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ON FREDHOLM PROPERTY OF MATRIX SINGULAR INTEGRAL OPERATORS IN THE HÖLDER SPACES WITH THE WEIGHT DEGENERATED ON SUBMANIFOLD OF SMALLER DIMENSION

Abstract

At the paper in terms of the symbol the problem on Fredholm property of matrix in the Hölder spaces is solved with the weighted generating on submanifold operators of Lipschitz manifold of smaller dimension. The element of matrix are multidimensional Mikhlin-Kalderon-Zygmund singular operators. In particular, in definite conditions on the symbol, a two-sided regularizer is constructed and it is proved that index of matrix operator in considered Hölder spaces with weight and in the space L_n^2 *is the same.*

At the paper in terms of symbol the problem on Fredholm property of matrix singular integral (SI) operators $A: u \to Au$,

$$Au(x) = a(x)u(x) + \int_{R^m} f(x,\theta)|x-y|^m u(y)dy$$

is solved in the Hölder spaces with the weight $\left(-H_{\alpha\beta}^{n\nu}\right)$, when the weight is degenerated on submanifold Lipschitz manifold of smaller dimension, where R^m is m-dimensional $(m \geq 2)$ Euclidean space, $S = \{x \in R^m : |x| = 1\}$, $a(x)$, $f(x,\theta)$ $(x \in R^m, \theta \in S)$ are matrices of dimension $n \times n$ elements of which are the scalar functions $a_{ij}(x)$, $f_{ij}(x,\theta)$ $\left(i,j = \overline{1,n}\right)$ and respectively $\int_S f_{ij}(x,\theta)d_\theta S = 0$, $\forall x \in R^m$. Particularly the two-sides regularization for the operator A is constructed and the problem on index is considered.

The results received at the paper generalize on matrix case (in particular at calculation of index they clearify) the corresponding results of papers [1], [2]. Let's note that in one-dimensional case the formulas for the calculation of index SI operator in terms of its symbol are known, both in case of continuous and discontinuous coefficients. In multidimensional case mainly the problem on index of SI operators is solved in context of pseudo-differential operators [3]. But application of main results of index theory as for example Atia-Zinger theorem is accompanied with difficulty. Sometimes insuperable especially in case of Hölder spaces. In connection with this we choose an other way. Applying the theorem of conservation of index of Noeter operator at crossing from one B space to other, to calculate the index of the matrix SI operator A in $H_{\alpha\beta}^{n\nu}$ we'll reduce to calculation of index in L_2. This approach is connected with the fact that the space L_2 is very suitable to calculate the index.

The arguments of the paper require to solve a problem on boundedness and on construction of two sided regularizer of the operator A also in the spaces L_p and $L_p(\omega)$.

The boundedness of the operator A in the spaces $H_{\alpha\beta}^{n\nu}\left(R^m \setminus \Gamma\right)$.

In further $C(G)$ (G is a closed set of finite-dimensional space) means that the space of continuous on G functions with the norm $\|u\|_\infty = \sup_{x \in G}|u(x)|$. $R^m = R^m \cup \{\infty\}$ is a compactified R^m. By the determination $u \in C(\dot{R}^m)$, if $u \in C(R^m)$ and there exists the final limit $u(\infty) = \lim_{x\to\infty} u(x)$. $C(R^m) - B$ is a space in the norm $u(\infty) = \sup_{x\in R^m} |u(x)|$.

Let's note that the other notations used at the given paper are taken from the books of S.G. Mikhlin [4] and from the papers by I.B. Simonenko [5,6].

Everywhere later on $\Gamma \subset R^m$ is a submanifold of some borderless (in R^m) Lipschitz manifold of dimension $k : 0 \le k \le m-1$, $k = 0$ means that Γ consists of one point. $r_\Gamma(x)$ is a distance from the point $x \in R^m$ to Γ, $\rho_{\alpha\beta}(x) \overset{def}{=} r_\Gamma^\alpha(x)(1+|x|)^\beta$, $d(x,y) \overset{def}{=}$ $\overset{def}{=} |x-y|/(1+|x|)(1+|y|)$.

Let $0 < \nu \le 1$, $\alpha > 0$, $b \in (-\infty, +\infty)$. The weight Hölder space $H_{\alpha\beta}^\nu(R^m \setminus \Gamma) - B$ is a space of continuous on $R^m \setminus \Gamma$ functions such that $(u\rho)(x)$ tends to zero when $x \to z \in \Gamma$ and $x \to \infty$, and the norm

$$\|u\| \overset{def}{=} \sup_{x,y \in R^m \setminus \Gamma} |(u\rho_{\alpha\beta})(x) - (u\rho_{\alpha\beta})(y)| d^{-\nu}(x,y)$$

is finite.

The following lemma admits to express the condition $u \in H_{\alpha\beta}^\nu$ by the inequality.

Lemma. *If*

$$0 < \nu < 1,\ \nu < \alpha < (m-k) + \nu,\ 0 < \beta + \alpha + \nu < m, \qquad (1)$$

then $u \in H_{\alpha\beta}^\nu$, *iff there exists* $C_1(u) > 0$, $C_2(u) > 0$ *such that*

a) $\forall x \in R^m \setminus \Gamma$, $|\rho(x)u(x)| \le C_1\left[r_\Gamma(x)(1+|x|)^{-2}\right]^\nu$;

b) $\forall x \in R^m \setminus \Gamma$, $\forall y \in \left\{y \in R^m : |x-y| \le \frac{r_\Gamma(x)}{2}\right\}$

$$\rho(x)\,|u(x) - u(y)| \le C_2(u) d^{-\nu}(x,y),$$

$$L_p \overset{def}{=} \left\{u - meas : \|u\| \overset{def}{=} \left(\int_{R^m} |u(x)|^p dx\right)^{1/p} < +\infty\right\},\ L_p(\omega) \overset{def}{=} \left\{u - meas : \|u\| \overset{def}{=} \|\omega u\|_{L_p} < +\infty\right\},$$

where $1 \le p < \infty$, ω *is almost everywhere positive in* R^m *functions.*

Definition 1. *Let* B *be some Banach space. We'll denote by* $B^n(B^{nn})$ *the totality of the vectors* $u = (u_1, \ldots, u_n)$ *(quadratic matrices* $a = \{a_{ij}\}$, $i, j = \overline{1,n}$*) with the components* $u_i \in B$ $(a_{ij} \in B)$.

$B^n(B^{nn})$ is a linear set relative to the ordinary operations of addition and multiplication on number and on Banach space in the norm

$$\|u\| = \left(\sum_{i=1}^n \|u_i\|^2\right)^{1/2}; \qquad \left(\|a\| = \left(\sum_{i,j=1}^n \|a_{ij}\|^2\right)^{1/2}\right).$$

According to this definition the spaces $H_{\alpha\beta}^{n\nu} \overset{def}{=} (H_{\alpha\beta}^\nu)^n$, L_p^n, $L_p^n(\omega)$ are introduced.

Let's introduce also the classes $C_{\delta\varepsilon}^{nn}$, $C_{\mu}^{nn}(S)$ and $C_{\delta\varepsilon}^{nn}H_l$.

Let $0\le\delta\le1$, $\varepsilon\le0$, $\mu\in(0,1]$ an

$$\varphi_{\delta\varepsilon}(x,y)=\left(|x-y|/(1+\max\{|x|,|y|\})\right)^{\delta}\left(1+\min\{|x|,|y|\}\right)^{-\varepsilon},\quad x,y\in R^m .$$

Let's denote by $C_{\delta\varepsilon}$ and $C_{\mu}(S)$ the totality of the functions $f(x,\theta)\in C(R^m\times S)$ with the finite semi-norms

$$K_{\delta\varepsilon}(f)=\sup_{x,y\in R^m}\left(\|f(x,\cdot)-f(y,\cdot)\|_{C(S)}\varphi_{\delta\varepsilon}^{-1}(x,y)\right)$$

and

$$K_{\mu}(f)=\sup_{\theta,w\in S}\left(\|f(\cdot,\theta)-f(\cdot,w)\|_{C(R^m)}|\theta-w|^{-\mu}\right)$$

respectively.

$-C_{\delta\varepsilon}H_l$ $(l>0)$ is a class of functions $f(x,\theta)\in C(R^m\times S)$ belonging to H_l uniformly by x, for which

$$K_{l\delta\varepsilon}(f)\overset{def}{=}\sup_{x,y\in R^m}\left(\|f(x,\cdot)-f(y,\cdot)\|_{H_l}\varphi_{\delta\varepsilon}^{-1}(x,y)\right)<\infty ,$$

where the space H_l is Sobolev-Slobodetsky on S.

According to definition 1 the classes $C_{\delta\varepsilon}^{nn}$, $C_{\mu}^{nn}(S)$, $C_{\delta\varepsilon}^{nn}H_l\overset{def}{=}(C_{\delta\varepsilon}H_l)^{nn}$ are introduced with the corresponding semi-norms denoted by $K_{\delta\varepsilon}^{nn}$, $K_{\mu}^{nn}(S)$, $K_{l\delta\varepsilon}^{nn}$.

Remark 1. The matrix operator $A=\{A_{ij}\}$ is bounded (completely continuous) in the space B^n iff the operators A_{ij} are bounded (completely continuous) in B and at this

$$\|A_{ij}\|\le\|A\|\le\left(\sum_{i,j}\|A_{ij}\|\right)^{1/2} .$$

The truthness of this remark follows from definition 1.

Theorem 1. *Let the condition (1) and* $f\in C_{\nu 0}^{nn}\cap C_{\mu}^{nn}(S)$, $\nu<\mu<1$ *be fulfilled. Then the matrix SI operator* $S_f\mapsto S_f u$ *,*

$$(S_f u)x=\int_{R^m}f(x,\theta)|x-y|^{-m}u(y)dy ,\quad x\in R^m\setminus\Gamma$$

is bounded in $H_{\alpha\beta}^{n\nu}(R^m\setminus\Gamma)$ *and*

$$\|S_f\|\le C\left(K_{\nu 0}^{nn}(f)+K_{\mu}^{nn}(f)+\|f\|_{\infty}^{nn}\right).$$

When Γ is non-bounded the proof of theorem 1 is reduced to the corresponding theorems from [2].

Let's take the function

$$\rho(x)=(1+|x|)^{\beta}\left(r_{\Gamma}(x)\left(1+r_{\Gamma}(x)^{-1}\right)\right)^{\alpha},$$

when Γ is compact. Let's note that the operators of multiplication by the functions $\rho(x)\rho_{\alpha\beta}^{-1}(x)$ and $\rho_{\alpha\beta}(x)\rho^{-1}(x)$ are bounded in $H_{\alpha\beta}^{\nu}$ if the conditions (1) are fulfilled. Subject to this by virtue of remark 1 the proof of the theorem follows from the corresponding results of paper [1].

Let $1<p<\infty$. Let's denote by W_p a class of weights ω for which the multidimensional singular integral of Mikhlin-Calderon-Zygmund is bounded in $L_p(\omega)$. Using the corresponding results of papers [1], [2] it is proved that if the conditions (1) are fulfilled, then there exists p and $\omega \in W_p$ such that $H^{\nu}_{\alpha\beta} \subset L_p(\omega)$ and imbedding is continuous.

Let's cite some properties of the classes $C^{nn}_{\delta\varepsilon}$ and $C^{nn}_{\delta\varepsilon}H_l$.

Using corresponding theorem on imbedding of spaces H_l in $C(S)$ (see [7]) the following is proved.

Theorem 2. *Let* $l>(m-1)/2$, $l_1 \ge (m-1)/2+\mu$, $\mu \in (0,1]$ *and* $f(x,\theta) \in C^{nn}_{\delta\varepsilon}H_l \cap \cap C^{nn}_{00}H_{l_1}$, *then* $f(x,\theta) \in C^{nn}_{\delta\varepsilon} \cap C^{nn}_{\mu}(S)$ *and* $K^{nn}_{\mu}(f) + \|f\|^{nn}_{\infty} \le C \sup\limits_{x\in R^m} \|f(x,\cdot)\|_{H_l}$, $K^{nn}_{\delta\varepsilon}(f) \le CK^{nn}_{l\delta\varepsilon}(f)$, *where* C *doesn't depend on* f.

Theorem 3. *Let* $f(x,\theta) \in C^{nn}_{\delta\varepsilon}H_l \; (l>(m-1)/2)$ *and* $\inf|\det f(x,\theta)|>0$, *then* $f^{-1}(x,\theta) \in C^{nn}_{\delta\varepsilon}H_l$, *where* $f^{-1}(x,\theta)$ *is an inverse matrix* $f(x,\theta)$.

Proof. The structure of the matrix $f^{-1}(x,\theta)$ subject to that H_l $(l>(m-1)/2)$ is a normed ring [8], admits to be restricted by the case $m=1$.

Let $f(x,\theta) \in C_{\delta\varepsilon}H_l$ and $\inf|f(x,\theta)|>0$, then for any $x,y \in R^m$ we have:

$$\|f^{-1}(x,\cdot) - f^{-1}(y,\cdot)\|_{H_l} = \left\| \frac{f(x,\cdot) - f(y,\cdot)}{f(x,\cdot)(y,\cdot)} \right\|_{H_l} \le$$

$$\le \left(\sup_{x\in R^m} \|f(x,\cdot)\|_{H_l} \right)^2 \|f(x,\cdot) - f(y,\cdot)\|_{H_l}.$$

Dividing on $\varphi_{\delta\varepsilon}(x,y)$ passing to $\sup$, from the last we get $f^{-1}(x,\cdot) \in C_{\delta\varepsilon}H_l$.

The symbol of the operator A we call the matrix

$$\Phi_A(x,\theta) = \{\Phi_{A_{ij}}(x,\theta)\},$$

where $\Phi_{A_{ij}}$ $(i,j=\overline{1,n})$ are symbols of the operators $A_{ij} = a_{ij}J + S_{f_{ij}}$.

As in scalar case

$$\Phi_A(x,\theta) = a(x) + \mathsf{F}\left(f(x,\cdot)r^{-m}\right), \quad r=|x-y|,$$

where F is a Fourier transformation multiplied by the unique matrix. The last shows that the symbol is determined locally, i.e. at every fixed x (see [4]).

Theorem 4. $\Phi_A \in C^{nn}_{\delta\varepsilon}H_l$, $\left(l>\dfrac{m}{2}\right)$ *iff* $a(x) \in C^{nn}_{\delta\varepsilon}$, $f(x,\theta) \in C^{nn}_{\delta\varepsilon}H_{l-\frac{m}{2}}$ *and at this*

$$c_1 K^{nn}_{\left(l-\frac{m}{2}\right)\delta\varepsilon}(f) \le K^{nn}_{l\delta\varepsilon}(\Phi_A - a) \le c_2 K^{nn}_{\left(l-\frac{m}{2}\right)\delta\varepsilon}(f),$$

where the constants c_1, c_2 *don't depend on* Φ_A, a *and* f.

The theorem is proved as in case $n=1$ (see [7]).

Remark 2. By means of this theorem it is particularly proved that if homogeneous zero degree by second argument of the function $\varphi(x,\theta)$ belongs to $C^{nn}_{\delta\varepsilon}H_l$,

$\left(l > \frac{m}{2}\right)$ then there exists $b(x) \in C^{nn}_{\delta\varepsilon}$ and $g(x,\theta) \in C^{nn}_{\delta\varepsilon} H_{l-\frac{m}{2}}$ such that $\int_S g(x,\theta) d_\theta S = 0$, $\forall x \in R^m$ and the operator symbol $R = bJ + S_g$ equals $\varphi(x,\theta)$.

The construction of two sided regularizer.

Theorem 5. *Let the conditions (1)* $a(x) \in C^{nn}_{\delta\varepsilon}$ *and* $f(x,\theta) \in C^{nn}_{\delta\varepsilon} H_l \cap C^{nn}_{00} H_{l_1}$ *be fulfilled where* $\nu < \delta$, $\nu < \mu < 1$, $l > (m-1)/2 + \mu$. *If*

$$\inf \left| \det \Phi_A(x,\theta) \right| > 0, \tag{2}$$

then the matrix SI operator A *admits the two sided regularization in* $H^{n\nu}_{\alpha\beta}$.

Proof. Let $RbJ + S_g$ be an operator, whose symbol is equal to $\Phi^{-1}_A(x,\theta)$. By virtue of remark 2 $b \in C^{nn}_{\delta\varepsilon}$, $g(x,\theta) \in C^{nn}_{\delta\varepsilon} H_l \cap C^{nn}_{00} H_{l_1}$ and $\int_S g(x,\theta) d_\theta S = 0$.

Let's suppose for every $z \in R^m$

$$A_z : u(x) \mapsto a(z)u(x) + \int_{R^m} f(z,\theta)|x-y|^{-m} u(y) dy .$$

Analogously we determine R_z.

By virtue of the condition (2) A_z and R_z mutually inverse operators in L^n_2 (see [6]), i.e.

$$A_z R_z = R_z A_z = J .$$

The equality holds in L_p and $L_p(\omega)$ $\left(1 < p < \infty,\ \omega \in W_p\right)$. Besides, $\forall u \in H^{n\nu}_{\alpha\beta}$ and $z \in R^m \setminus \Gamma$

$$(A_z u)(z) = A(u)(z),\ (R_z u)(z) = R(u)(z).$$

For $z \in R^m \setminus \Gamma$ we get

$$(RA)(z) = R(Au)(z) = R_z\left[(A_z + (A - A_z))u\right](z) = R_z(A_z u)(z) + R_z\left[(A_z)u\right](z) = u(z) + T_{RA},$$

where $T_{RA} = C_{LK} + {}_{La}$;

$$C_{LK} : u \mapsto (C_{LK}u)(z) \overset{def}{=} \int_{R^m} L(z, z-x) \left[\int_{R^m} (K(x, x-y) - K(z, x-y)) u(y) dy \right] dx ;$$

$$D_{La} : u \mapsto (D_{La}u)(z) \overset{def}{=} \int_{R^m} L(z, z-x)\left[a(x) - a(z)\right] u(x) dx ;$$

$$K(x, x-y) = f(x,\theta)|x-y|^{-m},\ L(x, x-y) = g(x,\theta)|x-y|^{-m} .$$

Analogously

$$AR = J + T_{AR},\ T_{AR} = C_{KL} + D_{Kb} .$$

Thus R is two sided regularizer of A in $H^{n\nu}_{\alpha\beta}\left(R^m \setminus \Gamma\right)$ if in this space the operators T_{RA} and T_{AR} are completely continuous that is proved by means of corresponding results from [1], [2].

Remark 3. If all the conditions of theorem 5 are fulfilled relative to a and f, then the operator R is a two sided regularizer A and in the spaces L^n_p and in $L^n_p(\omega)$ $\left(1 < p < \infty, \omega \in W_p\right)$. The case of the space $L^n_p(\omega)$ is considered in [10].

By that, it is proved that the matrix SI operator A is normally solvable and has a finite index (i.e. nöthers) in the spaces L_p^n and $L_p^n(\omega)\left(1<p<\infty, \omega \in W_p\right)$ and $H_{\alpha\beta}^{n\nu}$ if the conditions of theorem 5 are fulfilled.

On the index of the operator A.

Let $\{B_1, B_2\}$ be an interpolation pair of B spaces B_1 and B_2. As is known (see [9]) the space $B_1 \cap B_2$ with the norm

$$\|a\| = \max\left\{\|a\|_{B_1}, \|a\|_{B_2}\right\}$$

and the space $B_1 + B_2$ with the norm

$$\|a\|_{B_1+B_2} = \inf_{a=a_1+a_2}\left\{\|a\|_{B_1} + \|a\|_{B_2}\right\}$$

are Banach spaces.

Insignificant revision of theorem 10.6 from [7] leads to the proof of the following theorem.

Theorem 6. *Let $\{B_1, B_2\}$ be an interpolation pair of B spaces, $B_1 \cap B_2$ densely in $B_1 + B_2$, A, R are bounded in B_1 and B_2, $RA - J$ and $AR - J$ are completely continuous in B_1 and B_2, then*

$$Ind\ A(B_1 \to B_1) = Ind\ A(B_2 \to B_2).$$

Allowing for remark 3 and applying theorem 6 for the pair $\left(L_2^n, L_p^n(\omega)\right)$ and $\left(L_p^n(\omega), H_{\alpha\beta}^{n\nu}\right)$ we get:

Theorem 7. *Let all the conditions of theorem 5 be fulfilled then*

$$Ind\ A\left(H_{\alpha\beta}^{n\nu} \to H_{\alpha\beta}^{n\nu}\right) = Ind\ A\left(L_2^n \to L_2^n\right).$$

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