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# ON A DIFFERENTIATION OPERATION IN NORMED VECTOR LATTICES

**Abstract**

*In the paper scheme of defining of a differential operation in normed vector lattices is given. Properties of the introduced operation are studied. In particular mean value theorem and theorems on relation between total and partial derivatives are studied.*

**1.** $V-$**differenriable mappings.**

Questions on differential calculus in topological vector spaces were studied in papers of many mathematicians (see review of Averbukh V.I. and Smaljanov O.G. [1], [2] and also Balabanov's V.A. monograph [3], see also [4], [5]). The given paper is devoted to investigation of a differentiation operation in normed vector lattices.

Let $X, Y-$ be normed vector lattices [7]. Suppose that $V : X \to Y$ is homogeneous isotopic continuous at zero point mapping. Isotone property of mapping $V$ implies its positiveness.

**Definition 1.** *We'll call linear mapping $l : X \to Y$ $V$-bounded if there exists such non-negative number $M$ that*

$$|l(h)| \leqslant MV(|h|) \quad \text{for any } h \in X.$$

**Theorem 1.** *$V$-boundedness of linear mapping implies its continuity.*

**Proof.** Let $l : X \to Y$ be $V$ bounded linear mapping, then there exists a number $M \geq 0$ such that

$\forall h \in X \qquad |l(h)| \leq MV(|h|)$.

By virtue of monotonicity of norm in normed lattices we have:

$$\|l(h)\| \leq M \|V(|h|)\|.$$

Hence continuity at zero of mapping $V$ implies that mapping $l$ is continuous at zero. Since $l$ is linear then it is continuous in whole space $X$, Q.E.D.

**Definition 2.** *A number*

$$|l|_v = \inf\{M;\ M > 0,\ |l(h)| \leq MV(|h|) \quad \forall h \in X\}$$

*will be called a $V$-norm of $V$-bounded linear mapping $l : X \to Y$.*

The following relation holds

$$|l(h)| \leq |l|_v V(|h|) \quad \forall h \in X.$$

Let's denote by $L_v(X, Y)$ the set of linear $V$-bounded mappings from $X$ into $Y$. All the axiom of norm are satisfied on $L_v(X, Y)$, i.e. $L_v(X, Y)$ is a normed space.

**Definition 3.** *Mapping $f$ acting from $X$ into $L_v(X, Y)$ will be called continuous at the point $x_0 \in X$ if for any positive number $\varepsilon$ there exists $\delta > 0$ such that*

$$|f(x) - f(x_0)|_v \leq \varepsilon \text{ as soon as } \|x - x_0\| < \delta.$$

**Definition 4.** *Mapping* $r : X \to Y$ *will be called* $V$*-small if* $\forall \varepsilon > 0\ \exists \delta > 0$ *such that* $|r(h)| \le \varepsilon V(|h|)$ *as soon as* $\|h\| < \delta$.

We denote set of $V$-small mappings from the space $X$ into the space $Y$ by $R_v(X, Y)$.

**Theorem 2.** *The set of* $V$*-small mappings* $R_v(X, Y)$ *is a vector space.*

**Proof.** It's sufficient to prove that

$$r_i \in R_v(X, Y) \quad (i = 1, 2) \Rightarrow r_1 + r_2 \in R_v(X, Y);$$

$r_i \in R_v(X, Y)$ and $\lambda \in \mathbb{R} \Rightarrow \lambda \cdot r \in R_v(X, Y)$.

Let's assign number $\varepsilon > 0$. Let $r_i \in R_v(X, Y)$, i.e. there exists $\delta_i > 0$ such that

$$\|h\| < \delta_i \Rightarrow |r_i(h)| \le \frac{\varepsilon}{2} V(|h|) \quad (i = 1, 2).$$

Then for $\|h\| < \min\{\delta_1, \delta_2\}$ we have $|(r_1 + r_2)(h)| \le |r_1(h)| + |r_2(h)| \le \varepsilon V(|h|)$ so that $r_1 + r_2 \in R_v(X, Y)$.

Let now $r \in R_v(X, Y)$ and $\lambda \ne 0$. Then there exists $\delta > 0$ such that

$$\|h\| < \delta \Rightarrow |r(h)| \le \frac{\varepsilon}{|\lambda|} V(|h|),$$

i.e. $|\lambda| |r(h)| = |\lambda r(h)| \le \varepsilon V(|h|)$ which means that $\lambda \cdot r \in R_v(X, Y)$. For $\lambda = 0$ the statement is obvious, i.e. operator zero $\theta \in R_v(X, Y)$.

**Theorem 3.** $V$*-is small mapping* $r : X \to Y$ *is continuous at zero.*

**Definition 5.** *Mapping* $f : X \to Y$ *will be called* $V$*-differentiable at the point* $x \in X$ *if there exist mappings* $l \in L_v(X, Y)$ *and* $r \in R_v(X, Y)$ *such that for any* $h \in X$ *inequality* $f(x + h) - f(x) = l(h) + r(h)$ *holds.*

In that case linear $V$-bounded mapping $l$ will be called $V$-derivative of mapping $f$ at the point $x$ and denote by $f'(x)$.

**Theorem 4.** *There exists no more than one linear* $V$*-bounded mapping* $l : X \to Y$ *such that mapping* $r : X \to Y$ *defined by the equality* $r(h) = f(x + h) - - f(x) - l(h)$ *is* $V$*-small one.*

**Proof.** Suppose that there exist two mappings $l_1, l_2 \in L_v(X, Y)$ for which mapping $r_i$ defined by equality

$$r_i(h) = f(x + h) - f(x) - l_i(h) \quad (i = 1, 2),$$

is $V$-small. Then by virtue of Theorem 2 $r_1 - r_2 \in R_v(X, Y)$, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|h\| < \delta \Rightarrow |(l_2 - l_1)(h)| = |(r_1 - r_2)(h)| \le \varepsilon V(|h|).$$

By virtue of monotonicity of norm we have $\|(l_2 - l_1)(h)\| \le \varepsilon \|V(|h|)\|$. Since $\varepsilon$ is arbitrary, then $\|(l_2 - l_1)(h)\| = 0\ \forall h$. Consequently, $l_2 = l_1$. The theorem is proved.

**Theorem 5.** *If mapping* $f : X \to Y$ *is* $V$*-differentiable at the point* $x \in X$, *then it is continuous at this point.*

This follows from theorems 1 and 3.

**Theorem 6.** *If mapping* $f \in L_v(X, Y)$, *then it is* $V$*-differentiable at each point* $x \in X$ *and* $f'(x) = f$.

**Theorem 7.** *If mapping $f : X \to Y$ is constant, then it is $V$-differentiable, and at any point $x \in X$ its $V$-derivative $f'(x)$ is equal to operator zero.*

**Theorem 8.** *If mappings $f_i : X \to Y \quad (i = 1, 2)$ are $V$-differentiable at the point $x \in X$, then mapping $f = \lambda_1 f_1 + \lambda_2 f_2 \ \ (\lambda_1, \lambda_2 \in \mathbb{R})$ is also $V$-differentiable at this point and*

$$f'(x) = \lambda_1 f_1'(x) + \lambda_2 f_2'(x).$$

**Proof.** By the hypothesis of the theorem we have:

$$f_i(x+h) - f_i(x) = f_i'(x)(h) + r_i(h) \quad (i = 1, 2).$$

where $f_i'(x) \in L_v(X, Y)$, $r_i \in R_v(X, Y)$. Let $\lambda_1, \lambda_2 \in \mathbb{R}$. Then by virtue of Theorem 2 mapping

$$r(h) = (\lambda_1 r_1 + \lambda_2 r_2)(h) = (\lambda_1 f_1 + \lambda_2 f_2)(x+h) - (\lambda_1 f_1 + \lambda_2 f_2)(h) -$$

$$- \left(\lambda_1 f_1' + \lambda_2 f_2'\right)(x)(h) = f(x+h) - f(x) - \left(\lambda_1 f_1' + \lambda_2 f_2'\right)(x)(h)$$

is $V$-small.

Linear mapping $\lambda_1 f_1'(x) + \lambda_2 f_2'(x)$ is $V$-bounded. Consequently, $V$-derivative $f'(x)$ exists and equals to $\lambda_1 f_1'(x) + \lambda_2 f_2'(x)$. Q.E.D.

**Theorem 9.** *$V$-differentiability of mapping $f : X \to Y$ implies its Frechet differentiability.*

**Proof.** Let mapping $f : X \to Y$ be $V$-differentiable. Let's assign number $\varepsilon > 0$. Then $\exists \delta > 0$ such that $\|h\| < \delta \Rightarrow |r(h)| \le \frac{\varepsilon}{c} V(|h|)$ where $c = \frac{2}{\delta}$. Hence, by virtue of monotonicity of the norm in normed lattices we have

$$\|h\| < \delta \Rightarrow \|r(h)\| \le \frac{\varepsilon}{c} \|V(|h|)\|. \tag{1}$$

Since mapping $V : X \to Y$ is continuous at zero then $\exists \delta_1 > 0$ (without loss of generality we assume that $\delta_1 = \delta$) such that

$$\|h\| < \delta \Rightarrow \|V(|h|)\| \le 1.$$

Let's take arbitrary element $h \in X$, $h \ne 0_x$. Denote $\bar{h} = \frac{\delta \cdot h}{2\|h\|}$. Then $\|\bar{h}\| = \frac{\delta}{2} < \delta$. Therefore, we have $\left\|V\left(\left|\bar{h}\right|\right)\right\| = \frac{\delta}{2\|h\|} \|V(|h|)\| \le 1$. Hence,

$$\|V(|h|)\| \le c \|h\|. \tag{2}$$

Thus, taking into account (2) in (1) we have $\|h\| < \delta \Rightarrow \|r(h)\| \le \varepsilon \|h\|$, i.e. $\lim\limits_{h \to 0} \frac{\|r(h)\|}{\|h\|} = 0$, Q.E.D.

**Corollary.** *$V$-differentiability of mapping $f : X \to Y$ implies its weak differentiability.*

Let $Z-$ be a normed vector lattice, $V_1 : X \to Y$ and $V_2 : Y \to Z$ be homogeneous isotone mappings which are continuous at zero. Then mapping $V_3 = V_2 \circ V_1$ is homogeneous isotone mapping from space $X$ into $Z$ continuous at zero.

**Lemma 1.** *If $l_1 \in L_{V_1}(X, Y)$, $l_2 \in L_{V_2}(Y, Z)$, then $l_2 \circ l_1 \in L_{V_3}(X, Z)$.*

**Proof.** By assumption $\forall h \in X$, $k \in Y$ we have

$$|l_1(h)| \le |l_1|_{V_1} V_1(|h|), \tag{3}$$

$$|l_2(k)| \le |l_2|_{V_2} V_2(|k|) . \tag{4}$$

Since $V_2$ is isotone homogeneous mapping, then by using relation (3) in (4) we have:

$$|l_2(l_1(h))| \le |l_2|_{V_2} V_2(|l_1(h)|) \le |l_2|_{V_2} |l_1|_{V_1} V_3(|h|) \qquad \forall h \in X.$$

i.e. $l_2 \circ l_1$ is $V_3$-bounded mapping.

**Lemma 2.** *If* $l_2 \in L_{V_2}(Y,Z)$, $r_1 \in R_{V_1}(X,Y)$, *then* $l_2 \circ r_1 \in R_{V_3}(X,Z)$.

**Proof.** Let positive number $\varepsilon$ be given. Then according to lemma's condition there exists $\delta > 0$ such that

$$\|h\| < \delta \Rightarrow |r_1(h)| \le \frac{\varepsilon}{|l_2|_{V_2}} V_1(|h|) ; \tag{5}$$

$$|l_2(r_1(h))| \le |l_2|_{V_2} V_2(|r_1(h)|) \qquad \forall h \in X.$$

Since $V_2$ is isotone homogeneous mapping then by using relation (5) in the last inequality we'll obtain

$$\|h\| < \delta \Rightarrow |l_2(r_1(h))| \le |l_2|_{V_2} V_2(|r_1(h)|) \le \varepsilon V_3(|h|) , i.e. l_2 \circ r_1 \in R_{V_3}(X,Z) .$$

**Lemma 3.** *If* $r_1 \in R_{V_1}(X,Y)$, $l_1 \in L_{V_1}(X,Y)$, $r_2 \in R_{V_2}(Y,Z)$, *then*

$$r_2 \circ (l_1 + r_1) \in R_{V_3}(X,Z) .$$

**Proof.** Let's assign arbitrary number $\varepsilon > 0$. Let number $\varepsilon_1$ satisfy the following condition $0 < \varepsilon_1 \le \min\left\{\varepsilon \cdot \left(1 + |l_1|_{V_1}\right)^{-1}, 1\right\}$. By virtue of theorems 1 and 3 a mapping $l_1 + r_1$ is continuous at zero, i.e. $\forall \delta > 0 \ \exists \ \delta_1 > 0$ such that

$$\|h\| < \delta_1 \Rightarrow \|(l_1 + r_1)(h)\| < \delta. \tag{6}$$

By lemma's condition we have

$$\exists \quad \delta > 0 \quad \text{such that} \quad \|k\| < \delta \Longrightarrow |r_2(k)| \le \varepsilon_1 V_2(|k|) ; \tag{7}$$

$$\exists \quad \delta_2 > 0 \quad \text{such that} \quad \|h\| < \delta_2 \Longrightarrow |r_1(h)| \le V_1(|h|) ; \tag{8}$$

$$\forall h \in X \qquad |\ell_1(h)| \le |\ell_1|_{V_1} V_1(|h|) . \tag{9}$$

Denote $\bar{\delta} = \min(\delta_1, \delta_2)$.

Then from relation (6)-(9) by virtue of isotonity property and homogeneity of mapping $V_2$ we have

$$\|h\| < \bar{\delta} \Longrightarrow \|\ell_1(h) + r_1(h)\| < \delta \Longrightarrow$$

$$\Longrightarrow |r_2(\ell_1(h) + r_1(h))| \le \varepsilon_1 V_2(|\ell_1(h) + r_1(h)|) \le$$

$$\le \varepsilon_1 V_2\left(\left(|\ell_1|_{V_1} + 1\right) \cdot V_1(|h|)\right) = \varepsilon V_3(|h|) ,$$

i.e. $r_2 \circ (\ell_1 + r_1) \in R_{V_3}(X,Z)$.

**Theorem 10.** *Let mappings* $f : X \to Y$, $g : Y \to Z$ *be given. If the following* $f$ *is* $V_1$*-differentiable at the point* $a \in X$, *and* $g$ *is* $V_2$*-differentiable at the point*

$b = f(a)$, *then mapping* $p = g \circ f$ *is* $V_3 = V_2 \circ V_1$*-differentiable at the point* $a$, *and the following inequality holds*

$$p'(a) = g'(b) \cdot f'(a).$$

**Proof.** By the assumption we have

$$f(a+h) - f(a) = \ell_1(h) + r_1(h); \tag{10}$$

$$g(b+k) - g(b) = \ell_2(k) + r_2(k), \tag{11}$$

where

$$\begin{aligned} \ell_1 &= f'(a) \in L_{V_1}(X,Y), \quad r_1 \in R_{V_1}(X,Y), \\ \ell_2 &= f'(b) \in L_{V_2}(Y,Z), \quad r_2 \in R_{V_2}(Y,Z). \end{aligned}$$

In order to prove theorem it's sufficient to show that in equality

$$p(a+h) - p(a) - \ell_2(\ell_1(h)) = r(h),$$

$$r \in R_{V_3}(X,Z), \quad \ell_2 \circ \ell_1 \in L_{V_3}(X,Z).$$

By changing $b+k$ by $f(a+h)$ to $b$ by $f(a)$ in relation (11) we have

$$g \circ f(a+h) - g \circ f(a) - \ell_2(f(a+h) - f(a)) = r_2(f(a+h) - f(a)).$$

Putting (10) into the last equality we obtain 7

$$g \circ f(a+h) - g \circ f(a) - \ell_2(r_1(h) + \ell_1(h)) = r_2(\ell_1(h) + r_1(h)).$$

Hence,

$$p(a+h) - p(a) - \ell_2(\ell_1(h)) = \ell_2(r_1(h)) + r_2(\ell_1(h) + r_1(h)).$$

By virtue of lemma 1

$$p'(a) = \ell_2 \circ \ell_1 \in L_{V_3}(X,Z).$$

Further it follows from lemma 2 and 3 that the mapping

$$r = \ell_2 \circ r_1 + r_2 \circ \ell_1 + r_2 \circ r_1$$

is $V_3$-small.

The theorem is proved.

**Theorem 11.** *Let mappings* $f_i : X \to Y \quad (i = 1,2)$ *coincide in some neighbourhood* $D$ *of the point* $x_0 \in X$. *If mapping* $f_1$ *is* $V$*-differentiable at this point* $x_0$, *then mapping* $f_2$ *is also* $V-$*differentiable at this point and*

$$f_1'(x_0) = f_2'(x_0).$$

**Proof.** By virtue of hypothesis, for any $h \in X$ we have

$$f_1(x_0+h) = f_1(x_0) + f_1'(x_0)(h) + r_1(h)$$

where $r_1 \in R_V(X,Y)$. Put

$$r_2(h) = f_2(x_0+h) - f_2(x_0) - f_1'(x_0)(h). \tag{12}$$

Let's prove that $r_2$ is $V$-small mapping.

Since $f_1(x) = f_2(x)$ for $x \in D$, then we have$r_2(h) = r_1(h)$ for all $h \in D - x_0 = W$.

Let number $\varepsilon > 0$ be given. Since mapping $r_1$ is $V-$small, then exists $\delta > 0$ such that

$$\|h\| < \delta \Longrightarrow |r_1(h)| \le \varepsilon V(|h|).$$

Denote $U = \{y : \|y\| < \delta\}$. Consequently, for $h \in (W \cap U)$ we have

$$|r_2(h)| \le \varepsilon V(|h|).$$

It means that $r_2 \in R_V(X,Y)$. Therefore by virtue of (12) and Theorem 4 we conclude that $f_2$ is $V$-differentiable at the point $x_0$ and $f_2'(x_0) = f_1'(x_0)$, Q.E.D.

**2. Mean value theorem.**

Let $E$ be a normed vector lattice, $F$ be a locally convex lattice, $V : E \to F$ be a homogeneous isotone mapping continuous at zero.

**Definition 6.** *A mapping $f : \mathbb{R} \to F$ is called differentiable at the point if there exists limit*

$$f'(t) = \lim_{k\to 0} \frac{f(t+k) - f(t)}{k}.$$

**Lemma 4.** *If a mapping $\varphi : E \to F$ is weak differentiable at each point of the segment $S = [x, x+h] \subset E$, then mapping $f : [0,1] \to F$, where $f : [0,1] \to F$, $f(\xi) = \varphi(x+\xi h)$ is differentiable, moreover,*

$$f'(\xi) = \varphi'(x+\xi h).$$

**Proof.** If the mapping $\varphi$ is weak differentiable at the each point $x+\xi h$ $(0 \le \xi \le 1)$, then by virtue of the corollary to Theorem 9, we have

$$\lim_{\Delta\xi\to 0} \frac{f(\xi+\Delta\xi) - f(\xi)}{\Delta\xi} = \lim_{\Delta\xi\to 0} \frac{\varphi(x+\xi h+\Delta\xi h) - \varphi(x+\xi h)}{\Delta\xi} =$$

$$= \varphi'(x+\xi h)(h),$$

so,

$$f'(\xi) = \varphi'(x+\xi h)(h) = D\varphi(x+\xi h, h),$$

where $D\varphi$ is a weak differential. The lemma is proved.

**Theorem 12.** *Let $\varphi : E \to F$ be mapping of some neighbourhood of the segment $S = [x, x+h] \subset E$ into the space $F$, weak differentiable at the each point of this segment, where $D\varphi(x,h)$ is linear with respect to $h$ and*

$$D\varphi(x, \circ) \equiv \varphi'(x) \in L_V(E,F) \quad \forall x \in S.$$

*Then*

$$|\varphi(x+h) - \varphi(x)| \le MV(|h|),$$

*where*

$$M = \sup_{0\leq\xi\leq 1} \left|\varphi'(x+\xi h)\right|_V .$$

**Proof.** Let's introduce into consideration the mapping $f:[0,1]\to F$ assumming that $f(\xi)=\varphi(x+\xi h)$. Lemma 4 implies

$$f'(\xi)=\varphi'(x+\xi h)(h).$$

Since by hypothesis of the theorem $\varphi'(x+\xi h)\in L_V(E,F)$, then its obvious that

$$\left|f'(\xi)\right| = \left|\varphi'(x+\xi h)(h)\right| \leq \left|\varphi'(x+\xi h)\right|_V V(|h|) \leq MV(|h|),$$

where

$$M = \sup_{0\leq\xi\leq 1} \left|\varphi'(x+\xi h)\right|_V .$$

Let's denote $B=\{z:|z|\leq V(|h|)\}$. $B$ is a convex set. Now we'll show that $B$ is a closed set.

Let $z_n\in B$ and $z_n\to z$. Then by virtue of the continuity of lattice's operations we have $|z_n|\to|z|$. Since $|z_n|\leq V(|h|)$, then by virtue of the closure of positive cone in $F$ we have $|z|\leq V(|h|)$, i.e. $z\in B$. Thus, $B$ is a closed convex set.

So

$$f'(\xi)\in MB \quad (0\leq\xi\leq 1).$$

Thus, all the conditions of Freilicher-Boocher theorem ([6], p.54) are satisfied by virtue of which $f(1)-f(0)\in M\cdot B$, i.e.

$$\varphi(x+h)-\varphi(x)\in M\cdot B.$$

In other words,

$$|\varphi(x+h)-\varphi(x)|\in MV(|h|).$$

The theorem is proved.

**Theorem 13.** *If a mapping $\varphi:E\to F$ is $V$-differentiable in neighbourhood $D$ of the segment $S=[x_1,x_2]$, then for each point $x_0\in D$ we have*

$$\left|\varphi(x_1)-\varphi(x_2)-\varphi'(x_0)(x_1-x_2)\right| \leq \sup_{x\in S}\left|\varphi'(x)-\varphi'(x_0)\right|_V V(|x_1-x_2|).$$

**Proof.** Let's consider the mapping $x\to\varphi(x)-\varphi'(x_0)\cdot x$ of the segment $S$ into the space $F$.

This mapping is $V$-differentiable and has $V$-derivative

$$t\to\left(\varphi'(x)-\varphi'(x_0)\cdot t\right).$$

By virtue of the previous theorem,

$$\left|\varphi(x_1)-\varphi(x_2)-\varphi'(x_0)(x_1-x_2)\right| \leq \sup_{x\in S}\left|\varphi'(x)-\varphi'(x_0)\right|_V V(|x_1-x_2|).$$

**Definition 7.** *A mapping $\varphi:E\to F$ is called continuously $V$-differentiable at the point $x_0$, if it is $V$-differentiable at each point of neighbourhood $W$ of the point $x_0$, and the mapping $x\to\varphi'(x)$ is continuous at the point $x_0$.*

**Corollary.** *If a mapping $\varphi : E \to F$ is continuously $V$-differentiable at the point $x_0$, then $\forall \varepsilon > 0$ $\exists$ neighbourhood $D$ of the point $x_0$ such that*

$$\left|\varphi(x_1) - \varphi(x_2) - \varphi'(x_0)(x_1 - x_2)\right| \leq \varepsilon \cdot V(|x_1 - x_2|),$$

*as soon as* $x_1,\ x_2 \in D$.

**3. Partial derivatives.**

Let $E_1,\ E_2,\ F$ be normed vector lattices. Let's define on product $E_1 \times E_2$ the structure of normed vector lattice.

Put $\forall x_1, y_1 \in E_1;\ x_2,\ y_2 \in E_2;\ \lambda \in \mathbb{R}$

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2),$$

$$\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2),$$

$$\|(x_1, x_2)\| = \max(\|x_1\|, \|x_2\|).$$

We'll assume that

$$(x_1, x_2) \leq (y_1, y_2) \quad \text{if} \quad x_1 \leq y_1, \quad x_2 \leq y_2.$$

Then

$$\inf((x_1, x_2), (y_1, y_2)) = (\inf(x_1, y_1), \inf(x_2, y_2)),$$

$$\sup((x_1, x_2), (y_1, y_2)) = (\sup(x_1, y_1), \sup(x_2, y_2)),$$

in particular, $|(x_1, x_2)| = (|x_1|, |x_2|)$.

Let $V : E_1 \times E_2 \to F$ be a homogeneous isotone continuous at zero mapping. Let's denote natural embedding $x_1 \to (x_1, 0_{E_2}), x_2 \to (0_{E_1}, x_2)$ by $i_1$and $i_2$, respectively. Then $V(x_1, 0_{E_2}) = V_1(x_1),\ V(0_{E_1}, x_2) = V_2(x_2),$ where $V_1 = V \circ i_1$, $V_2 = V \circ i_2$. It's obvious that $V_1$ is a mapping of the space $E_1$ onto $F$, $V_2$ is a mapping of $E_2$ onto $F$..

Consider a mapping $f : E_1 \times E_2 \to F$. For each fixed point $(a_1, a_2) \in E_1 \times E_2$ we'll define partial mappings $x_1 \to f(x_1, a_2),\ x_2 \to f(a_1, x_2)$. If these mappings $V_1$ and $V_2$ are differentiable at the points $a_1 \in E_1$ and $a_2 \in E_2$ respectively then we'll say that $f$ has partial $V$-derivatives at the point $(a_1, a_2)$, these derivatives will be denoted by $D_1 f(a_1, a_2)$ and $D_2 f(a_1, a_2)$.

**Theorem 14.** *If a mapping $f : E_1 \times E_2 \to F$ is $V$-differentiable at the point $(a_1, a_2)$, then it has partial $V$-derivatives at this point and*

$$f'(a_1, a_2)(h_1, h_2) = D_1 f(a_1, a_2)(h_1) + D_2 f(a_1, a_2)(h_2).$$

*At that continuity of $f'(a_1, a_2)$ implies continuity of $D_1 f(a_1, a_2)$ and $D_2(a_1, a_2)$.*

**Proof.** By virtue of hypothesis of the theorem we have

$$f(a_1 + h_1, a_2 + h_2) = f(a_1, a_2) + \ell(h_1, h_2) + r(h_1, h_2),$$

where

$$\ell = f'(a_1, a_2) \in L_V(E_1 \times E_2, F), \quad r \in R_V(E_1 \times E_2, F).$$

By virtue of $V$-smallness of mapping $r$ $\forall\varepsilon > 0$ $\exists\delta > 0$ such that

$$\|(h_1, h_2)\| < \delta \Longrightarrow |r(h_1, h_2)| \le \varepsilon V(|h_1, h_2|).$$

We also have $|\ell(h_1, h_2)| \le |\ell|_V V(|h_1, h_2|)$ $\forall (h_1, h_2) \in E_1 \times E_2$.. Put

$$r_1(h_1) = r(h_1, 0_{E_2}), \quad \ell_1(h_1) = \ell(h_1, 0_{E_2}).$$

Then we have $\forall h_1 \in E_1$ $\ell_1(h_1) \le |\ell|_V V(|(h_1, 0_{E_2})|) = |\ell|_V V_1(|h_1|)$; $\forall\varepsilon > 0$ $\exists\delta > 0$ such that $\|h\| < \delta \Longrightarrow |r_1(h_1)| \le \varepsilon V(|(h_1, 0_{E_2})|) = \varepsilon V_1(|h_1|)$, i.e. $\ell_1 \in L_{V_1}(E_1, F)$, $r_1 \in R_{V_1}(E_1, F)$. Besides we have

$$f(a_1 + h_1, a_2) = f(a_1, a_2) + \ell_1(h_1) + r_1(h_1),$$

which implies that mapping has partial $V$-derivative with respect to the first argument at the point $(a_1, a_2)$ and $D_1 f(a_1, a_2)(h_1) = \ell_1(h_1) = f'(a_1, a_2)(h_1, 0_{E_2})$.

The existence of partial $V$-derivative with respect to the second argument and equality $D_2 f(a_1, a_2)(h_2) = f'(a_1, a_2)(0_{E_1}, h_2)$ are established analogously. Further

$$D_1 f(a_1, a_2)(h_1) + D_2 f(a_1, a_2)(h_2) = f'(a_1, a_2)(h_1, h_2).$$

The theorem is proved.

**Remark.** Since $(|h_1|, 0_{E_2}) \le (|h_1|, |h_2|)$, then by virtue of isotonic property of mapping $V$ we have

$$V_1(|h_1|) \le V(|(h_1, h_2)|), \quad V_2(|h_2|) \le V(|(h_1, h_2)|) \;\; \forall h_1 \in E_1, \; h_2 \in E_2.$$

**Theorem 15.** *Let mapping $f : E_1 \times E_2 \to F$ have partial $V$-derivatives in the neighbourhood $W$ of the point $(a_1, a_2)$, and mapping $D_2 f : W \to L_{V_2}(E_2, F)$ be continuous at the point $(a_1, a_2)$. Then is $V$ -differentiable at the point $(a_1, a_2)$ and*

$$f'(a_1, a_2)(h_1, h_2) = D_1 f(a_1, a_2)(h_1) + D_2 f(a_1, a_2)(h_2).$$

*At that continuity of mappings $D_1 f(a_1, a_2)(h_1)$ and $D_2 f(a_1, a_2)(h_2)$ implies continuity of $f'(a_1 a_2)$.*

**Proof.** We have

$$f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) = f(a_1 + h_1, a_2 + h_2) -$$

$$-f(a_1 + h_1, a_2) + f(a_1 + h_1, a_2) - f(a_1, a_2). \tag{13}$$

Let's assign number $\varepsilon > 0$. By virtue of $V_1$-differentiability of the mapping $x_1 \to f(x_1, a_2)$at the point $_{a_1 \in E_1}$ $\exists\delta_1 > 0$, such that for $\|h_1\| < \delta_1$ and $h_2 \in E_2$ .

$$|f(a_1 + h_1, a_2) - f(a_1, a_2) - D_1 f(a_1, a_2)(h_1)| \le \frac{\varepsilon}{3} V(|h_1, h_2|). \tag{14}$$

By virtue of $V_2$-differentiability of the mapping $x_2 \to f(a_1 + h_1, x_2)$ at the point $a_2 \in E_2$ $\exists\delta_2 > 0$ such that for $\|h_2\| < \delta_2$and $h_1 \in E_1$

$$|f(a_1 + h_1, a_2 + h_2) - f(a_1 + h_1, a_2) - D_2 f(a_1 + h_1, a_2)(h_2)| \le \frac{\varepsilon}{3} V(|(h_1, h_2)|). \tag{15}$$

Since the mapping $D_2f(a_1,a_2)$ is continuous at the point $(a_1,a_2)$, then by definition 3 there exists $\delta_3>0$ such that

$$|D_2f(a_1+h_1,a_2)-D_2f(a_1,a_2)|_{V_2}<\frac{\varepsilon}{3} \text{ as soon as } \|h_1\|<\delta_3.$$

Therefore, since $D_2f(a_1+h_1,a_2)-D_2f(a_1,a_2)\in L_{V_2}(E_2,F)$, then for $\|h_1\|<\delta_3$ and $h_2\in E_2$ we have

$$|D_2f(a_1+h_1,a_2)(h_2)-D_2f(a_1,a_2)(h_2)|\leq$$
$$|D_2f(a_1+h_1,a_2)-D_2f(a_1,a_2)|_{V_2}V_2(|h_2|)\leq\frac{\varepsilon}{3}V(|(h_1,h_2)|). \tag{16}$$

Let's denote $\delta=\min(\delta_1,\delta_2,\delta_3)$. By virtue of relations (13)-(16) we have

$$|f(a_1+h_1,a_2+h_2)-f(a_1,a_2)-D_1f(a_1,a_2)(h_1)-D_2f(a_1,a_2)(h_2)|=$$
$$=|(f(a_1+h_1,a_2+h_2)-f(a_1+h_1,a_2)-D_2f(a_1+h_1,a_2)(h_2))+$$
$$+(D_2f(a_1+h_1,a_2)(h_2)-D_2f(a_1,a_2)(h_2))+$$
$$+(f(a_1+h_1,a_2)-f(a_1,a_2)-D_1f(a_1,a_2)(h_1))|\leq\varepsilon V(|(h_1,h_2)|),$$

as soon as $\|(h_1,h_2)\|<\delta$.

By the hypothesis of the theorem linear mappings $D_1f(a_1,a_2)$ and $D_2f(a_1,a_2)$ are $V_1$ and $V_2$-bounded respectively, so that

$$|D_1f(a_1,a_2)(h_1)|\leq|D_1f(a_1,a_2)|_{V_1}V(|(h_1,h_2)|)\quad\forall h_1\in E_1,$$
$$|D_2f(a_1,a_2)(h_2)|\leq|D_2f(a_1,a_2)|_{V_2}V(|(h_1,h_2)|)\quad\forall h_2\in E_2,$$

Hence,

$$|D_1f(a_1,a_2)(h_1)+D_2f(a_1,a_2)(h_2)|\leq$$
$$\leq\left(|D_1f(a_1,a_2)|_{V_1}+|D_2f(a_1,a_2)|_{V_2}\right)V(|(h_1,h_2)|).$$

for any $(h_1,h_2)\in E_1\times E_2$.

Thus, $\forall\varepsilon>0\ \exists\,\delta>0$ such that

$$\|(h_1,h_2)\|<\delta\Rightarrow$$
$$\Rightarrow\left|f(a_1+h_1,a_2+h_2)-f(a_1,a_2)-f'(a_1,a_2)(h_1,h_2)\right|\leq\varepsilon V(|(h_1,h_2)|),$$

where $f'(a_1,a_2)(h_1,h_2)=D_1f(a_1,a_2)(h_1)+D_2f(a_1,a_2)(h_2)$ −is the linear $V$-bounded mapping of $E_1\times E_2$ into $F$, i.e. the mapping $f$ is $V$-differentiable at the point $(a_1,a_2)$.

The theorem is proved.

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