Bala A. ISKENDEROV, Vugar G. SARDAROV

MIXED PROBLEM FOR BOUSSINESKA EQUATION IN A CYLINDRICAL DOMAIN AND BEHAVIOR OF ITS SOLUTION AT $t \to +\infty$

Abstract

The existence and uniqueness of a mixed problem for Boussineska equation was proved in multidimensional cylindric domain, convergence to zero of the solution of mixed problem for $t \to +\infty$ when the longitudinal dimension of cylinder is a unit was shown.

Introduction. Boussineska equation appears by describing longitudinal waves in bars in the theory of long waves in water, and also by describing waves in plasma [1-3].

Cauchy problem and various questions mixed and connected with it for a class of Sobolev type equations were studied in [3-6] in which there is a large reference. We note also paper [7] where main initial-boundary value problems were investigated, and the existence of wave front is established for the equation describing the dynamics of one-dimensional flow.

The uniqueness, existence and behaviour at great values of time of the solution of mixed problem for Boussineska equation in multi-dimensional cylindrical domain is studied. The results of the paper are new.

§1. Definition, notations and uniqueness of solution of mixed problem for Boussineska equations.

Let $R_m(y)$ be a m-dimensional Euclidean space with elements $y = (y_1, y_2, ..., y_m)$, and $R_n(x)$ is the similar space with elements $x = (x_1, x_2, ..., x_n)$. Denote by $\Pi = R_n \times \Omega$ a cylindrical domain in $R_n(x) \times R_m(y)$, where Ω is a bounded domain in $R_m(y)$ with sufficiently smooth boundary $\partial\Omega$. Consider in $\Pi \times (0, \infty)$ the following mixed problem

$$\left(\sigma^2 \Delta_{n+m} - 1\right) D_t^2 u(x, y, t) + \gamma^2 \Delta_{n+m} u(x, y, t) = 0 \tag{1.1}$$

with initial condition

$$u(x, y, 0) = \psi_0(x, y), \qquad u'_t(x, y, 0) = \psi_1(x, y) \tag{1.2}$$

and with boundary condition

$$u(x, y, t)\,|_{\partial\Pi \times (0,\infty)} = 0 \tag{1.3}$$

Here Δ_{n+m} is a Laplace operator on variables (x, y), $\partial\Pi$ is a lateral surface of cylinder Π, $\psi_0(x, y)$, $\psi_1(x, y) \in C_0^{(\mu,v)}(\Pi)$ is a space of finite, continuously differentiable with respect to (x, y) functions in Π up to the order μ with respect to x and up to order v with respect to y, μ and v we'll define below. By

$C^{(p,q,r)}(\Pi\times[0,\infty))$ denote a class of functions determined for $(x,y,t)\in\Pi\times[0,\infty)$, such that $D_x^{\alpha}D_y^{\beta}D_z^{\gamma}u(x,y,t)\in C^{(0,0,0)}(\Pi\times[0,\infty))$ and

$$\left|D_x^{\alpha}D_y^{\beta}D_z^{\gamma}u(x,y,t)\right|\le Ce^{\varepsilon t-c_0|x|} \tag{1.4}$$

uniformly on $y\in\Omega$, where $D_x^{\alpha}=D_{x_1}^{\alpha_1}...D_{x_n}^{\alpha_n}, D_{x_j}=\dfrac{\partial}{\partial x_j}$, $D_y^{\beta}=D_{y_1}^{\beta_1}...D_{y_n}^{\beta_n}$, $D_t=\dfrac{\partial}{\partial t}$, $0\le|\alpha|\le p$, $0\le|\beta|\le q$, $\gamma\le r$, $|\alpha|=\alpha_1+...+\alpha_n$, $|\beta|=\beta_1+...+\beta_n$, ε is sufficiently small number, c_0 are some constants, $C^{(0,0,0)}(\Pi\times[0,\infty))$ a space of functions continuous in $\Pi\times[0,\infty)$.

Definition. *The function $u(x,y,t)$ will be called a classic solution of problem (1.1)-(1.3), if $u(x,y,t)\in C^{2,2,2}(\Pi\times(0,\infty))\cap C^{1,1,1}\left(\overline{\Pi}\times[0,\infty)\right)$ and satisfies the equation, initial and boundary conditions in an ordinary sense.*

Theorem 1. *Classic solution of problem (1.1)-(1.3) is unique, if it exists.*

Proof. Show that the solution of homogeneous problem corresponding to problem (1.1)-(1.3) is only trivial solution. Multiplying equation (1.1) by $u_t(x,y,t)$ and integrating with respect to $\Pi\times[0,t)$, we get

$$\int_0^t\int_{\Pi}\left[\left(\sigma^2\Delta_{n+m}-1\right)D_t^2u+\gamma^2\Delta_{n+m}u\right]u_t d\Pi dt=0 \tag{1.5}$$

Denote by $\sigma_R(x)$ a ball of radius R with a center at the origin of coordinates in $R_n(x)$ and $\Pi=\Omega\times\sigma_R(x)$ by Green's first formula

$$\int_{\Pi_R}\left(\Delta_{n+m}D_t^2u\right)u_t d\Pi=$$

$$=-\int_{\Pi_R}\left[\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(D_t^2u\right)\frac{\partial u_t}{\partial x_i}+\sum_{j=1}^{m}\frac{\partial}{\partial y_j}\left(D_t^2u\right)\frac{\partial u_t}{\partial y_j}\right]+\int_{\partial\Pi_R}u_t\frac{\partial}{\partial n}\left(D_t^2u\right)ds\ ,$$

$$\int_{\Pi_R}\left(\Delta_{n+m}u\right)u_t d\Pi=-\int_{\Pi_R}\left[\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial u_t}{\partial x_i}+\sum_{j=1}^{m}\frac{\partial u}{\partial y_j}\frac{\partial u_t}{\partial y_j}\right]d\Pi+\int_{\partial\Pi_R}u_t\frac{\partial}{\partial n}ds \tag{1.6}$$

where

$$\partial\Pi_R=\partial\Omega\times\sigma_R(x)\cup\Omega\times\partial\sigma_R(x)\ ,$$

and ds is an element of the surface $\partial\Pi_R$.

Then by virtue of condition (1.3)

$$\begin{aligned}&\int_{\Pi_R}u_t\tfrac{\partial}{\partial n}\left(D_t^2u\right)ds=\int_{\Omega\times\partial\sigma_R(x)}u_t\tfrac{\partial}{\partial n}\left(D_t^2u\right)ds\ ,\\&\int_{\Pi_R}u_t\tfrac{\partial u}{\partial n}ds=\int_{\Omega\times\partial\sigma_R(x)}u_t\tfrac{\partial u}{\partial n}ds\ ,\end{aligned} \tag{1.7}$$

For $R \to \infty$ by virtue of condition (1.4) the integrals in (1.7) tend to zero. Passing in (1.6) to the limit for $R \to \infty$ and taking into account the above said we get

$$\int\limits_{\Pi} \left(\Delta_{n+m} D_t^2 u\right) u_t d\Pi =$$

$$= -\int\limits_{\Pi_R} \left[\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(D_t^2 u\right) \frac{\partial u_t}{\partial x_i} + \sum_{j=1}^{m} \frac{\partial}{\partial y_j} \left(D_t^2 u\right) \frac{\partial u_t}{\partial y_j} \right] d\Pi =$$

$$= -\frac{1}{2} \int\limits_{\Pi} D_t \left[\sum_{i=1}^{n} \left(\frac{\partial u_t}{\partial x_i}\right)^2 + \sum_{j=1}^{m} \left(\frac{\partial u_t}{\partial y_j}\right)^2 \right] d\Pi = \qquad (1.8)$$

$$= -\frac{1}{2} D_t \int\limits_{\Pi} \left[\sum_{i=1}^{n} \left(\frac{\partial u_t}{\partial x_i}\right)^2 + \sum_{j=1}^{m} \left(\frac{\partial u_t}{\partial y_j}\right)^2 \right] d\Pi \ .$$

Similarly

$$\int\limits_{\Pi} \left(\Delta_{n+m} u\right) u_t d\Pi = -\frac{1}{2} D_t \int\limits_{\Pi} \left[\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 + \sum_{j=1}^{m} \left(\frac{\partial u}{\partial y_j}\right)^2 \right] d\Pi \qquad (1.9)$$

Introduce notations

$$\int\limits_{\Pi} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 d\Pi = ||\nabla_x u||_{L_2(\Pi)}^2 \ ,$$

$$\int\limits_{\Pi} \sum_{i=1}^{m} \left(\frac{\partial u}{\partial y_i}\right)^2 d\Pi = ||\nabla_y u||_{L_2(\Pi)}^2 \ . \qquad (1.10)$$

Transform the second addend in (1.5)

$$\int\limits_{\Pi} \left(D_t^2 u\right) u_t d\Pi = \frac{1}{2} \int\limits_{\Pi} D_t \left(D_t u\right)^2 d\Pi = \frac{1}{2} D_t \int\limits_{\Pi} \left(D_t u\right)^2 d\Pi = \frac{1}{2} D_t \, ||D_t u||_{L_2}^2 \qquad (1.11)$$

we get from (1.5), (1.6), (1.8)-(1.11)

$$\int\limits_{0}^{t} D_t \left\{ \frac{1}{2} \left[\sigma^2 \left(||\nabla_x u_t||_{L_2(\Pi)}^2 + ||\nabla_y u_t||_{L_2(\Pi)}^2 \right) + ||u_t||_{L_2(\Pi)}^2 \right] + \right. \qquad (1.12)$$

$$\left. + \gamma^2 \left(||\nabla_x u||_{L_2(\Pi)}^2 + ||\nabla_y u||_{L_2(\Pi)}^2 \right) \right\} dt = 0$$

Denote energy integral of problem (1.1)-(1.3) by $E(t)$

$$E(t) = \frac{1}{2} \left[\sigma^2 \left(||\nabla_x u_t||_{L_2(\Pi)}^2 + ||\nabla_y u_t||_{L_2(\Pi)}^2 \right) + \right.$$

$$+\left.||u_t||^2_{L_2(\Pi)}\right] + \gamma^2\,||\nabla_x u||^2_{L_2(\Pi)} + ||\nabla_y u||^2_{L_2(\Pi)}$$

Then we get from (1.12)

$$E\,(t) = E\,(o)\,.$$

since for a homogeneous problem $E\,(0) = 0$, then

$$E\,(t) \equiv 0 \quad for \quad t > 0$$

Hence it follows that $u\,(x, y, t) \equiv 0$. The theorem is proved.

§2. Construction of Green's function for the stationary problem.

By virtue of estimation (1.4) there exist Fourier transformation with respect to x and Laplace transformation with respect to t of function $u\,(x, y, t)$ and its derivatives. Therefore performing Laplace transformation with respect to t in problem (1.1)-(1.3) we get

$$\left(\sigma^2\Delta_{n+m} - 1\right) k^2 V\,(x, y, k) + \gamma^2\Delta_{n+m} V\,(x, y, k) = \Phi\,(x, y, k)\,, \qquad (2.1)$$

$$V\,(x, y, k)\,|_{\partial\Pi} = 0 \qquad (2.2)$$

where

$$\Phi\,(x, y, k) = \left(\sigma^2\Delta_{n+m} - 1\right)(\psi_1\,(x, y) + k\psi_0\,(x, y)) \equiv f_1\,(x, y) + kf_2\,(x, y) \qquad (2.3)$$

Further performing Fourier transformation in problem (2.1)-(2.2) with respect to x, we get the following boundary value problem

$$\left(\sigma^2 k^2 + \gamma^2\right)\Delta_m \widetilde{V}\,(s, y, k) - \left[|s|^2\left(\sigma^2 k^2 + \gamma^2\right) + k^2\right]\widetilde{V}\,(s, y, k) = \widetilde{\Phi}\,(s, y, k) \qquad (2.4)$$

$$\widetilde{V}\,(s, y, k)\,|_{\partial\Omega} = 0 \qquad (2.5)$$

where $\widetilde{V}\,(s, y, k)$ and $\widetilde{\mathcal{F}}\,(s, y, k)$ is a Fourier transformation with respect to x of the functions $V\,(x, y, k)$ and $\mathcal{F}\,(x, y, k)$, $\operatorname{Re} k > 0$.

Consider a differential operator L, generalized by differential expression $L = \Delta_m$ with domain of definition

$$D\,(L) = \left\{w\,(y) : w\,(y) \in C^{(2)}\,(\Omega) \cap C\left(\overline{\Omega}\right),\;\; \Delta_m w\,(y) \in L_2\,(\Omega)\,,\;\; w\,(y)\,|_{\partial\Omega} = 0\right\}$$

Operator $\widetilde{L}$ is a negatively defined self-adjoint operator. It is known that [8, p.177-178], a spectrum of this operator is discrete and for its eigenvalues λ_l it holds the inequality

$$0 > \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_t \geq ..., \qquad \lim_{l\to\infty}\lambda_l = -\infty \qquad (2.6)$$

Eigenfunctions $\varphi_l\,(Y)$ of the operator L corresponding to eigenvalues λ_l forms a basis in space $L_2\,(\Omega)$. Using the abovesaid we prove the following theorem.

Let

$$C_\delta = \mathbb{C}\backslash\left[O_\delta\left(i\frac{\gamma}{\sigma}\right) \cup O_\delta\left(-i\frac{\gamma}{\sigma}\right)\right]\;,$$

where $O_\delta\,(k)$ is a circle of radius δ with a centre at the point k and

$$\mathbb{C}^+ = \{k : k \in \mathbb{C},\ \operatorname{Re} k > 0\}$$

$\mathbb{C}$ is a space of complex number.

Theorem 2. *Green's function of problem (2.1)-(2.2) is an analytical function of a complex parameter k excluding the points $k_{1,2}^{(1)} = \pm i\sqrt{\frac{\gamma}{\sigma}}$ and $k_{1,2}^{(2)} = \pm i\sqrt{\frac{\gamma^2}{\sigma^2+|\lambda_l|^{-1}}}$ which are singular points and for it the representation holds*

$$G(x,y,z,k) = -\frac{i}{4}\frac{(2\pi)^{-\frac{n}{2}}}{(k^2\sigma^2+\gamma^2)}|x|^{1-\frac{n}{2}}\sum_{l=1}^{\infty}\sqrt{\lambda_l - \frac{k^2}{k^2\sigma^2+\gamma^2}}^{\frac{n}{2}-1} \times$$
$$\times H_{\frac{n}{2}-1}^{(1)}\left(|x|\sqrt{\lambda_l - \frac{k^2}{k^2\sigma^2+\gamma^2}}\right)\varphi_l(Y)\varphi_l(z)\ , \tag{2.7}$$

where $H_{\frac{n}{2}-1}^{(1)}(z)$ is the first-kind Hankel function of the first genus of order $\frac{n}{2}-1$. For $|x| \geq \delta_1 > 0$ series in (2.7) uniformly converges with respect to (k,x,y,z) in every compact $K \subset \overline{\Pi} \times \overline{C}_\delta \cap C^+$.

Proof. For constructing Green's function of problem (1.1)-(2.2) we'll apply the method of [9]. Using theorem 3.6 from [8, p.177], for the solution of problem (2.4)-(2.5) we have

$$\widetilde{V}(s,y,k) = \sum_{l=1}^{\infty}\frac{C_l(s,k)\varphi_l(y)}{(k^2\sigma^2+\gamma^2)\lambda_l - [|s|^2(\sigma^2k^2+\gamma^2)+k^2]} \tag{2.8}$$

where

$$C_l(s,k) = \int_{\Omega}\widetilde{\Phi}(s,y,k)\varphi_l(y)\,dy\ .$$

The solution of the problem (2.2)-(2.3) is defined as the inverse Fourier transformation from $\widetilde{V}(s,y,k)$

$$V(x,y,k) = \frac{1}{(2\pi)^n}\sum_{l=1}^{\infty}\varphi_l(y)\int_{R_n}\frac{C_l(s,k)e^{-i(s,x)}ds}{(k^2\sigma^2+\gamma^2)\lambda_l - [|s|^2(\sigma^2k^2+\gamma^2)+k^2]}, \tag{2.9}$$

here term by term integration is valid by virtue of uniform convergence of series (2.8) [10, p.253] and convergence of series (2.9) in $\overline{C}_\delta$. Note that $C_l(s,k)$ sufficiently fast decrease over l and $|s|$ by virtue of the fact that $\psi_0(x,y)$, $\psi_1(x,y)$ are finite and sufficiently smooth functions of (x,y). Allowing for

$$\widetilde{\Phi}(s,y,k) = \mathcal{F}(\Phi(x,y,k))\ ,$$

where $\mathcal{F}$ is a Fourier transformation with respect to x from (2.9) we get

$$V(x,y,k) = \frac{1}{(2\pi)^n}\sum_{l=1}^{\infty}\varphi_l(y)\times$$
$$\times\int_{R_n}\Phi_l(\xi,k)\left[\int_{R_n}\frac{e^{-i(s,\xi-x)}ds}{(k^2\sigma^2+\gamma^2)\lambda_l - [|s|^2(\sigma^2k^2+\gamma^2)+k^2]}\right]d\xi \tag{2.10}$$

where

$$\Phi_l(\xi,k)=\int\limits_{\Omega}\Phi(\xi,y,k)\,\varphi_l(y)\,dy$$

Here change of integration order is valid, and it is performed as in [11, p.377-382]. Moreover, it is required that functions $\psi_0(x,y),\psi_1(x,y)$ have absolutely summable on the whole space R_n derivatives with respect to x up to the order $\left[\frac{n}{2}\right]+1$. Calculate intrinsic integral in (2.10). For this aim denote $\tau=\xi-x$ and

$$\mathcal{J}_l(\tau,k)=\frac{1}{(2\pi)^n}\lim_{N\to\infty}\int\limits_{|s|\le N}\frac{e^{i(s,\tau)}}{(k^2\sigma^2+\gamma^2)\lambda_l-\left[|s|^2(\sigma^2k^2+\gamma^2)+k^2\right]}\equiv$$

$$\equiv\frac{1}{(2\pi)^n}\lim_{N\to\infty}\mathcal{J}_{l,N}(\tau,k) \tag{2.11}$$

Passing to spherical coordinates in (2.11) and allowing for spherical symmetry of integrand in (2.11) we get

$$\mathcal{J}_{l,N}(\tau,k)=(2\pi)^{-\left(\frac{n}{2}+1\right)}|\tau|^{1-\frac{n}{2}}\int\limits_0^N\frac{|s|^{\frac{n}{2}}J_{\frac{n}{2}-1}(|\tau|\,|s|)\,d|s|}{(k^2\sigma^2+\gamma^2)\lambda_l-\left[|s|^2(\sigma^2k^2+\gamma^2)+k^2\right]} \tag{2.12}$$

where $J_{\frac{n}{2}-1}(z)$ is a Bessel function of order $\frac{n}{2}-1$.

Applying a residue method, we calculate the integral in (2.12). Let n be an odd number. Then $z^{\frac{n}{2}}J_{\frac{n}{2}-1}(z)$ is an even function. Therefore

$$\mathcal{J}_{l,N}(\tau,k)=\frac{1}{2}(2\pi)^{-\left(\frac{n}{2}+1\right)}|\tau|^{1-\frac{n}{2}}\times$$

$$\times\int\limits_{-N}^{N}\frac{|s|^{\frac{n}{2}}J_{\frac{n}{2}-1}(|\tau|\,|s|)\,d|s|}{(k^2\sigma^2+\gamma^2)\lambda_l-\left[|s|^2(\sigma^2k^2+\gamma^2)+k^2\right]} \tag{2.13}$$

Now, using formula [12, p.175]

$$J_{\frac{n}{2}-1}(z)=\frac{1}{2}\left(H^{(1)}_{\frac{n}{2}-1}(z)+H^{(2)}_{\frac{n}{2}-1}(z)\right) \tag{2.14}$$

we get from (2.13)

$$\mathcal{J}_{l,N}(\tau,k)=\frac{1}{4}(2\pi)^{-\left(\frac{n}{2}+1\right)}|\tau|^{1-\frac{n}{2}}\times$$

$$\times\int\limits_{-N}^{N}\frac{|s|^{\frac{n}{2}}\left[H^{(1)}_{\frac{n}{2}-2}(|\tau|\,|s|)+H^{(2)}_{\frac{n}{2}-2}(|\tau|\,|s|)\right]d|s|}{(k^2\sigma^2+\gamma^2)\lambda_l-\left[|s|^2(\sigma^2k^2+\gamma^2)+k^2\right]} \tag{2.15}$$

The poles of integrand in (2.15) are at the points

$$|S|^{(l)}_{1,2}=\pm\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}} \tag{2.16}$$

here we take a branch for the root, for which $\sqrt{-1}=i$. For $\operatorname{Re} k>0$ the roots are equally arranged in upper and lower half-planes symmetrically with respect to origin of coordinates. Allowing for the analyticity of integrand in (2.15) and asymptotics of Hankel functions at $z\to\infty$ (for $H^{(1)}_{\frac{n}{2}-1}(z)$ at $\operatorname{Im} z>0$ and for $H^{(2)}_{\frac{n}{2}-1}(z)$ at $\operatorname{Im} z<0$) and using a residue method we get

$$\mathcal{J}_l(\tau,k)=-\frac{(2\pi)^{-\frac{n}{2}}\,i}{8\left(k^2\sigma^2+\gamma^2\right)}|\tau|^{1-\frac{n}{2}}\times$$

$$\times\left[\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}}^{\,\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}\left(|\tau|\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}}\right)-\right.$$

$$\left.-\left(-\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}}\right)^{\frac{n}{2}-1}H^{(2)}_{\frac{n}{2}-1}\left(-|\tau|\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}}\right)\right] \tag{2.17}$$

allowing for [12, p.218]

$$H^{(2)}_{\frac{n}{2}-1}(-z)=(-1)^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}(z), \tag{2.18}$$

for $\mathcal{J}_l(\tau,k)$ we get from (2.17)

$$\mathcal{J}_l(\tau,k)=-\frac{(2\pi)^{-\frac{n}{2}}\,i\,|\tau|^{1-\frac{n}{2}}}{4\left(k^2\sigma^2+\gamma^2\right)}\times$$

$$\times\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}}^{\,\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}\left(|\tau|\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}}\right) \tag{2.19}$$

Now let n be an even number. Then $z^{\frac{n}{2}}J_{\frac{n}{2}-1}(z)$ is an odd function. Expressing the Bessel function by Hankel function according to formula (2.14), in addition performing a cut $(-\infty,0)$, since Hankel functions have a logarithmic branching point at the point $z=0$ for entire indices, and allowing for (2.18) we get

$$\mathcal{J}_{l,N}(\tau,k)=\frac{(2\pi)^{-\left(\frac{n}{2}+1\right)}}{2}|\tau|^{1-\frac{n}{2}}\int\limits_0^N\frac{|s|^{\frac{n}{2}}\left[H^{(1)}_{\frac{n}{2}-1}(|\tau|\,|s|)+H^{(2)}_{\frac{n}{2}-1}(|\tau|\,|s|)\right]ds}{\left(k^2\sigma^2+\gamma^2\right)\lambda_l-\left[|s|^2\left(\sigma^2k^2+\gamma^2\right)+k^2\right]}$$

Using (2.18) we get

$$\mathcal{J}_{l,N}(\tau,k)=\frac{(2\pi)^{-\left(\frac{n}{2}+1\right)}}{2}|\tau|^{1-\frac{n}{2}}\int\limits_{-N}^N\frac{|s|^{\frac{n}{2}}H^{(1)}_{\frac{n}{2}-1}(|\tau|\,|s|)\,ds}{\left(k^2\sigma^2+\gamma^2\right)\lambda_l-\left[|s|^2\left(\sigma^2k^2+\gamma^2\right)+k^2\right]}. \tag{2.20}$$

Applying a residue method to integral (2.20), going out to upper half-plane, and tending $N\to\infty$ we get

$$\mathcal{J}_l(\tau,k)=\frac{(2\pi)^{-\frac{n}{2}}\,i}{4}|\tau|^{1-\frac{n}{2}}\frac{\sqrt{\lambda_l-\dfrac{k^2}{k^2\sigma^2+\gamma^2}}^{\,\frac{n}{2}-1}}{k^2\sigma^2+\gamma^2}\times$$

$$\times H^{(1)}_{\frac{n}{2}-1}\left(|\tau|\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}}\right) \tag{2.21}$$

Thus, for $\mathcal{J}_l(\tau,k)$ at even and odd n we got the same expression (2.19), (2.21).

Substituting this expressions into (2.10) and changing the order of integration and summation, for the solution of problem (2.1)-(2.2) we get

$$V(x,y,k)=-\frac{(2\pi)^{-\frac{n}{2}}i}{4}\frac{i}{k^2\sigma^2+\gamma^2}\int\limits_{\Pi}|x-\xi|^{1-\frac{n}{2}}\sum_{l=1}^{\infty}\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}}^{\,\frac{n}{2}-1}\times$$
$$\times H^{(1)}_{\frac{n}{2}-1}\left(|x-\xi|\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}}\right)\varphi_l(y)\,\varphi_l(z)\,\Phi(\xi,z,k)\,d\Pi \tag{2.22}$$

Hence for the Green's function of problem (2.1)-(2.2) for $\operatorname{Re}k>0$ we get the following expression

$$G(x,y,z,k)=-\frac{(2\pi)^{-\frac{n}{2}}\,i\,|x|^{1-\frac{n}{2}}}{4(k^2\sigma^2+\gamma^2)}\sum_{l=1}^{\infty}\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}}^{\,\frac{n}{2}-1}\times$$
$$\times H^{(1)}_{\frac{n}{2}-1}\left(|x|\sqrt{\lambda_l-\frac{k^2}{k^2\sigma^2+\gamma^2}}\right)\varphi_l(y)\,\varphi_l(z)\ .$$

Now study the convergence of series in (2.7) and its derivatives up to the second order. To this end we prove the following lemma.

Lemma 1. *At sufficiently large l and $k\in\overline{C}_{\delta_1}$ the asymptotics*

$$\sqrt{-\lambda_l+\frac{k^2}{k^2\sigma^2+\gamma^2}}=\sqrt{-\lambda_l-}\,(1+o(1)) \tag{2.23}$$

holds.

Represent the left hand side of (2.23) in the form

$$\sqrt{-\lambda_l+\frac{k^2}{k^2\sigma^2+\gamma^2}}=\sqrt{-\lambda_l\left(1+\frac{k^2}{\lambda_l(k^2\sigma^2+\gamma^2)}\right)} \tag{2.24}$$

For $k\in\mathbb{C}_{\delta_1}$

$$\left|\frac{k^2}{k^2\sigma^2+\gamma^2}\right|\le M\ ,$$

M is some number. Therefore for $l\to\infty$ by virtue of (2.6) we have

$$\left|\frac{k^2}{\lambda_l(k^2\sigma^2+\gamma^2)}\right|=o(1)\ .$$

Then we get from (2.24)

$$\sqrt{-\lambda_l+\frac{k^2}{k^2\sigma^2+\gamma^2}}=\sqrt{-\lambda_l}\,(1+o(1))\ .$$

Lemma 1 is proved.

Now continue the proof of theorem 1. In [9] it is shown that

$$||\varphi_l(Y)||_{H^{([\frac{m}{2}]+1)}_{(\Omega)}} \le C\,|\lambda_l|^{([\frac{m}{2}]+1)/2}$$

where parenthesis $[\tau]$ means the entire part of τ.

Hence by means of Sobolev's embedding theorem we get

$$||\varphi_l(Y)||_{C(\overline{\Omega})} \le C\,|\lambda_l|^{([\frac{m}{2}]+1)/2} \tag{2.25}$$

It is known that [10, p.190]

$$c_0 l^{\frac{2}{m}} \le |\lambda_l| \le c_1 l^{\frac{2}{m}} \tag{2.26}$$

where c_0, c_1 are constants not depending on l. Then it follows from (2.25) and (2.26)

$$||\varphi_l(Y)||_{C(\overline{\Omega})} \le C l^{([\frac{m}{2}]+1)/m} \tag{2.27}$$

Since $\Delta^v\varphi_l(y)\ \ (v \ge 1)$ is also an eigenfunction of the operator L with eigenvalue λ_l^v, then as above, we can show that

$$||\varphi_l(Y)||_{C^{(v)}(\overline{\Omega})} \le C l^{([\frac{m}{2}]+1)/m} \tag{2.28}$$

Now prove a uniform convergence of the series (2.7) with respect to (x, y, z, k) in each compact $K \subset \overline{\Pi} \times \overline{C}_\delta$ for $|x| \ge \delta_1 > 0$.

Estimating on modulus, we get

$$|G(x,y,z,k)| \le C_0 \left[1 + \sum_{l=l_0}^{\infty} \left|\sqrt{\lambda_l - \frac{k^2}{k^2\sigma^2+\gamma^2}}\right|^{\frac{n}{2}-1} \times \right.$$

$$\left. \times\, H^{(1)}_{\frac{n}{2}-1}\left(|x|\sqrt{\lambda_l - \frac{k^2}{k^2\sigma^2+\gamma^2}}\right) ||\varphi_l(Y)||^2_{C(\overline{\Omega})}\right], \tag{2.29}$$

C_0 is a constant and l_0 is sufficiently large number.

Further, using the asymptotics of Hankel function $H^{(1)}_{\frac{n}{2}-1}(z)$ for $z \to \infty$ lemma 1 and estimates (2.25), (2.27) we get from (2.29)

$$||G(x,y,z,k)||_{C(K)} \le C_0 \left[1 + \sum_{l=l_0}^{\infty} |\lambda_l|^{\frac{n+1}{4}+[\frac{m}{2}]} e^{-\delta_1\sqrt{\frac{-\lambda_l}{2}}}\right] \le$$

$$\le C_0 \left[1 + \sum_{l=l_0}^{\infty} l^{\frac{n+1}{2m}+1} e^{-\delta_1\sqrt{\frac{c_0}{2}}\, l^{\frac{1}{m}}}\right].$$

Hence, it follows a uniform convergence of series in (2.7) in the compact K for $|x| \ge \delta_1 > 0$. Using the estimate (2.26) we can show as above that the series in (2.7) may be term-by-term differentiated with respect to (x, y, z) for $|x| \geq \delta_1 > 0, \quad k \in \overline{C}_\delta \cap C^+$.

Theorem 2 is proved.

Corollary 1. *The Green function $G(x, y, z, k)$ is an even function with respect to k. Therefore we can evenly continue it to the left half-plane. Thus, $G(x, y, z, k)$ is defined with respect to k on the all complex plane with singular points $k = \pm\frac{i\delta}{\sigma}$, $k = \pm i\delta\left(\sigma^2 - \frac{1}{\lambda_l}\right)^{-\frac{1}{2}}$.*

Putting the expression $\Phi(x, y, k)$ from (2.3) to (2.22), for the solution of the problem (2.1)-(2.2) we get

$$\begin{gathered} V(x, y, k) = \int\limits_{\Pi} G(x - \xi, y, z, k) f_1(\xi, z)\, d\Pi + \\ + k\int\limits_{\Pi} G(x - \xi, y, z, k) f_2(\xi, z)\, d\Pi \equiv V_1(x, y, k) + kV_2(x, y, k) \end{gathered} \tag{2.30}$$

§3. Behaviour of solution of mixed problem for Boussineska equation. Solution $u(x, y, k)$ of nonstationary problem (1.1)-(1.3) is defined as the inverse Laplace transformation with respect to k from $V(x, y, k)$. Then we have form (2.33)

$$u(x, y, t) = u_1(x, y, t) + u_2(x, y, t), \tag{3.1}$$

where $u_j(x, y, t)$ is the inverse Laplace transformation with respect to k from $V_j(x, y, k)$, $j = 1, 2$, in this $u_1(x, y, k)$ is a solution of problem (1.1)-(1.3) with initial data

$$u_1(x, y, 0) = \psi_0(x, y), \quad u'_{1t}(x, y, 0) = 0 \tag{3.2$_1$}$$

and $u_2(x, y, k)$ is a solution of problem (1.1)-(1.3) with initial data

$$u_2(x, y, 0) = 0, \quad u'_{2t}(x, y, 0) = \psi_1(x, y) \tag{3.2$_2$}$$

Now we get estimate (1.4) for the solution of problem (1.1)-(1.3). To this end the following lemmas are necessary

Lemma 2. *For $|k| \geq N$ for all l*

$$\mathrm{Re}\sqrt{-\lambda_l + \frac{k^2}{k^2\sigma^2 + \gamma^2}} > \sqrt{-\lambda_l} > \sqrt{-\lambda_1},$$

where N is a sufficiently large number.

Proof. Using the formula for a real part of a quadratic root of complex number, and relation (2.6) we have

$$\begin{gathered} \mathrm{Re}\sqrt{-\lambda_l + \frac{k^2}{k^2\sigma^2 + \gamma^2}} = \mathrm{Re}\sqrt{-\lambda_l + |\mu(k)|\, e^{i\theta}} > \\ > \sqrt{-\lambda_l + |\mu(k)|\cos\theta} > \sqrt{-\lambda_l} > \sqrt{-\lambda_1}, \end{gathered}$$

where

$$\mu(k) = \left(\sigma^2 + \frac{\gamma^2}{k^2}\right)^{-1}, \quad \theta = \arg\mu(k) \tag{3.3}$$

Since $|k| \geq N$ where N is sufficiently large number, then θ will be sufficiently small angle. Lemma 2 is proved.

Lemma 3. *For* $\mathrm{Re}\, k \geq \varepsilon > 0$ *for all* l *the estimate*

$$\mathrm{Re}\sqrt{-\lambda_l + \frac{k^2}{k^2\sigma^2 + \gamma^2}} \geq c_2 > 0$$

is valid, where c_2 *is a constant independent of* l.

Proof. For $l > l_0$ or for $|k| > N$, where l_0, N are sufficiently large numbers, the proof of lemma follows from lemmas 1 and 2 respectively. Therefore we'll assume that $l < l_0$ and $|k| \leq N$. Then for $\mathrm{Re}\, k \geq \varepsilon > 0$

$$-\frac{\pi}{2} + \delta \leq \arg k \leq \frac{\pi}{2} - \delta$$

and for the points k^2 and $k^2\sigma^2 + \gamma^2$ we have

$$0 \leq \arg\left(k^2\sigma^2 + \gamma^2\right) \leq \arg k^2 < \pi - 2\delta, \quad 0 < \theta \leq \pi - 2\delta$$

for Im $k \geq 0$ and for Im $k \leq 0$

$$-\pi + 2\delta \leq \arg k^2 \leq \arg\left(k^2\sigma^2 + \gamma^2\right) \leq 0, \quad -\pi + 2\delta \leq \theta < 0$$

where $\delta = \delta(\varepsilon)$ and $\delta(\varepsilon) \to 0$ for $\varepsilon \to 0$, θ is defined in (3.3). Denote

$$\theta_1 = \arg\left(-\lambda_l + \frac{k^2}{k^2\sigma^2 + \gamma^2}\right)$$

analogously

$$\begin{aligned} 0 &\leq \theta_1 < \pi - 2\delta \quad \mathit{for} \quad \mathrm{Im}\ k \geq 0, \quad \mathrm{Re}\, k \geq \varepsilon > 0, \\ -\pi + 2\delta &< \theta_1 \leq 0, \quad \mathit{for} \quad \mathrm{Im}\, k < 0\ , \quad \mathrm{Re}\, k \geq> 0\ . \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{Re}\sqrt{-\lambda_l + \tfrac{k^2}{k^2\sigma^2+\gamma^2}} &= \left|-\lambda_l + \tfrac{k^2}{k^2\sigma^2+\gamma^2}\right|^{\frac{1}{2}} \cos\tfrac{\theta_1}{2} \geq \\ \geq \left|-\lambda_l + \tfrac{k^2}{k^2\sigma^2+\gamma^2}\right|^{\frac{1}{2}} \cos\left(\tfrac{\pi}{2} - \delta\right) &= \left|-\lambda_l + \tfrac{k^2}{k^2\sigma^2+\gamma^2}\right|^{\frac{1}{2}} \sin\delta \end{aligned} \tag{3.4}$$

The zeros of the function

$$F(l,k) = \left|-\lambda_l + \frac{k^2}{k^2\sigma^2 + \gamma^2}\right|$$

are on the imaginary axis at points

$$k_{1,2} = \pm i\sqrt{\frac{-\lambda_l\gamma^2}{1 - \lambda_l\sigma^2}}$$

since $\lambda_l < 0$. Therefore for $k \leq N$ $(\mathrm{Re}\, k \geq \varepsilon > 0)$ and $l \leq l_0$ there exists a number M_0, such that

$$F(l,k) \geq M_0 \tag{3.5}$$

Then it follows form (3.4), (3.5) that

$$\mathrm{Re}\sqrt{-\lambda_l + \frac{k^2}{k^2\sigma^2 + \gamma^2}} \geq M_0^{1/2}\sin\delta$$

If we assume

$$c_2 = \min\left\{|\lambda_1|^{1/2}, M_0^{1/2}\sin\delta\right\},$$

then we get the proof of lemma 3 from lemmas 1,2 and (36).

Theorem 3. *If* $\psi_0(x,y)$, $\psi_1(x,y) \in C_0^{2,\mu}(\Pi)$, *where* $\mu = \left[\frac{m}{2}\right] + m + \frac{n+3}{2}$ *then for solution of the problem it holds estimate (1.4).*

Proof. By integrating the series term-by-term in (2.9) we get

$$u_j(x,y,t) = -\tfrac{i}{4}(2\pi)^{-\frac{n}{2}}\textstyle\sum_{l=1}^{\infty}\varphi_l(y)\times$$

$$\times\int\limits_{R_n}|x-\xi|^{1-\frac{n}{2}}\left[\frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty}\sqrt{\lambda_l - \tfrac{k^2}{k^2\sigma^2+\gamma^2}}^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}\times\right. \tag{3.7}$$

$$\left.\times\left(|x-\xi|\sqrt{\lambda_l - \tfrac{k^2}{k^2\sigma^2+\gamma^2}}\right)\tfrac{k^{i-1}}{k^2\sigma^2+\gamma^2}e^{kt}dk\right]f_{jl}(\xi)\,d\xi\ ,$$

where

$$f_{jl}(\xi) = \int\limits_{\Omega} f_j(\xi,z)\,\varphi_l(z)\,dz, \qquad j=1,2\ .$$

here term by term integration is valid by virtue of uniform convergence of series (2.9) and (3.7) that will be shown later. Denote

$$B_{jl}(\eta,t) = \frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty}\sqrt{\lambda_l - \frac{k^2}{k^2\sigma^2+\gamma^2}}^{\frac{n}{2}-1}\times$$

$$\times H^{(1)}_{\frac{n}{2}-1}\left(\eta\sqrt{\lambda_l - \frac{k^2}{k^2\sigma^2+\gamma^2}}\right)\frac{k^{i-1}}{k^2\sigma^2+\gamma^2}dk, \tag{3.8}$$

where $\eta = |x-\xi|$, $j=1,2$. Estimate $B_{jl}(\eta,t)$ at large l and η. To this end we introduce the following contour

$$\Gamma_\varepsilon = L_\varepsilon^- \cup (\varepsilon - iN,\ \varepsilon + iN) \cup L_\varepsilon^+,$$

where L_ε^- is a ray starting from the point $\varepsilon - iN$ and composing with negative imaginary semi-axis the angle -$\frac{\pi}{6}$ and L_ε^+ is a ray starting form the point $\varepsilon + iN$ and composing with positive imaginary semi-axis the angle $+\frac{\pi}{6}$. Further by Cauchy theorem we substitute in the expression $B_{jl}(\eta,t)$ an integration contour into Γ_ε, along which integrand for $k\to\infty$ decreases exponentially. Estimating by modulus

$B_{jl}(\eta,t)$ and its derivatives with respect to t, assuming here η sufficiently large, taking into account asymptotics of Hankel function $H^{(1)}_{\frac{n}{2}-1}(z)$ for $z\to\infty$ and lemmas 1-3, we get

$$|D_t^v B_{jl}(\eta,t)|\le C|\lambda_l|^{\frac{n-3}{4}}e^{\varepsilon t-c_2\eta}, \tag{3.9}$$
$$v=0,1,2;\quad j=1,2;\quad l=1,2,...$$

Estimating (3.7) by modulus, where integration contour with respect to k is substituted into Γ_ε and using estimate (3.9) we get

$$|u_j(x,y,t)|\le Ce^{\varepsilon t-c_2|x|}\sum_{l=1}^{\infty}||\varphi_l(y)||_{C(\overline{\Omega})}|\lambda_l|^{\frac{n-3}{4}}\times$$
$$\times\int_{Q_j}e^{c_2|\xi|}|x-\xi|^{1-\frac{n}{2}}|f_{jl}(\xi)|d\xi \tag{3.10}$$

Q_j is a support of the function $\psi_j(\xi,\eta)$ with respect to ξ. Using estimate (2.25), we get from (3.10)

$$|u_j(x,y,t)|\le Ce^{\varepsilon t-c_2|x|}\sum_{l=1}^{\infty}|\lambda_l|^{\left[\frac{m}{2}\right]+m+\frac{n-1}{2}}\int_{Q_j}e^{c_2|\xi|}|x-\xi|^{1-\frac{n}{2}}|f_{jl}(\xi)|d\xi \tag{3.11}$$

Represent (3.11) in the form

$$|u_j(x,y,t)|\le Ce^{\varepsilon t-c_2|x|}\times$$
$$\times\left[\sum_{l=1}^{\infty}|\lambda_l|^{-m}+\sum_{l=1}^{\infty}|\lambda_l|^{\left[\frac{m}{2}\right]+m+\frac{n-1}{2}}\left(\int_{Q_j}e^{c_2|\xi|}|x-\xi|^{1-\frac{n}{2}}|f_{jl}(\xi)|d\xi\right)^2\right] \tag{3.12}$$

Applying Cauchy-Bunyakovski inequality to inequality (3.12) taking into account, that at large $|x|$

$$\int_{Q_j}e^{2c_2|\xi|}|x-\xi|^{2-n}d\xi\le C|x|^{2-n}$$

then we get

$$|u_j(x,y,t)|\le Ce^{\varepsilon t-c_2|x|}\left[\sum_{l=1}^{\infty}|\lambda_l|^{-m}+\sum_{l=1}^{\infty}|\lambda_l|^{\left[\frac{m}{2}\right]+m+\frac{n-1}{2}}\int_{Q_j}f_{jl}^2(\xi)d\xi\right] \tag{3.13}$$

By B.Levi theroem [13, p.142] we get from (3.13)

$$|u_j(x,y,t)|\le Ce^{\varepsilon t-c_2|x|}\left[\sum_{l=1}^{\infty}|\lambda_l|^{-m}+\int_{Q_j}\sum_{l=1}^{\infty}|\lambda_l|^{\left[\frac{m}{2}\right]+m+\frac{n-1}{2}}f_{jl}^2(\xi)d\xi\right] \tag{3.14}$$

Since the functions $f_j(\xi,z)$, $j=1,2$ with respect to z satisfy the conditions of theorem [8] from [10, p.253], then

$$\sum_{l=1}^{\infty}|\lambda_l|^{\mu}f_{jl}^2(\xi)=||f_j(\xi,z)||^2_{H^{\mu}(\Omega)},\qquad \mu=\left[\frac{m}{2}\right]+m+\frac{n-1}{2} \tag{3.15}$$

From (3.14) and (3.15) it follows

$$|u_j(x,y,t)|\le Ce^{\varepsilon t-c_2|x|}\left[\sum_{l=1}^{\infty}|\lambda_l|^{-m}+\int\limits_{Q_j}||f_j(\xi,z)||\,d\xi\right],\quad j=1,2 \tag{3.16}$$

Series $\sum_{l=1}^{\infty}|\lambda_l|^{-m}$ in (3.16) converges by virtue of estimate (2.26).

We have from (3.1) and (3.16)

$$|u(x,y,t)|\le Ce^{\varepsilon t-c_2|x|} \tag{3.17}$$

By virtue of estimate (2.28), smoothness of functions $f_l(\xi)$ in the same way as above we can get estimate (3.17) of derivatives $u(x,y,t)$ contained in equation (1.1). For this in formula (3.7) $\varphi_l(y)$should be substituted into $D_y^{\beta}\varphi_l(y)$ and $f_l(\xi)$ into $D_{\xi}^{\alpha}f_l(\xi)$. Thus, estimate (1.4) for the solution of problem (1.1)-(1.3) is proved.

Theorem 4. *Let $n=1$ $\psi_0(x,y)$, $\psi_1(x,y)\in C_0^{(2,\mu)}(\Pi)$, $\mu=\left[\frac{m}{2}\right]+m+1$. Then at $t\to+\infty$ for the solution of problem (1.1)-(1.3) it holds asymptotic estimate*

$$u(x,y,t)=o(1)$$

uniformly with respect to (x,y) in each compact from Π.

Proof. To study the asymptotics of solution of problem (1.1)-(1.3) at $t\to+\infty$ it is sufficient to study an asymptotics of integrals (3.8) at $t\to+\infty$. The integrand in (3.8) have singular points $k_{1,2}^{(1)}=\pm i\frac{\gamma}{\sigma}$, $k_{1,2}^{(2)}=\pm i\sqrt{\frac{\gamma^2}{\sigma^2-\lambda_l^{-1}}}$. Let's perform the cut $\left(k_1^{(1)},k_2^{(1)}\right)$ on the plane k. By $C_{\varepsilon}^{(1),(2)}$ denote a circle of radius ε with a center at points $k_{1,2}^{(1)}$ and by $C_{\varepsilon}^{(3),(4)}$ a circle of radius ε with a center at points $k_{1,2}^{(2)}$.

We also denote

$$\begin{aligned} L_{\varepsilon}^{(1)} &= C_{\varepsilon}^{(1)}\cup J_{1\varepsilon}^{+}\cup C_{\varepsilon}^{(3)}\cup J_{2\varepsilon}^{+}\cup C_{\varepsilon}^{+(4)}\cup J_{3\varepsilon}^{+}\cup C_{\varepsilon}^{(2)}\ , \\ L_{\varepsilon}^{(2)} &= J_{3\varepsilon}^{-}\cup C_{\varepsilon}^{-(1)}\cup J_{2\varepsilon}^{-}\cup C_{\varepsilon}^{-(3)}\cup J_{1\varepsilon}^{-}\ , \end{aligned}$$

where $J_{1\varepsilon}^{+\pm}, J_{2\varepsilon}^{\pm}, J_{3\varepsilon}^{\pm}$ compose the left and right banks of the cut $\left(k_1^{(1)},k_2^{(1)}\right)$, respectively, and $C_{\varepsilon}^{\pm(3),(4)}$ -semi-circles of circles $C_{\varepsilon}^{(3),(4)}$, arranged in the right and left half-planes k respectively.

Assuming in (3.8) $n=1$, taking into account the obvious from of $H_{-\frac{1}{2}}^{(1)}(z)$ and that integrand decreases exponentially at $\operatorname{Re}k<0$, applying Cauchy theorem we get

$$B_{jl}(\eta,t)=-\frac{\left(\sqrt{\lambda_l+\frac{1}{\sigma^2}}\right)^{-1}}{\sqrt{2}\pi^{3/2}\sigma^2\eta^{1/2}}\left(\int\limits_{L_{\varepsilon}^{(1)}}+\int\limits_{L_{\varepsilon}^{(2)}}\right)\sqrt{k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}}^{\,-1}\frac{k^{j-1}e^{kt}}{\left(k^2+\frac{\gamma^2}{\sigma^2}\right)^{1/2}}$$

$$\exp\left[-\eta\sqrt{|\lambda_l|+\frac{1}{\sigma^2}}\sqrt{\frac{k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}}{k^2+\frac{\gamma^2}{\sigma^2}}}\right]dk \qquad (3.18)$$

Let

$$\theta_2=\frac{1}{2}\arg\frac{k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}}{k^2+\frac{\gamma^2}{\sigma^2}}.$$

Consider the change of θ_2 at moving k in the positive direction along the banks of the cut $\left(k_1^{(1)},k_2^{(1)}\right)$. Then

$$\theta_2=\begin{cases} from\ -\frac{\pi}{2}\ to\ \frac{\pi}{2}, & at\quad k\in C_\varepsilon^{(1)}\cup J_{1\varepsilon}^{+}, \\ from\ \frac{\pi}{2}\ to\ 0, & at\quad k\in C_\varepsilon^{+(3)}\cup J_{2\varepsilon}^{+}, \\ from\ 0\ to\ -\frac{\pi}{2}, & at\quad k\in C_\varepsilon^{+(4)}\cup J_{3\varepsilon}^{+}, \\ from\ -\frac{\pi}{2}\ to\ \frac{\pi}{2}, & at\quad k\in C_\varepsilon^{(2)}\cup J_{3\varepsilon}^{-}, \\ from\ \frac{\pi}{2}\ to\ 0, & at\quad k\in C_\varepsilon^{-(4)}\cup J_{2\varepsilon}^{-}, \\ from\ 0\ to\ -\frac{\pi}{2}, & at\quad k\in C_\varepsilon^{-(3)}\cup J_{1\varepsilon}^{-}, \end{cases} \qquad (3.19)$$

Allowing for (3.19) we get that at $k\in L_\varepsilon^{(1)}\cup L_\varepsilon^{(2)}$

$$\mathrm{Re}\sqrt{\frac{k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}}{k^2+\frac{\gamma^2}{\sigma^2}}}\geq 0$$

and integrand in (3.18) has a summable singularity. Therefore in (3.18) we can pass to the limit at $\varepsilon\to 0$. Then integrals along circles $C_\varepsilon^{(1),(2)}$ and semicircles $C_\varepsilon^{+(3),(4)}$ tend to zero. Then integral in (3.18) will be on contour $L^{(1)}\cup L^{(2)}$ where $L^{(1)}=\sum\limits_{\tau=1}^{3}J_\tau^{+}$, $L^{(2)}=\sum\limits_{\tau=1}^{3}J_\tau^{-}$, $J_\tau^{\pm}=\lim\limits_{\varepsilon\to 0}J_{\tau\varepsilon}^{\pm}$.

Now consider the change

$$\theta_3=-\frac{1}{2}\arg\left(k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}\right)\left(k^2+\frac{\gamma^2}{\sigma^2}\right)$$

at passing from J_τ^{-} to J_τ^{+} $(\tau=1,2,3)$ that is necessary at estimating integrals on the banks of the cut $\left(k_1^{(1)},k_2^{(1)}\right)$

$$\begin{array}{llll} on\ J_1^{-} & \theta_3=-\frac{3\pi}{2}; & on\ J_1^{+} & \theta_3=-\frac{\pi}{2}; \\ on\ J_2^{-} & \theta_3=-2\pi; & on\ J_2^{+} & \theta_3=0; \\ on\ J_3^{-} & \theta_3=-\frac{\pi}{2}; & on\ J_3^{+} & \theta_3=\frac{\pi}{2}; \end{array} \qquad (3.20)$$

Consider the integrals on J_1^{-} and J_1^{+}. Allowing for (3.19), (3.20) for $\varepsilon=0$ we get from (3.18)

$$Q_{jl}(t)=\left(\int\limits_{J_1^{-}}+\int\limits_{J_1^{+}}\right)k^{j-1}e^{kt}\sqrt{\left(k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}\right)\left(k^2+\frac{\gamma^2}{\sigma^2}\right)}^{\,-1}\times$$

$$\times \exp\left[-\eta\sqrt{|\lambda_l|+\frac{1}{\sigma^2}}\sqrt{\frac{k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}}{k^2+\frac{\gamma^2}{\sigma^2}}}\right]=$$

$$=2i\int\limits_{J_1^-}k^{j-1}e^{kt}\sqrt{\left(k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}\right)\left(k^2+\frac{\gamma^2}{\sigma^2}\right)}^{-1}\times$$

$$\times \sin\eta\sqrt{|\lambda_l|+\frac{1}{\sigma^2}}\left|\sqrt{\frac{k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}}{k^2+\frac{\gamma^2}{\sigma^2}}}\right|dk$$

Since the integrand in the expression Q_{1j} has a summable singularity, then by Riemann-Lebesque lemma at $t\to+\infty$

$$Q_{1j}(t)=o(1),\quad j=1,2 \tag{3.21}$$

Analogously, allowing for (3.19), (3.20) we prove that at $t\to+\infty$

$$Q_{3j}(t)=o(1),\quad j=1,2 \tag{3.22}$$

It follows from (3.20) that

$$Q_{2j}(t)=2\int\limits_{J_2^-}dk=-2\int\limits_{k_2^{(1)}}^{k_2^{(2)}}k^{j-1}e^{kt}\sqrt{\left(k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}\right)\left(k^2+\frac{\gamma^2}{\sigma^2}\right)}^{-1}\times$$
$$\times\exp\left[-\eta\sqrt{|\lambda_l|+\frac{1}{\sigma^2}}\left|\sqrt{\frac{k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}}{k^2+\frac{\gamma^2}{\sigma^2}}}\right|\right]dk,\quad j=1,2 \tag{3.22}$$

Since in (3.22) $k=i\tau$, τ is a real variable, and assuming

$$c(l)=\eta\sqrt{|\lambda_l|+\frac{1}{\sigma^2}},\qquad \tau_{1,2}=\pm\sqrt{k^2+\frac{\gamma^2}{\sigma^2+\frac{1}{|\lambda_l|}}}$$

we get

$$Q_{2j}(t)=-2(i)^j\int\limits_{-\tau_1}^{\tau_1}\frac{\tau^j e^{i\tau t}}{\sqrt{\frac{\gamma^2}{\sigma^2}-\tau^2}}(\tau_1-\tau)^{-\frac{1}{2}}(\tau_1+\tau)^{-\frac{1}{2}}\times$$
$$\times\exp\left[-\frac{c(l)}{\sqrt{\frac{\gamma^2}{\sigma^2}-\tau^2}}(\tau_1-\tau)^{\frac{1}{2}}(\tau_1+\tau)^{\frac{1}{2}}\right]d\tau \tag{3.23}$$

where $\tau_1<\frac{\gamma}{\sigma}$. We estimate the integral in (3.23) by the following way: dividing it to intervals $(-\tau_1,-\tau_1+\delta_1)$, $(-\tau_1+\delta_1,\tau_1-\delta_1)$, $(\tau_1-\delta_1,\tau_1)$, where δ_1 is a sufficiently small number, estimating the first and third integral by modulus, and once integrating the second ones by parts, and then estimating by modulus, at $t \to +\infty$ we get

$$Q_{2j}(t) = |\lambda_l|^{\frac{1}{2}} \eta o(1) \tag{3.24}$$

It follows from (3.18), (3.21), (3.22), 3.24) that at $t \to +\infty$

$$B_{jl}(\eta, t) = \eta^{\frac{1}{2}} o(1) \tag{3.25}$$

Putting asymptotics (3.25) in (3.7) for $u_j(x, y, t)$ at $t \to +\infty$ we receive, that

$$u_j(x, y, t) = o(1) \sum_{l=1}^{\infty} \varphi_l(y) \int_{-\infty}^{\infty} |x - \xi| f_{jl}(\xi) d\xi, \quad j = 1, 2. \tag{3.26}$$

Since functions $f_{jl}(\xi)$ are finite and sufficiently smooth, then acting as at receiveing estimation (1.4) we show that series in (3.26) converges uniformly with respect to $y \in \overline{\Omega}$. We get from (3.1) and (3.26) that at $t \to +\infty$

$$u(x, y, t) = o(1)$$

uniformly with respect to (x, y) at each compact from Π.

Theorem 4 is proved.

Remark. Behaviour of solution of mixed problem (3.1)-(3.3) at $t \to +\infty$ and $n \geq 2$ will be obtained in another paper.

In conclusion the authors express their gratitude to corresponding member of NAS of Azerbaijan prof. Mamedov Yu.A. for useful discussions of the results.

References

[1]. Wisem J. *Linear and nonlinear waves.* M.: Mir, 1977 (Russian)

[2]. Ikezi Kh. *Experimental investigation of solutions in plasma* // Solutions in action. M.: Mir, 1981, p.163-184 (Russian)

[3]. Demidenko G.V., Uspenskii S.V. *Equations and systems unsolved with respect to higher derivative.* Novosibirsk, Nauchnaya kniga, 1998 (Russian)

[4]. Gabov S.A., Sveshnikov A.G. *Problems of dynamics of stratified liquids.* M.: Nauka, 1986 (Russian)

[5]. Gabov S.A., Sveshnikov A.G. *Linear problems of the theory of nonstationary intrinsic waves.* M.: Nauka, 1990 (Russian)

[6]. Mamedov Y.A., Can M. *Application of the residue method to a mixed problem* // Tirkish Journal of mathematics, 1996, v.20, No3, p.305-321.

[7]. Gabov S.A., Orazov B.B. *On the equation* $\frac{\partial^2}{\partial t^2}[u_{xx} - u] + u_{xx} = 0$ *and some related problem* // Zhurnal vychislitelnoi matem. i matem. fiziki, 1986, v.26, 1, p.92-102. (Russian)

[8]. Mizohata S. *Theory of partial differential equations.* M.: Mir, 1977 (Russian)

[9]. Iskenderov B.A. *Principles of radiation for elliptic equation in the cylindrical domain* // Colloquia mathematica sociotatis Janos Bolyai. Szeged, Hungary, 1988, p.249-261

[10]. Mikhaylov V.P. *Partial differential equations.* M.: Nauka, 1983 (Russian)

[11]. Shilov G.E. Mathematical analysis. Special course. M., Nauka, 1974.

[12]. Nikoforov A.V., Uvarov V.V. *Special functions of mathematical physics.* M.: Nauka, 1974 (Russian)

[13]. Natanson P. *Theory of real variable functions.* M.: Nauka, 1987 (Russian)

Bala A. Iskenderov, Vugar G. Sardarov
Institute of Mathematics & Mechanics of NAS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 38-72-50 (off.)

Received June 17, 2002; Revised November 20, 2002.
Translated by Aliyeva E.T.