,r.$$

On the basis of the theorem there exists the matrix

$$S_m(\lambda) = \begin{pmatrix} S_{11}^m(\lambda) & S_{12}^m(\lambda) & \ldots & S_{1k}^m(\lambda) \\ S_{21}^m(\lambda) & S_{22}^m(\lambda) & \ldots & S_{2k}^m(\lambda) \\ \ldots & \ldots & \ldots & \ldots \\ S_{n-k,1}^m(\lambda) & S_{n-k,2}^m(\lambda) & \ldots & S_{n-k,k}^m(\lambda) \end{pmatrix}, \tag{8}$$

such that

$$S_m(\lambda)\begin{pmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_k \end{pmatrix} = \begin{pmatrix} B_{k+1}^m \\ B_{k+2}^m \\ \cdot \\ \cdot \\ \cdot \\ B_n^m \end{pmatrix}, (\ m = 1,2,...,r). \tag{9}$$

We'll call matrix $S(\lambda) = (S_1(\lambda), S_2(\lambda), ..., S_r(\lambda))$ the scattering matrix for system (1) on a semiaxis.

The inverse scattering problem for the system of differential equations (1) at the given scattering matrix is, generally speaking, overdetermined one. Since by the scattering matrix $S(\lambda) = (S_1(\lambda), S_2(\lambda), ..., S_r(\lambda))$ containing $r\left(kn-k^2\right)$ of the given functions on a whole axis $(-\infty < \lambda < +\infty)$ or $2r\left(kn-k^2\right)$ functions on a semiaxis $(0 \leqslant \lambda < +\infty)$ it's necessary to find potential

$$c(x) = \begin{pmatrix} 0 & c_{12}(x) & \ldots & c_{1,n-1}(x) & c_{1n}(x) \\ c_{21}(x) & 0 & \ldots & c_{2,n-1}(x) & c_{2n}(x) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{n1}(x) & c_{n2}(x) & \ldots & c_{n,n-1}^{(x)} & 0 \end{pmatrix}$$

in system (1) containing $n^2 - n$ unknown functions $c_{ij}(x)$, $i,j = 1,2,...n; i \neq j$. Therefore, the following inequalities hold

$$2rk(n-k) \geq n^2 - n.$$

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