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ON ONE DISTURBANCES THEORY PROBLEM FOR BOUNDARY VALUE PROBLEMS OF OPERATOR-DIFFERENTIALS EQUATIONS OF THE SECOND ORDER

Abstract

At the paper the theorem on existence of holomorphic solutions of one class of boundary value problem for operator- differential equation of the second order is got, when the boundary conditions contains the disturbance operator.

Let H be a separable Hilbert space, A be a normal operator with completely continuous inverses $A^{-1}, \left(A^{-1}\right)^* A^{-1} = A^{-1}\left(A^{-1}\right)^*$ and if $\lambda_1, \lambda_2, ..., \lambda_n, ...$ $\left(|\lambda_1| \le |\lambda_2| \le ... \le |\lambda_n| \le ...\right)$ are eigen-values of the operator A, and $e_1, e_2, ..., e_n, ...$ is corresponding orthonormal system of eigen-elements of the operator A, then A is represented by the following form

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n , \qquad x \in D(A).$$

Let's denote by

$$Cx = \sum_{n=1}^{\infty} |\lambda_n| (x, e_n) e_n , \qquad x \in D(A).$$

Let's determine further a Hilbert scale generated by the operator C , i.e.

$$H_\gamma = D\left(A^\gamma\right) = D\left(C^\gamma\right), \qquad (x, y)_\gamma = \left(C^\gamma x, C^\gamma y\right), \ \ \gamma \ge 0 .$$

Let $L_2(R_+ : H)$ be a Hilbert space of the vector-function $f(t)$ with the values from H measurable and integrable by Bokhner square [1]

$$L_2(R_+ : H) = \left\{ f(t) : \| f(t) \| = \left(\int_0^\infty \| f(t) \|_H^2 dt \right)^{\frac{1}{2}} < \infty \right\}.$$

Let's denote by S_α the following sector in surface

$$S_\alpha = \left\{ z : |\arg z| < \alpha \right\}, \qquad 0 < \alpha < \frac{\pi}{2}$$

and let's denote by $H_2(\alpha : H)$ (see [2]) the space of vector-function $f(z)$ holomorphic in S_α and for which

$$\sup_{|\varphi| < \alpha} \int_0^\infty \left\| f\left(te^{i\varphi}\right) \right\|^2 dt < \infty .$$

The functions from $H_2(\alpha : H)$ have the boundary values in sense $L_2(R_+ : H)$ (and almost everywhere)

$$f_\alpha(t) = f\left(te^{i\alpha}\right) \text{ and } f_{-\alpha}(t) = f\left(te^{-i\alpha}\right).$$

The space $H_2(\alpha : H)$ is a Hilbert space with respect to scalar product

$$(f, g)_\alpha = \frac{1}{2}\left(f_\alpha(t), g_\alpha(t) \right)_{L_2(R_+ : H)} + \frac{1}{2}\left(f_{-\alpha}(t), g_{-\alpha}(t) \right)_{L_2(R_+ : H)} .$$

Let's denote further by $W_2^2(\alpha : H)$ a space of vector-functions

$$W_2^2(\alpha:H)=\{u(z)|u''(z)\in H_2(\alpha:H),\ A^2u(z)\in H_2(\alpha:H)\}$$

with scalar product

$$((u(z),\upsilon(z)))_\alpha=(u''(z),\upsilon''(z))_\alpha+(A^2u(z),A^2\upsilon(z))_\alpha .$$

The space $W_2^2(\alpha:H)$ is also a Hilbert space and the theorem on intermediate products and the theorem on traces hold in this space, i.e. if $u(z)\in W_2^2(\alpha:H)$, then $A^{2-j}u^{(j)}(z)\in H_2(\alpha:H)$

$$\left\|A^{2-j}u\right\|_\alpha\le const\left|\left\|u\right\|\right|_\alpha\ ,\quad j=0,1,2\ ,$$

$$\left\|u^{(j)}(0)\right\|_{2-j-\frac{1}{2}}\le const\left|\left\|u\right\|\right|_\alpha .$$

Here $\left|\left\|\ \right\|\right|_\alpha$ is a norm in the space $W_2^2(\alpha:H)$.

At the given paper the following boundary value problem is considered

$$-\frac{d^2u(z)}{dz^2}+A^2u(z)=f(z),\quad z\in S_\alpha, \tag{1}$$

$$u(0)-Ku=0\ , \tag{2}$$

where the operator A is normal with completely continuous inverse, and the operator $K:W_2^2(R_+:H)\to H_{2/3}$ is bounded.

Let's denote that many boundary value problems for operator-differentials equations are investigated when operator-differentials equation has the disturbed part and the boundary conditions haven't such disturbed parts (see [4]).

Let's denote that the equation (1) with the boundary condition $u(0)=0$ is solvable at some conditions on the spectrum of the operator A.

We are interested in the problem, at which conditions on smallness of the norm of the operator K the problem (1), (2) is also solvable. Let's denote that such problems are in book [3] for ordinary differential operators.

First of all let's give some definitions

Definitions 1. *If the vector-function $u(z)\in W_2^2(\alpha:H)$ satisfies the condition (1) in S_α identically then we'll call it a regular solution of the equation (1).*

Definition 2. *If at any $f(z)\in H_2(\alpha:H)$ there exists a regular solution of the equation (1) which satisfies the boundary condition (2) in the sense*

$$\lim_{\substack{|z|\to 0\\ |\arg z|\le\alpha}}\left\|u(z)-Ku\right\|_{\frac{3}{2}}=0$$

and holds the inequality

$$\left|\left\|u\right\|\right|_\alpha\le const\left\|f\right\|_\alpha ,$$

then we call the problem (1), (2) regularity solvable.

First of all let's prove the following lemma.

Lemma 1. *Let the operator A be normal with completely continuous inverse A^{-1} whose spectrum is contained in a corner sector*

$$S_\varepsilon=\{\lambda:|\arg\lambda|\le\varepsilon\}\ ,\quad 0\le\varepsilon<\frac{\pi}{2}$$

and the number $0<\alpha+\varepsilon<\frac{\pi}{2}$. *Then a semi-group of linear bounded operators* $e^{-zA}:H_{\frac{3}{2}}\to W_2^2(\alpha:H)$ *is a continuous operator with the norm no more than the numbers* $(\cos(\alpha+\varepsilon))^{-\frac{1}{2}}$.

Proof. Let $\varphi\in H_{\frac{3}{2}}$. Then

$$\left\|\left\|e^{-zA}\psi\right\|\right\|_\alpha^2=\left\|A^2e^{-zA}\psi\right\|_\alpha^2+\left\|A^2e^{-zA}\psi\right\|_\alpha^2=2\left\|A^2e^{-zA}\psi\right\|_\alpha^2=$$
$$=\left\|A^2e^{-te^{i\alpha}}\psi\right\|_{L_2(R_+:H)}^2+\left\|A^2e^{-te^{-i\alpha}}\psi\right\|_{L_2(R_+:H)}^2. \tag{3}$$

Let's estimate the first summand in the equality (3). The second summand is determined analogously.

Using the spectral expansion of the operator A we have

$$\left\|A^2e^{-te^{i\alpha}}\psi\right\|_{L_2(R_+:H)}^2=\left\|\sum_{n=1}^\infty\lambda_n^2e^{-te^{i\alpha}\lambda_n}(\psi,e_n)e_n\right\|_{L_2(R_+:H)}^2=$$
$$=\int_0^\infty\sum_{n=1}^\infty|\lambda_n|^4\left|e^{-te^{i\alpha}\lambda_n}\right|^2|(\psi,e_n)|^2dt=\int_0^\infty\sum_{n=1}^\infty|\lambda_n|^4\left|e^{-te^{i\alpha}|\lambda_n|e^{i\varphi_n}}\right|^2|(\psi,e_n)|^2dt=$$
$$=\int_0^\infty\sum_{n=1}^\infty|\lambda_n|^4e^{-2t|\lambda_n|\cos(\alpha+\varphi_n)}|(\psi,e_n)|^2dt\le\int_0^\infty\sum_{n=1}^\infty|\lambda_n|^4e^{-2t|\lambda_n|\cos(\alpha+\varphi_n)}|(\psi,e_n)|^2dt\le$$
$$\le\sum_{n=1}^\infty|\lambda_n|^4|(\psi,e_n)|^2\int_0^\infty e^{-2t|\mu_n|\cos(\alpha+\varepsilon)}dt=\sum_{n=1}^\infty\frac{|\lambda_n|^3}{2\cos(\alpha+\varepsilon)}|(\psi,e_n)|^2=$$
$$=\frac{1}{2\cos(\alpha+\varepsilon)}\sum_{n=1}^\infty|\lambda_n|^3|(\psi,e_n)|^2=\frac{1}{2\cos(\alpha+\varepsilon)}\|\psi\|_{3/2}^2.$$

Analogously we've that

$$\left\|A^2e^{-te^{-i\alpha}}\psi\right\|_{L_2(R_+:H)}^2\le\frac{1}{2\cos(\alpha+\varepsilon)}\|\psi\|_{3/2}^2.$$

Allowing for these inequalities in the equality (3) we get

$$\left\|\left\|e^{-zA}\psi\right\|\right\|_\alpha^2\le\frac{1}{\cos(\alpha+\varepsilon)}\|\psi\|_{3/2}^2,\ \text{i.e.}\ \left\|\left\|e^{-zA}\psi\right\|\right\|_\alpha\le(\cos(\alpha+\varepsilon))^{-\frac{1}{2}}\|\psi\|_{3/2}.$$

Lemma is proved.

Let's prove now the following theorem on a regular solvability of the problem (1), (2).

Theorem. *Let* A *be a normal operator with completely continuous inverse* A^{-1} *whose spectrum is contained in corner sector*

$$S_\varepsilon=\{\lambda:|\arg\lambda|\le\varepsilon\},$$

where $0<\varepsilon<\frac{\pi}{2}$. *If the number* $0<\alpha+\varepsilon<\frac{\pi}{2}$ *and the norm of the operator* K *is less than* $(\cos(\alpha+\varepsilon))^{\frac{1}{2}}$ *then the problem (1), (2) is regularly solvable.*

Proof. First of all let's prove that the homogeneous problem i.e. the problem (1), (2) has only null regular solution when $f(z)=0$.

Since the general regular solution of the equation

$$-u''(z)+A^2u(z)=0$$

has the form

$$u_0(z)=e^{-zA}\psi ,$$

where $\psi\in H_{3/2}$, then from the boundary condition (2) follows that

$$\psi-K\left(e^{-zA}\psi\right)=0 ,\quad \psi\in H_{3/2}$$

or

$$\left(E-Ke^{-zA}\right)\psi=0 ,\quad \psi\in H_{3/2} ,$$

and E is an unit operator at the space $H_{3/2}$.

Since

$$\left\|Ke^{-zA}\psi\right\|_{3/2}\le\left\|K\right\|_{W_2^2(\alpha:H)\to H_{3/2}}\cdot\left\|e^{-zA}\psi\right\|_{W_2^2(\alpha:H)}\le$$

$$\le\left\|K\right\|_{W_2^2(\alpha:H)\to H_{3/2}}\cdot\left\|e^{-zA}\right\|_{H_{3/2}\to W_2^2(\alpha:H)}\cdot\left\|\psi\right\|_{3/2},$$

then applying the lemma 1 we have

$$\left\|Ke^{-zA}\psi\right\|_{3/2}\le\left(\cos(\alpha+\varepsilon)\right)^{-1/2}\left\|K\right\|_{W_2^2(\alpha:H)\to H_{3/2}} .$$

From the condition of the theorem it follows that

$$\chi=\left\|Ke^{-zA}\right\|_{H_{3/2}\to H_{3/2}}<1 .$$

Therefore the operator $\left(E-Ke^{-zA}\right)$ is inverse $H_{3/2}$, consequently $\psi=0$, i.e. $u_0(z)=0$.

Now show that for any $f(z)\in H_{3/2}$ there exists a regular solution $u(z)\in W_2^2(\alpha:H)$.

It is easy to see that for any $f(z)\in H_2(\alpha:H)$ the vector-function

$$u_1(z)=\frac{1}{2\pi i}\int\limits_{\Gamma_{(\pi/2+\alpha)}}\left(-\lambda^2E+A^2\right)^{-1}e^{-\lambda z}\hat f(\lambda)d\lambda-\frac{1}{2\pi i}\int\limits_{\Gamma_{(-\pi/2-\alpha)}}\left(-\lambda^2E+A^2\right)^{-1}e^{-\lambda z}\hat f(\lambda)d\lambda$$

satisfies the equation (1) identically in S_α . Here $\hat f(\lambda)$ is a Laplacian transformation of a vector-function $f(z)$ from the class $H_2(\alpha:H)$ (see [5]), and

$$\Gamma_{\pm(\pi/2+\alpha)}=\left\{\lambda\middle|\arg\lambda=\pm\left(\pi/2+\alpha\right)\right\}.$$

On these beams it holds the inequality

$$\left\|A^2\left(-\lambda^2E+A^2\right)^{-1}\right\|\le const,\quad\left\|\lambda^2\left(-\lambda^2E+A^2\right)^{-1}\right\|\le const . \qquad (4)$$

Really for example when $\lambda\in\Gamma_{(\pi/2+\alpha)}$ $\left(\lambda=re^{i(\pi/2+\alpha)},r>0\right)$

$$\left\|A^2\left(-\lambda^2E+A^2\right)^{-1}\right\|=\left\|A^2\left(-r^2e^{i(\pi+2\alpha)}+A^2\right)^{-1}\right\|=\left\|A^2\left(r^2e^{i2\alpha}+A^2\right)^{-1}\right\|=$$

$$= \sup_{\lambda_n \in \sigma(A)} \left| \lambda_n^2 \left(r^2 e^{2i\alpha} + \lambda_n^2 \right)^{-1} \right| = \sup_{\lambda_n \in \sigma(A)} \left(|\lambda_n|^2 \left(r^4 + |\lambda_n|^4 + 2|\lambda_n|^2 r^2 \cos 2(\arg \lambda_n + \alpha) \right)^{-1/2} \right) \le$$

$$\le \sup_{\lambda_n \in \sigma(A)} \left(|\lambda_n|^2 \left(r^4 + |\lambda_n|^4 + 2|\lambda_n|^2 r^2 \cos 2(\varepsilon + \alpha) \right)^{-1/2} \right). \tag{5}$$

Since when $0 < \alpha + \varepsilon \le \pi/4$ $\cos 2(\alpha + \varepsilon) \ge 0$, then from the inequality (5) it follows that in this case

$$\left\| A^2 \left(-\lambda^2 E + A^2 \right)^{-1} \right\| \le \sup_{\lambda_n \in \sigma(A)} \left(|\lambda_n|^2 \left(|\lambda_n|^4 + r^4 \right)^{-1/2} \right) \le 1, \tag{6}$$

and when $\pi/4 < \alpha + \varepsilon \le \pi/2$ $\cos 2(\alpha + \varepsilon) < 0$. Therefore using the Cauchy's inequality in the inequality (5) we'll get

$$\left\| A^2 \left(-\lambda^2 E + A^2 \right)^{-1} \right\| \le \sup_{\lambda_n \in \sigma(A)} \left(|\lambda_n|^2 \left(r^4 + \lambda_n^4 + \left(r^4 + \lambda_n^4 \right) \right) \cos 2(\alpha + \varepsilon)^{-1/2} \right) =$$

$$= \sup_{\lambda_n \in \sigma(A)} \left(|\lambda_n|^2 \left(r^4 + |\lambda_n^4| \right)^{-1/2} \left(1 + \cos 2(\alpha + \varepsilon) \right)^{-1/2} \right) \le \left(2 \cos 2(\alpha + \varepsilon) \right)^{-1/2}. \tag{7}$$

From the inequality (6) and (7) it follows the first inequality from (4). The second inequality from (4) is proved analogously. From the inequality (4) follows that $u(z) \in W_2^2(\alpha : H)$. Since a general regular solution of the equation (1) is represented in the following form

$$u(z) = u_1(z) + e^{-zA}\psi, \tag{8}$$

where $\psi \in H_{3/2}$ then from the boundary condition (2) it follows that

$$u_1(0) + \psi = K\left(u_1(z) + e^{-zA}\psi \right)$$

or

$$\psi - Ke^{-zA}\psi = Ku_1(z) - u_1(0).$$

Since $\psi_1 = Ku_1(z) - u_1(0)$, then from the last equation we get that

$$\left(E - Ke^{-zA} \right)\psi = \psi_1.$$

As shown that the operator $E - Ke^{-zA}$ is inverse in $H_{3/2}$ then $\psi = \left(E - Ke^{-zA} \right)^{-1} \psi_1 \in H_{3/2}$. Thus $u(z)$ is regular solution of the problem (1), (2).

On the other hand

$$\left\| u''(z) - A^2 u(z) \right\|_\alpha^2 \le 2\left(\left\| u''(z) \right\|_\alpha^2 + \left\| A^2 u(z) \right\|_\alpha^2 \right) = 2 \left| \left\| u \right\| \right|_\alpha^2,$$

then by the Banach theorem on the inverse operator it follows that it holds the inequality

$$\left| \left\| u \right\| \right|_\alpha^2 \le const \left\| f \right\|_\alpha.$$

Theorem is proved.

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