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APPROXIMATION OF FUNCTIONS BY THE FAMILY OF INTEGRAL OPERATORS WITH POSITIVE KERNELS*

Abstract

The problem of pointwise approximation of functions by the family of singular integrals of non-convolution type are solved.

This paper is devoted to the problem of approximation of functions by the family of integral operators with positive kernels in fixed characteristic points of integrable functions. This problem were investigated by many authors and we refer to the papers [1]-[4] and to the monographs [5]-[7]. Note that the well known results in this direction have deals with the convolution type integrals, which are, in general, the generalizations of classical singular integrals of Fejer type, having the kernels $\lambda K(\lambda(t-x))$.

The more general results, concerning to kernels $K_\lambda(t,x)$ also is applicable, as a roole, to mentioned type kernels. By this reason in this paper we will give more general theorems on convergence of integral operators of type

$$L_\lambda(f;x) = \int_A^B f(t)K_\lambda(t,x)dt; \qquad x\in[a,b], \tag{1}$$

where between the sets $[A,B]$ and $[a,b]$ may be different embeddings.

First of all we give the following definition.

Definition 1. *We will say that the function $K_\lambda(t,x)$ of two variables $t\in[A,B]$, $x\in[a,b]$, depending on real parameter $\lambda>0$, satisfy the condition (A), if the following are holds:*

a) $$\lim_{\lambda\to\infty}\int_A^B K_\lambda(t,x)dt = 1, \tag{2}$$

for any fixed $x\in[a,b]$.

b) For any fixed $x\in[a,b]$ there exist a point $\xi_x\in[A,B]$ such that for any $t\neq\xi_x$

$$(\xi_x - t)\cdot\frac{\partial K_\lambda(t,x)}{\partial t} > 0; \tag{3}$$

c) For any fixed $x\in[a,b]$ and any fixed $\delta>0$

$$\lim_{\lambda\to\infty} K_\lambda(\xi_x \pm \delta, x) = 0 \tag{4}$$

holds.

Our main result is the following

Theorem 1. *Let $f\in L_1(A,B)$ and the non-negative kernel $K_\lambda(t,x)$ satisfy the condition (A). If the point ξ_x in b) of definition 1 is the Lebesgue point of function f, then for integral operators (1) in fixed point $x\in[a,b]$*

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$$\lim_{\lambda\to\infty} L_\lambda(f;x) = f(\xi_x)$$

holds.

Proof. By the definition of Lebesgue point we can write

$$\lim_{h\to 0}\frac{1}{h}\int_{\xi_x}^{\xi_x+h}|f(t)-f(\xi_x)|dt = 0$$

$$\lim_{h\to 0}\frac{1}{h}\int_{\xi_x-h}^{\xi_x}|f(t)-f(\xi_x)|dt = 0 .$$

Therefore, given $\xi > 0$, there exist a $\delta > 0$ such that for all $h,\ 0 < h \le \delta$, the inequalities

$$\int_{\xi_x}^{\xi_x+h}|f(t)-f(\xi_x)|dt < \varepsilon h \tag{5}$$

and

$$\int_{\xi_x-h}^{\xi_x}|f(t)-f(\xi_x)|dt < \varepsilon h . \tag{6}$$

holds.

Fixed this δ and using (2) and positivity of kernel $K_\lambda(t,x)$, we have

$$|L_\lambda(f;x)-f(\xi_x)| \le \left\{\int_A^{\xi_x-\delta}+\int_{\xi_x-\delta}^{\xi_x}+\int_{\xi_x}^{\xi_x+\delta}+\int_{\xi_x+\delta}^{B}+\right\}\cdot|f(t)-f(\xi_x)|\cdot K_\lambda(t,x)dt+|f(\xi_x)|\left|\int_A^B K_\lambda(t,x)dt-1\right| =$$

$$= I_{1,\lambda}+I_{2,\lambda}+I_{3,\lambda}+I_{4,\lambda}+I_{5,\lambda}. \tag{7}$$

Consider $I_{1,\lambda}$ and $I_{4,\lambda}$. By the condition b) of definition 1, $K_\lambda(t,x)$ is increasing, as a function of t, in $[A,\xi_x-\delta]$ and decreasing in $[\xi_x+\delta,B]$.

Therefore, from the inequalities

$$I_{1,\lambda} < K_\lambda(\xi_x-\delta,x)\int_A^{\xi_x-\delta}|f(t)-f(\xi_x)|dt \le \left(\|f\|_{L_1}+|f(\xi_x)|(B-A)\right)K_\lambda(\xi_x-\delta,x),$$

$$I_{4,\lambda} < K_\lambda(\xi_x+\delta,x)\int_{\xi_x+\delta}^{B}|f(t)-f(\xi_x)|dt \le \left(\|f\|_{L_1}+|f(\xi_x)|(B-A)\right)K_\lambda(\xi_x+\delta,x),$$

we obtain

$$I_{1,\lambda}+I_{4,\lambda} < \left(\|f\|_{L_1}+|f(\xi_x)|(B-A)\right)K_\lambda(\xi_x-\delta,x)+K_\lambda(\xi_x+\delta,x).$$

Using (4), we have

$$\lim_{\lambda\to\infty}\left(I_{1,\lambda}+I_{4,\lambda}\right)=0 . \tag{8}$$

Consider $I_{3,\lambda}$. Denoting

$$F(t)=\int_{\xi_x}^{t}|f(y)-f(\xi_x)|dy ,$$

we see, that, by (5),

$$F(t) \le \varepsilon(t-\xi_x) \tag{9}$$

if $t-\xi_x \le \delta$. Hence, integrating by part and using (9), we have

$$I_{3,\lambda} = \int_{\xi_x}^{\xi_x+\delta} K_\lambda(t,x)dF(t) = F(\xi_x+\delta)\cdot K_\lambda(\xi_x+\delta,x) + \int_{\xi_x}^{\xi_x+\delta} F(t)d_t[-K_\lambda(t,x)] \le$$

$$\le \varepsilon\cdot\delta\cdot K_\lambda(\xi_x+\delta,x) + \varepsilon\int_{\xi_x}^{\xi_x+\delta}(t-\xi_x)d_t[-K_\lambda(t,x)] =$$

$$= \varepsilon\cdot\delta\cdot K_\lambda(\xi_x+\delta,x) - \varepsilon\cdot\delta\cdot K_\lambda(\xi_x+\delta,x) + \varepsilon\int_{\xi_x}^{\xi_x+\delta} K_\lambda(t,x)dt$$

and since $K_\lambda(t,x)$ is positive

$$I_{3,\lambda} \le \varepsilon\int_A^B K_\lambda(t,x)dt$$

By the same way we also obtain

$$I_{2,\lambda} < \varepsilon\int_A^B K_\lambda(t,x)dt$$

and therefore

$$I_{2,\lambda} + I_{3,\lambda} < 2\varepsilon\int_A^B K_\lambda(t,x)dt\,. \tag{10}$$

Finitely, (7), (8), (10) and (2) give the desired result.

Theorem 2. *Let* $f \in L_1(-\infty,\infty)$ *and non-negative kernel* $K_\lambda(t,x)$ *satisfy the condition* (A) *and the condition: For any* $\delta > 0$

$$\lim_{\lambda\to\infty}\int_{\xi_x+\delta}^{\infty} K_\lambda(t,x)dt = 0,$$
$$\lim_{\lambda\to\infty}\int_{-\infty}^{\xi_x+\delta} K_\lambda(t,x)dt = 0. \tag{11}$$

If ξ_x *is the Lebesgue point of function* f *then for integral operators (1) with* $(A,B)=(-\infty,\infty)$

$$\lim_{\lambda\to\infty} L_\lambda(f;x) f(\xi_x)$$

holds.

Proof. As above we can write (7). Now

$$I_{1,\lambda} < \int_{-\infty}^{\xi_x-\delta} K_\lambda(\xi_x-\delta,x)\cdot|f(t)-f(\xi_x)|dt \le \|f\|_{L_1} K_\lambda(\xi_x-\delta,x) + |f(\xi_x)|\int_{-\infty}^{\xi_x-\delta} K_\lambda(t,x)dt$$

and also

$$I_{4,\lambda} < \|f\|_{L_1} K_\lambda(\xi_x + \delta, x) + |f(\xi_x)| \int_{-\infty}^{\xi_x+\delta} K_\lambda(t,x)dt .$$

Therefore, by (11)

$$\lim_{\lambda\to\infty}(I_{1,\lambda} + I_{4,\lambda}) = 0 .$$

Theorem 3. *Let* $x \in [a,b]$ *is fixed,*

$$N = \lim_{\lambda\to\infty} \int_A^{\xi_x} K_\lambda(t,x)dt$$

and the non-negative kernel $K_\lambda(t,x)$ *satisfy the condition* (A).

If there exists

$$\lim_{y\to\xi_x-0} f(y) = f(\xi_x - 0), \qquad \lim_{y\to\xi_x+0} f(y) = f(\xi_x + 0)$$

then for integral operators (1)

$$\lim_{\lambda\to\infty} L_\lambda(f;x) = Nf(\xi_x - 0) + (1-N) f(\xi_x + 0)$$

hold.

Proof. Since

$$L_\lambda(f;x) = \int_A^{\xi_x} [f(t) - f(\xi_x - 0)] K_\lambda(t,x)dt + \int_{\xi_x}^{B} [f(t) - f(\xi_x + 0)] K_\lambda(t,x)dt +$$

$$+ f(\xi_x - 0) \int_A^{\xi_x} K_\lambda(t,x)dt + f(\xi_x + 0) \int_{\xi_x}^{B} K_\lambda(t,x)dt ,$$

it sufficient to show that the first two integrals in right hand side tends to zero as $\lambda\to\infty$.

Consider, for example, the integral

$$I_\lambda = \int_{\xi_x}^{B} [f(t) - f(\xi_x + 0)] K_\lambda(t,x)dt .$$

Obviusly by the condition of the theorem we have

$$|f(t) - f(\xi_x + 0)| < \varepsilon \quad \text{if} \quad |t - \xi_x| < \delta .$$

Therefore

$$|I_\lambda| \le \int_{\xi_x}^{\xi_x+\delta} |f(t) - f(\xi_x + 0)| K_\lambda(t,x)dt + \int_{\xi_x+\delta}^{B} |f(t) - f(\xi_x + 0)| K_\lambda(t,x)dt <$$

$$< \varepsilon \int_A^B K_\lambda(t,x)dt + K_\lambda(\xi_x + \delta, x)\left(\|f\|_{L_1} + |f(\xi_x + 0)|(B - A)\right)$$

and (2) and (4) gives the proof.

Corollary 1. *If there exist* $\lim_{y\to\xi_x} f(y)$ *then*

$$\lim_{\lambda\to\infty} L_\lambda(f;x) = \lim_{y\to x} f(y) .$$

In conclution we will give some examples of kernels, satisfying the condition (A).

1. Let $(A,B) = (-\infty,\infty)$, $[a,b) = [1,\infty)$

$$K_\lambda(t,x)=\frac{1}{\sqrt{\pi}}e^{-\lambda^2(tx-1)^2}\cdot\left(\lambda x+\frac{x^2}{\lambda}\right).$$

Then $\xi_x=\frac{1}{x}$.

2. Let $(A,B)=(a,b)=(-\infty,\infty)$

$$K_\lambda(t,x)=\frac{\sqrt{\varphi(x)}}{\lambda\sqrt{\pi}}\cdot\frac{1}{\lambda^2+\varphi(t)\cdot t^2}.$$

Then $\xi_x=\sqrt{\varphi(x)}$.

3. $(A,B)=[0,\infty),\quad [a,b)=[1,\infty)$

$$K_\lambda(t,x)=\frac{2x\left(1+x+\frac{x}{\lambda}\right)\cdot\sqrt{\lambda}}{\sqrt{\pi}}\cdot e^{-x\lambda\left((1+x)t+b(x)\right)^2}$$

Then $\xi_x=-\frac{b(x)}{1+x}$.

4. $[A,B)=[a,b)=[0,\infty)$

$\psi_\lambda(t:x)$ *is increasing in* $[0,1]$ *and decreasing in* $[1,\infty)$ *and*

$$\lim_{\lambda\to\infty}\int_0^\infty\psi_\lambda(t;x)dt=1.$$

Then for a kernel

$$K_\lambda(t,x)=\frac{1}{\varphi(x)}\cdot\psi_\lambda\left(\frac{t}{\varphi(x)},x\right)$$

where $\varphi(x)$ *is positive function on* $[0,\infty)$, *we have* $\xi_x=\varphi(x)$.

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