## APPLIED PROBLEMS OF MATHEMATICS AND MECHANICS

**GASANOV H.G.**

# HEAT TRANSFER IN A BOUNDARY LAYER OF LIQUID IN A CYLINDRICAL PIPE

**Abstract**

*Taking surrounding medium temperature as constant, non- stationary problem of heat transfer in liquid boundary layer in cylindrical pipe and between two concentric cylindrical pipes are solved. It is shown, for this approach temperature of liquid near sides keeps unchangeable during all the process. Presence of transverse temperature gradient between concentric cylindrical pipes is established, if temperature of he pipes are different, thereto this gradient changes its sign along pipe length.*

### §1. Introduction.

Heat transfer processes in liquid are complicated by the motion of different parts of irregular heated liquid. We assume that temperature differences existing in liquid are sufficiently small for its properties can be considered, as independent on temperature. On the other hand, these differences will be assumed so large that in comparison with them one could ignore temperature changes, stipulated by heat emission, concerned with energy dissipation by means of internal friction. Under assumptions made, in heat equation [1] the term containing viscosity can be omitted, so it remains

$$\frac{\partial T}{\partial t}+\vec{v}\overrightarrow{\nabla T}=a^2\cdot\Delta T-\frac{2\alpha}{R}\left(T-T_0\right) \qquad (1)$$

this equation is written for the case of heat exchange with surrounding medium, which has constant temperature $T_0$, i.e. assume that $T_0=T_0(z, R, t)$, moreover $T>T_0$, the rest of notions are generally accepted ones.

Being of interest liquid temperature distribution at very large Reynold's numbers detects features, analogous to those which has speed distribution itself. Very large values of $\mathrm{Re}$ are equivalent to very small viscosity. But since Prandtle $\mathrm{Pr}=\nu/a^2$ number $a$ is not very small then together with $\nu$ thermal diffusivity a must also be considered as a small one. This corresponds to the fact, that at sufficiently large speeds of motion liquid can be approximately considered as an ideal one, in ideal liquid both internal friction processes and heat conduction processes must be absent.

Such consideration however, will be non-applicable in wall eager of liquid, since neither boundary adhesion condition, nor condition of identity of temperatures of liquid and pipe will not be satisfied.

As a result in boundary layer along with rapid decreasing of speed also occurs rapid change of temperature of liquid to the value, which is equal to the temperature of cylindrical pipe surface. The boundary layer is characterized by presence of large gradients both of speed and temperature.

Consider heat exchange process in laminar boundary layer. For that we write heat conduction equation in an open form subject to the fact, that process itself has cylindrical symmetry, i.e. all derivatives $\partial^i T/\partial\varphi^i$ $(i=1,2)$ are considered to be zero

$$\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+v_z\frac{\partial T}{\partial z}=a^2\cdot\left\{\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right\}-\frac{2\alpha}{R}(T-T_0), \tag{2}$$

$v_r$ and $v_z$ are radial and longitudinal components of velocity of liquid flow in considered layer respectively, for simplicity we assume $v_r$ and $v_z$ to be constant, $T=T(z,r,t)$.

### §2. Solution of heat conduction equation for a cylindrical pipe.

First of all we make change of desired function

$$T(z,r,t)-T_0=\Phi(z,r,t).$$

Subject to last change the equation (2) will turn into

$$\frac{\partial \Phi}{\partial t}+v_r\frac{\partial \Phi}{\partial r}+v_z\frac{\partial \Phi}{\partial z}=a^2\cdot\left\{\frac{\partial^2 \Phi}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi}{\partial r}+\frac{\partial^2 \Phi}{\partial z^2}\right\}-\frac{2\alpha}{R}\Phi\,. \tag{3}$$

We'll solve this equation by separation of variables method. Represent function $\Phi(z,r,t)$ in the following form

$$\Phi(z,r,t)=\Phi_z(z)\Phi_{r,t}(r,t) \tag{4}$$

after that instead of (3) we obtain two equations

$$\begin{cases}\dfrac{\partial \Phi_{r,t}}{\partial t}+v_r\dfrac{\partial \Phi_{r,t}}{\partial r}=a^2\cdot\left\{\dfrac{\partial^2 \Phi_{r,t}}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial \Phi_{r,t}}{\partial r}\right\}+\left(\omega-\dfrac{2\alpha}{R}\right)\cdot\Phi_{r,t}=0\,,\\ a^2\dfrac{d^2\Phi_z}{dz^2}-v_z\dfrac{d\Phi_z}{dz}-\omega\cdot\Phi_z=0\,.\end{cases}$$

Since the last two equations will be solved separately, then its rational to set initial and boundary conditions in the following form:

1) relative to the function $\Phi_{r,t}(r,t)$

$$\Phi_{r,t}(r,t=0)=\Phi_1=const\neq g(r),\quad \Phi_{r,t}(r=R,t)=\Phi_2=const$$

$$\left.\frac{\Phi_{r,t}}{\partial r}\right|_{r=r_0}=0\,;\quad r_0\le r\le R$$

$r_0$ is a radius of liquid flow, $R$ is a radius of cylindrical pipe.

2) relative to $\Phi_z(z)$

$$\Phi_z(z=0)=\Phi_3=const\,,\quad \Phi_z(z\to\infty)=0$$

at that $\omega$ is to be some constant, subjected to empirical definition (in some cases, e.g. solving inverse problems the mentioned constant $\omega$ can be defined by setting additional boundary condition). At first we solve an equation for $\Phi_z$; it is so-called free oscillation equation. Form of its solution is defined by the value:

$$\lambda=\sqrt{v_z^2+4\omega a^2}\;;$$

since $\lambda$, as it follows from this expression, passes the essentially positive value, then function $\Phi_z(z)$ has a solution in the form

$$\Phi_z(z)=C_1\exp\left\{\frac{v_z-\lambda}{2a^2}\cdot z\right\}+C_2\exp\left\{\frac{v_z+\lambda}{2a^2}\cdot z\right\}.$$

The second addend gives divergent at large $z$ values of function $\Phi_z(z)$. In order to avoid such ambiguity we assume $C_2=0$, correspondingly

$$\Phi_z(z)=C_1\exp\left\{\frac{v_z-\lambda}{2a^2}\cdot z\right\}\ .$$

using boundary conditions for $z$-component, we find $C_1=\Phi_3$ or ultima analysi for $\Phi_z(z)$ we obtain

$$\Phi_z(z)=\Phi_3\exp\left\{\frac{v_z-\lambda}{2a^2}\cdot z\right\}. \qquad (5)$$

Now we pass on to solving equation for the component $\Phi_{r,t}(r,t)$. It also my be solved by separation of variables, but this will lead to appearance of one more constant demanding experimental determination. Considering such approach inexpedient, we solve the pointed equation by operational method. Applying time (by $t$) Laplace transformation we obtain in images subject to initial condition

$$\frac{d^2\Phi^*_{r,t}}{d\,r^2}+\left(\frac{1}{r}-\frac{v_r}{a^2}\right)\cdot\frac{d\Phi^*_{r,t}}{d\,r}-\frac{A_1}{a^2}\cdot\Phi^*_{r,t}=\frac{\Phi_1}{a^2}\ ; \qquad (6)$$

here were used notions

$$\Phi^*_{r,t}=\Phi^*_{r,t}(r,s)=\int_0^\infty\Phi_{r,t}(r,t)\cdot e^{-st}\,dt\ ,\quad A_1=\left(\frac{2\alpha}{R}+s-\omega\right).$$

Boundary conditions in images look like

$$\left.\frac{\partial\Phi^*_{r,t}(r,s)}{\partial r}\right|_{r=r_0}=0\ ;\ \Phi^*_{r,t}(r=R,s)=\Phi_2/s\ .$$

Before solving (6) we make change

$$\Phi^*_{r,t}(r,s)=P^*_{r,t}(r,s)-\frac{\Phi_1}{A_1}\ ,$$

after which this equation will turn into homogeneous

$$\frac{d^2P^*_{r,t}}{dr^2}+\left(\frac{1}{r}-\frac{v_r}{a^2}\right)\cdot\frac{dP^*_{r,t}}{d\,r}-\frac{A_1}{a^2}\cdot P^*_{r,t}=0\ .$$

At first multiply this equation by $r$ and then by means of change

$$P^*_{r,t}(r,s)=e^{v_r\cdot r/2a^2}\cdot\varphi(r,s)$$

we reduce it to non-degenerate hypergeometric equation relative to some new function $\varphi(r,s)$

$$r\varphi''(r,s)+\varphi'(r,s)+(A_3-A_2\cdot r)\varphi(r,s)=0\ ,$$

whose solution is Pohhammer function, introduced in notations of paper [2]

$$\varphi(rs)=\frac{C_3}{\sqrt{r}}\cdot y\left(\frac{A_3}{2\sqrt{A_2}};\ 0\ ;\ 2r\sqrt{A_2}\right), \qquad (7)$$

where introduced denotations

$$A_2=\frac{A_1}{a^2}+\frac{v_r^2}{4a^4}\ ,\quad A_3=\frac{v_r}{2a^2}\ .$$

For hypergeometric function $y(k,m,x)$ given by (7), values of function $y(k,m,x)$ itself at $m=0$, corresponding to considered case, coincide by virtue of which both solutions of second order equation relative to $\varphi(r,s)$ will be identical (in details this question is posed in [2]). Finally for desired function in images we have

$$\Phi^*_{r,t}(r,s)=\frac{C_3}{\sqrt{r}}\cdot e^{v_r\cdot r/2a^2}\cdot y\left(\frac{A_3}{2\sqrt{A_2}};0;2r\sqrt{A_2}\right)-\frac{\Phi_1}{A_1}.$$

For facilitation of returning to original consider extreme cases, which are of interest.

1) Moment of time corresponding to the beginning of process, i.e. $t\to 0$ (in a such limit $s\to\infty$). Then for the constants $A_1, A_2$ and $A_3$ we obtain the following values

$$A_1=s,\ A_2=\frac{s}{a^2},\ A_3=\frac{v_r}{2a^2}$$

according to which for $\Phi^*_{r,t}(r,s)$ we have

$$\Phi^*_{r,t}(r,s)=\frac{C_3}{\sqrt{r}}e^{v_r\cdot r/2a^2}\cdot y(0;0;\infty)-\frac{\Phi_1}{s}.$$

Now we use properties of Pohhammer function [3]

$$y(0;0;\infty)=y(0;0;-\infty)=e^{-2r\sqrt{s}/a}.$$

Finally, for image function we obtain

$$\Phi^*_{r,t}(r,s)=\frac{C_3}{\sqrt{r}}e^{v_r\cdot r/2a^2}\cdot e^{-2r\sqrt{s}/a}-\frac{\Phi_1}{s}.$$

We define constant $C_3$ from boundary conditions

$$C_3=\frac{\sqrt{R}}{s}(\Phi_1+\Phi_2)\cdot e^{v_r\cdot r/2a^2}\cdot e^{-2r\sqrt{s}/a}.$$

Now function $\Phi^*_{r,t}(r,s)$ can be written in explicit form at the limit of small $t$

$$\Phi^*_{r,t}(r,s)=\frac{1}{s}\sqrt{\frac{R}{r}}\cdot\psi e^{-2(r-R)\sqrt{s}/a}-\frac{\Phi_1}{s}; \tag{8}$$

here were introduced the following denotation

$$\psi=(\Phi_1+\Phi_2)\cdot e^{v_r\cdot(R-r)/2a^2}.$$

Transition from (8) o the original gives

$$\Phi_{r,t}(r,s)=\sqrt{\frac{R}{r}}\cdot\psi\, erfc\left(\frac{r-R}{a\sqrt{t}}\right)-\Phi_1. \tag{9}$$

Since function $erfc(x)$ is an even one, i.e.

$$erfc\left(\frac{r-R}{a\sqrt{t}}\right)=erfc\left(\frac{R-r}{a\sqrt{t}}\right),$$

then, finally, for the desired function we have

$$\Phi_{r,t}(r,t)=\sqrt{\frac{R}{r}}\cdot\psi\ erfc\left(\frac{R-r}{a\sqrt{t}}\right)-\Phi_1,$$

indicating decreasing of temperature of liquid in boundary layer.

Complete expression for the temperature in this limit

$$T(z,r,t\to 0)=T_0+\Phi_3\left[\sqrt{\frac{R}{r}}\cdot\psi\ erfc\left(\frac{R-r}{a\sqrt{t}}\right)-\Phi_1\right]\cdot\exp\left\{\frac{v_z-\lambda}{2a^2}z\right\}. \tag{10}$$

We find value of temperature in immediate proximity to the internal surface of the pipe, i.e. at $r \approx R$. At the condition

$$\psi \approx \Phi_1 + \Phi_2, \quad erfc\left(\frac{R-r}{a\sqrt{t}}\right)\Bigg|_{r\approx R} \approx 1,$$

and from formula (10) immediately we obtain

$$T(z, r \approx R, t \to 0) = T_0 + \Phi_3 \Phi_2 \cdot \exp\left\{\frac{v_z - \lambda}{2a^2} z\right\}.$$

2) Consider case of great times after beginning of process, i.e. $t \to \infty$ (in images it corresponds to $s \to 0$). In this limit subject to the fact that the coefficient $a$ is sufficiently small value (so, approximation $A_1 << v_r^2 / 2a^2$ takes place) for constants $A_1$, $A_2$ and $A_3$ we obtain

$$A_1 = s + \frac{2\alpha}{R} - \omega, \quad A_2 = \frac{v_r^2}{4a^4}, \quad A_3 = \sqrt{A_2} = \frac{v_r}{2a^2}$$

respectively.

Then from solution of hypergeometric equation for $\Phi^*_{r,t}(r,s)$ we obtain the following expression

$$\Phi^*_{r,t}(r,s) = \frac{C_4}{\sqrt{r}} e^{v_r \cdot r / 2a^2} \cdot У(A_4; 0; rA_5) - \frac{\Phi_1}{A_1}.$$

Constant $C_4$ is to be determined from boundary conditions. This procedure gives

$$C_4 = \left(\frac{\Phi_2}{s} + \frac{\Phi_1}{A_1}\right) \cdot \sqrt{R}\, e^{-v_r \cdot R / 2a^2} \cdot \frac{1}{У(A_4;\, 0;\, RA_5)};$$

at that we have view, that

$$A_4 = \frac{v_r}{4a\theta}, \quad A_5 = \frac{2\theta}{a}, \quad \theta = \sqrt{A_3}.$$

Finally for image function we obtain

$$\Phi^*_{r,t}(r,s) = \frac{\Phi_2}{s} \cdot G(r) + \frac{\Phi_1}{A_1} \cdot [G(r) - 1], \tag{11}$$

where $G(r)$ is some space function which doesn't depend on

$$G(r) = \sqrt{\frac{R}{r}}\, e^{-v_r \cdot (R-r) / 2a^2} \cdot \frac{У(A_4;\, 0;\, r\,A_5)}{У(A_4;\, 0;\, R\,A_5)}.$$

Passing from (11) to original gives

$$\Phi_{r,t}(r,t) = \Phi_2 G(r) + \Phi_1 \cdot e^{-\left(\frac{2\alpha}{R} - \omega\right)t} [G(r) - 1]. \tag{12}$$

Since we consider the process after the lapse of sufficiently large period of time (by the condition $t \to \infty$), then the second addend in (12) tends to zero asymptotically and temperature actually is defined by stationary term

$$\Phi_{r,t}(r, t \to \infty) = \Phi_2 G(r).$$

Complete expression for the temperature in this limit is

$$T(z, r, t \to \infty) = T_0 + \Phi_3 \Phi_2 G(r) \cdot \exp\left\{\frac{v_z - \lambda}{2a^2} z\right\}. \tag{13}$$

For immediate proximity the walls of pipe at $r \approx R$, taking into account that $G(r \approx R)$ we obtain

$$T(z, r \approx R, t \to \infty) = T_0 + \Phi_3 \Phi_2 \cdot \exp\left\{ \frac{v_z - \lambda}{2a^2} z \right\},$$

which coincides with the corresponding value of temperature $T(z, r \approx R, t \to 0)$. So, under those conditions for which heat conduction problem is solved in this paragraph, the temperature near the walls of pipes during the whole process $0 \le t < \infty$ practically doesn't change. Heat transfer, obviously, is realized at the most boundary layer $r_0 \le r < R$.

### §3. Heat transfer between two concentric cylindrical pipes.

In §2 we considered the heat transfer problem for liquid moving in a cylindrical pipe, i.e. there was one boundary layer. We investigate problem on liquid motion between two concentrically located cylindrical pipes, at that temperatures of these pipes are generally speaking different. In such a case for the considered liquid there are two boundary layers (by convention call internal one, which is adjoining to internal cylinder and external one, respectively), for each of which its own heat transfer problem must be solved. For convention we assume that temperature of internal cylinder is greater than both temperature of external one and liquid. Then for external boundary layer the problem solved in §2 is applicable, and for internal one we have to solve the following equation

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z} = a^2 \cdot \left\{ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right\} - \frac{2\alpha}{R_{вн}} \left( T_{0вн} - T \right); \qquad (14)$$

here it was already taken into account that temperature of medium adjoining to the internal cylinder is $T_{0вн} > T = T(z, r, t)$. Formally after changing

$$T_{0вн} - T(z, r, t) = \Phi(z, r, t)$$

solutions of (14) will coincide with corresponding solutions of equation (2), but its necessary to note, that heat transfer process in internal boundary layer will be defined by the correlation

$$T(z, r, t) = T_{0вн} - \Phi(z, r, t),$$

which demonstrates decreasing of temperature in layer in domain $r_1 \le r \le R_{вн}$ ($r_1$ is internal radius of liquid flow). For example, in immediate proximity to the wall of the internal pipe calculations give

$$T(z, r \approx R_{вн}, t) = T_0 - \Phi_3 \Phi_4 \cdot \exp\left\{ \frac{v_z - \lambda}{2a^2} z \right\}, \qquad (15)$$

where $\Phi_4$ is defined by the boundary condition

$$\Phi_{r,t}(r = R_{вн}, t) = \Phi_4 = const.$$

### §4. Conclusion.

The problem, solved in the present paper, is of interest of practical case of liquid motion in pipe at constant surrounding medium temperature, for example, at transportation of liquid, boring in bounded interval of depth and etc. The obtained results show that liquid temperature near the pipe walls at $r \approx R$ practically during the whole process doesn't change and all temperature changes occur in boundary layer with thickness $3 \div 5$ mm. In other words, namely in the last one all space-time temperature variations occur, described by formulas (9)-(13). On the other hand, temperature near wall changes along the pipe length. For example, under the same conditions when the given problem was solved, temperature near the wall of cylindrical pipe changed from $T_0 + \Phi_3\Phi_2$ at $z = 0$ to $T_0$ at $z = \infty$, at that such decreasing $T \ (r \approx R)$ is characteristic only for domains near the external cylinder, since for $T \ (r \approx R_{вн})$ temperature growth takes place according to (15) from $T_{0вн} - \Phi_3\Phi_4$ at $z = 0$ up to $T_0$ at $z = \infty$.

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**Hikmet G. Gasanov**
Azerbaijan State Oil Research and Project Institute.
39, Aganeymetulla str., 370033, Baku, Azerbaijan.
Tel.: 66-59-59(off.).

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