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DIRECT AND INVERSE PROBLEMS FOR DIFFERENTIAL OPERATORS WITH SINGULARITY AND DISCONTINUITY CONDITIONS INSIDE THE INTERVAL

Abstract

In this paper some aspect of direct and inverse problems for differential operators with singularity and discontinuity conditions inside the interval are investigated. The completeness of the system of eigen and adjoint functions of the given operator is proved.

1. Introduction. Consider the differential operator with non-integrable singularity

$$ly := -y'' + \left(\frac{\nu_0}{x^2} + q(x)\right)y,\ 0 < x < T,$$

on a finite interval. Here $q(x)$ is a complex-valued function, ν_0 is a complex number. Let $\nu_0 = \nu^2 - \frac{1}{4}$ and for definiteness $\mathrm{Re}\,\nu > 0,\ \nu \notin \mathrm{N}$. We'll assume, that $q(x) \cdot x^{\min(0,1-2\mathrm{Re}\,\nu)} \in L(0,T)$.

The not self-adjoint boundary value problem L of the form

$$l\,y = \lambda\, y,\quad 0 < x < T, \tag{1}$$

$$y(x) = O\left(x^{\nu+1/2}\right),\ \ x \to 0, \tag{2}$$

$$y(T) = 0, \tag{3}$$

$$\begin{bmatrix} y \\ y' \end{bmatrix}(a_i + 0) = A_i \begin{bmatrix} y \\ y' \end{bmatrix}(a_i - 0),\qquad A_i = \begin{bmatrix} a_{11}^{(i)} & a_{12}^{(i)} \\ a_{21}^{(i)} & a_{22}^{(i)} \end{bmatrix},\ a_i \in (0,T),\ i = \overline{0,p} \tag{4}$$

is considered with discontinuity conditions (4) in interval points $x = a_i$, $i = \overline{0,p}$ of the interval $(0,T)$, where $a_0 = 0,\ a_{p+1} = T$. Here $a_{jk}^{(i)}$ are complex numbers, $\det A_i \neq 0,\ i = \overline{1,p}$.

The aim of the work is the investigation of direct and inverse spectral analysis problems for the boundary problem L. Boundary value problems with singularities and discontinuity conditions appear in different branches of mathematics, mechanics, radio electronics, geophysics and other spheres of natural sciences and techniques. For example, discontinuity conditions inside the interval are connected with discontinuous and non-smooth properties of medium [1,2]. Such type inverse problems are also connected with investigation of discontinuous solutions of some non-linear equations of mathematical physics.

For classical Sturm-Liouville operators, Shrödinger equation and hyperbolic equations, direct and inverse problems are sufficiently completely studied (see [3-6] and references). Existence of singularity and discontinuity conditions inside the interval introduces qualitative changes in investigation.

Some aspects of direct and inverse problems for differential operators with discontinuity conditions were studied in [7-9]. In paper [18] some aspects of the operator L were studied when in solution there is only one discontinuity point inside the interval. As distinct from [18] in this paper the operator L is studied when in solution there are any finite number $p \ge 2$ of discontinuity points inside the interval and the uniqueness theorem of solutions of the inverse problem by two spectrums is proved. Inverse problems for equations with singularity without discontinuity conditions were considered in [10, 11] and etc. In the given paper properties of eigen and adjoint functions of the problem are studied and inverse problem of restoration of L by the data its spectral characteristics is investigated.

For definiteness we restrict ourselves with the most important particular case $a_{12}^{(i)} = 0$, $i = \overline{1, p}$. The general case is considered analogously.

Remark 1. If $\operatorname{Re} \nu \ge \frac{1}{2}$, then condition (2) is equivalent to condition $y(0) = 0$.

Remark 2. Problem (1)-(4) will be self-adjoint iff $\nu, q(x), a_{jk}^{(i)}$ are real and $\det A_i = 1$.

2. Fundamental systems of solutions. 2.1. At first consider differential equation

$$l_0 y := -y'' + \frac{\nu_0}{x^2} y = y \tag{5}$$

in a complex x-plane. Denote by Π_- the x-plane with the cut $x \le 0$. Let numbers c_{j0}, $j = 1,2$ be such that

$$c_{20} \cdot c_{10} = \frac{1}{2\nu}.$$

Functions

$$C_j(x) = x^{\mu_j} \sum_{k=0}^{\infty} c_{jk} x^{2k}, \quad \mu_j := (-1)^j \nu + \frac{1}{2}, \quad j = 1,2, \tag{6}$$

$$c_{jk} = (-1)^j c_{j0} \left(\prod_{s=1}^{k} \left((2s + \mu_j)(2s + \mu_j - 1) - \nu_0 \right) \right)^{-1}$$

are solutions of equation (5). Here and later on $x^{\mu} = \exp(\mu(\ln|x| + i \arg x))$, $\arg x \in (-\pi, \pi]$. Functions $C_j(x)$ are regular in Π_- and

$$\det\left[c_j^{(m-1)}(x)\right]_{j,m=2} \equiv 1. \tag{7}$$

Denote $\varepsilon_k = (-1)^{k-1} \cdot i$. Equation (5) has solutions $e_k(x)$, $k = 1,2$, $(-1)^{k-1} \operatorname{Im} x \ge 0$, satisfying the integral equations

$$e_k(x) = \exp(\varepsilon_k x) + \frac{1}{2i} \int_x^{\infty} \left(\exp(i(t - x)) - \exp(i(x - t)) \right) \frac{\nu_0}{t^2} e_k(t) dt$$

(here $\arg t = \arg x$, $|t| > |x|$).

Using fundamental system of solutions $\{C_j(x)\}_{j=1,2}$ we can write

$$e_k(x) = \sum_{j=1}^{2} \beta_{kj}^0 C_j(x). \tag{8}$$

In particular, it gives the analytical extension for function $e_k(x)$ in Π_-. Denote $\Omega_{1,\delta} = \{x : \arg x \in [-\pi + \delta, \pi]\}$, $\Omega_{2,\delta} = \{x : \arg x \in [-\pi, \pi - \delta]\}, \delta > 0$. We can show (see[12-14], that

$$e_k^{(m-1)}(x)=\varepsilon_k^{(m-1)}\exp(\varepsilon_k x)\left(1+O\left(\frac{1}{x}\right)\right),\ |x|\to\infty,\ k,m=1,2 \tag{9}$$

uniformly in $\Omega_{k,\delta}$ at each fixed $\delta>0$. Since according to Ostrogradsky-Liouville theorem Wronskian $\det\left[e_k^{(m-1)}(x)\right]_{k,m=1,2}$ doesn't depend on v, then using (9) we find

$$\det\left[e_k^{(m-1)}(x)\right]_{k,m=1,2}\equiv -2i. \tag{10}$$

Lemma 1. *The following equalities take place*

$$\beta_{2j}^0=\beta_{1j}^0\exp(i\pi\mu_j),\quad j=1,2, \tag{11}$$

$$\beta_{11}^0\beta_{12}^0=\frac{i}{\sin\pi\nu}. \tag{12}$$

Proof. From construction it follows, that

$$e_1(x)=e_2(-x),\qquad \operatorname{Im}x>0. \tag{13}$$

Since at $\operatorname{Im}x>0\,(-x)^\mu=x^\mu\exp(-i\pi\mu)$, then by virtue of (6) and (8) we have

$$e_2(-x)=\sum_{j=1}^{2}\beta_{2j}^0\exp(-i\pi\mu_j)C_j(x). \tag{14}$$

Substituting (8) and (14) into (13) and equating coefficients at $C_j(x)$, we obtain (11).

Further, according to (7), (8) and (10) $\det\left[\beta_{kj}^0\right]_{k,j=1,2}=-2i$. From here and from (11) follows (12).

2.2. Consider now the differential equation

$$l_0 y=\lambda y,\ x>0. \tag{15}$$

Let $\lambda=\rho^2$. Obviously if $y(x)$ is the solution of equation (5), then $y(\rho x)$ satisfies (15). Functions

$$C_j(x,\lambda):=\rho^{-\mu_j}C_j(\rho x)=x^{\mu_j}\sum_{k=0}^{\infty}c_{jk}(\rho x)^{2k},\quad j=1,2,\ x>0$$

are integer by λ solutions of equation (15), where

$$\det\left[C_j^{(m-1)}(x,\lambda)\right]_{j,m=1,2}\equiv 1.$$

Denote $S_{\pm1}^*=\{\rho:\pm\operatorname{Im}\rho\ge 0\},S_{k_0}=\left\{\rho:\arg\rho\in\left(k_0\frac{\pi}{2},(k_0+1)\frac{\pi}{2}\right)\right\}$, $k_0=\overline{-2,1}$. In each sector S_{k_0} roots of the equation $R^2+1=0$ can be enumerated such that $\operatorname{Re}(\rho R_1)<\operatorname{Re}(\rho R_2)$, $\rho\in S_{k_0}$. It's clear, that $R_k=\varepsilon_k$ for S_0 and S_1 and $R_k=\varepsilon_{3-k}$ for S_{-1} and S_2. For definiteness let $\operatorname{Re}\rho\ge 0$, i.e. $\rho\in\overline{S_0\cup S_{-1}}$.

Using the results of p.2.1 we obtain that in each sector S_{k_0} equation (15) has a fundamental system of solutions $\{y_k(x,\rho)\}_{k=1,2}$ such, that

$$y_k(x,\rho)=y_k(\rho x),$$

$$y_k^{(m-1)}(x,\rho)=(\rho R_k)^{m-1}\exp(\rho R_k x)\left(1+O\left(\frac{1}{\rho x}\right)\right),\ \rho\in\overline{S_{k_0}},\ |\rho|x\to\infty,\ k,m=1,2,$$

$$\det\left[y_k^{(m-1)}(x,\rho)\right]_{k,m=1,2}\equiv -2\rho R_1,\ y_k(x,\rho)=\sum_{j=1}^{2}b_{kj}^0\rho^{\mu_j}C_j(x,\lambda),$$

where $b_{kj}^0 = \beta_{jk}^0$ for S_{+1}^* and $b_{kj}^0 = \beta_{3-k,j}^0$ for S_{-1}^*. Note that $y_k(x,\rho) = e_k(\rho x)$ for S_{+1}^* and $y_k(x,\rho) = e_{3-k}(\rho x)$ for S_{-1}^*.

2.3. Now we pass to the investigation of equation (1) and construct for it corresponding systems of solutions by the perturbation method.

At $x \in (0,T)$ equation (1) has integer by λ solutions $S_j(x,\lambda)$, $j = 1,2$, satisfying the integral equations

$$S_j(x,\lambda) = C_j(x,\lambda) + \int_0^x \left(C_1(t,\lambda)C_2(x,\lambda) - C_2(t,\lambda)C_1(x,\lambda)\right) q(t) S_j(t,\lambda)\,dt,$$

where

$$S_j^{(m)}(x,\lambda) = O\left(x^{\mu_j - m}\right),\ \left(S_j(x,\lambda) - C_j(x,\lambda)\right)x^{-\mu_j} = o\left(x^{2\nu}\right),\ x \to 0,$$

uniformly by λ on compacts. Besides,

$$\det\left[S_j^{(m-1)}(x,\lambda)\right]_{j,m=1,2} \equiv 1, \tag{16}$$

$$\left|S_j^{(m)}(x,\lambda)\right| \le c\left|x^{\mu_j - m}\right|,\ |\rho| x \le 1. \tag{17}$$

Here and later on by the same symbol C we'll denote different positive constants in estimations, independent on x, λ.

In [15] the fundamental system of solutions of equation (1) $\{Y_k(x,\rho)\}_{k=1,2}$, $x \in (0,T]$, $\rho \in S_{k_0}$ was constructed, which has the following properties:

1) for each $x \in (0,T]$ functions $Y_k^{(m)}(x,\rho), m = 0,1$ are regular at $\rho \in S_{k_0}$, $|\rho| \ge \rho_*$ and continuous at $\rho \in \overline{S}_{k_0}$, $|\rho| \ge \rho_*$;
2) functions $Y_k(x,\rho)$ satisfy the integral equations

$$Y_k(x,\rho) = y_k(x,\rho) + \frac{1}{2i\rho}\sum_{j=1}^{2}\int_{\varepsilon_{jk}}^{x} (-1)^{j-1} y_j(x,\rho) y_{3-i}(t,\rho) q(t) Y_k(t,\rho)\,dt,$$

where $\varepsilon_{jk} = 0$, at $j \le k$, $\varepsilon_{21} = T$;
3) the following correlations hold:

$$\left|Y_k^{(m)}(x,\rho)(\rho R_k)^{-m}\exp(-\rho R_k x) - 1\right| \le \frac{C}{|\rho| x},\ \rho \in \overline{S}_{k_0}, x \in (0,T], |\rho| x \ge 1,\ m = 0,1, \tag{18}$$

$$\det\left[Y_k^{(m-1)}(x,\rho)\right]_{k,m=1,2} = -2\rho R_1\left(1 + O\left(\frac{1}{\rho}\right)\right), \tag{19}$$

$$Y_k(x,\rho) = \sum_{j=1}^{2} b_{kj}(\rho) S_j(x,\lambda), \tag{20}$$

where

$$b_{kj}(\rho) = b_{kj}^0 \rho^{\mu_j}\left(1 + O\left(\frac{1}{\rho}\right)\right),\ |\rho| \to \infty,\ \rho \in \overline{S}_{k_0}. \tag{21}$$

Note, that later on the asymptotic (21) of Stock's multipliers $b_{kj}(\rho)$ is of great importance.

Using (18)-(21), we'll study the asymptotic of solutions $S_j(x,\lambda)$. Denote $d_1^0=\beta_{12}^0/2i$, $d_2^0=-\beta_{11}^0/2i$. Then

$$d_1^0 d_2^0=-\frac{1}{4i\sin\pi\nu}.$$

Lemma 2. *At* $|\rho|\to\infty, |\rho|x\geq 1,\ x\in(0,T],\ j=1,2,\ m=0,1$ *the following asymptotic formula holds:*

$$S_j^{(m)}(x,\lambda)=d_j^0\rho^{-\mu_j}\Big((i\rho)^m\exp(-i\pi\mu_j)\exp(i\rho x)[1]_0+ \\ +(-i\rho)^m\exp(-i\rho x)[1]_0\Big). \tag{22}$$

Here and later on the following denotation is used

$$[1]_0=1+O\left(\frac{1}{|\rho|x}\right),\ |\rho|x\geq 1$$

(i.e. equality $f(x,\rho=[1]_0)$ *notes that* $|f(x,\rho)-1|\leq\frac{C}{|\rho|x}$, $|\rho|x\geq 1$*).*

Proof. Solving (20) with respect to $S_j(x,\lambda)$ we obtain

$$S_j(x,\lambda)=\sum_{k=1}^{2}d_{jk}(\rho)Y_k(x,\rho),\quad j=1,2, \tag{23}$$

where $[d_{jk}(\rho)]_{j,k=1,2}=\left([b_{kj}(\rho)]_{k,j=1,2}\right)^{-1}$.

By virtue of (21) we have

$$d_{kj}(\rho)=d_{jk}^0\rho^{-\mu_j}[1],\ |\rho|\to\infty,\ \rho\in\bar{S}_{k_0}, \tag{24}$$

where $[d_{jk}^0(\rho)]_{j,k=1,2}=\left([b_{jk}^0(\rho)]_{k,j=1,2}\right)^{-1}$ and the denotation $[\theta]=\theta+O(1/\rho)$ is used. We'll write equation (18) in the form

$$Y_k^{(m)}(x,\rho)=(\rho R_k)^m\exp(\rho R_k x)[1]_0,\ \rho\in\bar{S}_{k_0},\ |\rho|x\geq 1;\ m=0,1,\ k=1,2. \tag{25}$$

Let for definiteness $\rho\in\bar{S}_0$ (for other ρ calculations are analogous). We have $R_k=\varepsilon_k$, $b_{kj}^0=\beta_{kj}^0$ and, consequently, using (11), we find

$$\begin{bmatrix} d_{11}^0 & d_{12}^0 \\ d_{21}^0 & d_{22}^0\end{bmatrix}=\begin{bmatrix} d_1^0\exp(-i\pi\mu_1) & d_1^0 \\ d_2^0\exp(-i\pi\mu_2) & d_2^0\end{bmatrix}. \tag{26}$$

Substituting (24), (25) into (23) and taking into account (26) we obtain (21).

3. Properties of spectrum. Consider functions

$$\varphi_j(x,\lambda)=\begin{cases} S_j(x,\lambda) & x<a_1, \\ \omega_{j1}^{(i)}(\lambda)S_1(x,\lambda)+\omega_{j2}^{(i)}(\lambda)S_2(x,\lambda), & a_i<x<a_{i+1},\ i=\overline{1,p}.\end{cases} \tag{27}$$

Here $\omega_{j1}^{(i)}(\lambda)$ and $\omega_{j2}^{(i)}(\lambda)$ are defined from the following recursive correlations:

$$\begin{bmatrix}\omega_{j1}^{(i)}(\lambda)\\ \omega_{j2}^{(i)}(\lambda)\end{bmatrix}=\begin{bmatrix} a_{11}^{(i)} & 0 \\ a_{21}^{(i)} & a_{22}^{(i)}\end{bmatrix}\begin{bmatrix}\omega_{j1}^{(i-1)}(\lambda)\\ \omega_{j2}^{(i-1)}(\lambda)\end{bmatrix}\begin{bmatrix} s_1(a_{i-1},\lambda) & s_2(a_{i-1},\lambda) \\ s_1'(a_{i-1},\lambda) & s_2'(a_{i-1},\lambda)\end{bmatrix}\begin{bmatrix} s_1(a_i,\lambda) & s_2(a_i,\lambda) \\ s_1'(a_i,\lambda) & s_2'(a_i,\lambda)\end{bmatrix}^{-1},\ i=\overline{2,p},$$

where the first approximation is defined by the formula:

$$\begin{bmatrix}\omega_{j1}^{(1)}(\lambda)\\ \omega_{j2}^{(1)}(\lambda)\end{bmatrix}=\begin{bmatrix}a_{11}^{(1)} & 0\\ a_{21}^{(1)} & a_{22}^{(1)}\end{bmatrix}\begin{bmatrix}s_j(a_1,\lambda)\\ s'_j(a_1,\lambda)\end{bmatrix}\begin{bmatrix}s_1(a_1,\lambda) & s_2(a_1,\lambda)\\ s'_1(a_1,\lambda) & s'_2(a_1,\lambda)\end{bmatrix}^{-1}.$$

By the construction of function $\varphi_j(x,\lambda),\ j=1,2$ they are solutions of equation (1) at $a_{i-1}<x<a_i$, $i=\overline{1,p+1}$ and satisfy the conditions:

$$\begin{bmatrix}\varphi_j(a_i+0)\\ \varphi'_j(a_i+0)\end{bmatrix}=\begin{bmatrix}a_{11}^{(i)} & 0\\ a_{21}^{(i)} & a_{22}^{(i)}\end{bmatrix}\begin{bmatrix}\varphi_j(a_i-0)\\ \varphi'_j(a_i-0)\end{bmatrix}_{i=\overline{1,p}}. \tag{28}$$

According to (16) and (28) we have

$$\det\left[\varphi_j^{(m-1)}(x,\lambda)\right]_{j,m=1,2}=\begin{cases}1, & x<a_1,\\ \prod\limits_{k=1}^{i}\det A_k, & a_i<x<a_{i+1},\ i=\overline{1,p}.\end{cases} \tag{29}$$

Denote

$$b_i^{\pm}=\frac{1}{2}\left(a_{11}^{(i)}\pm a_{22}^{(i)}\right)$$

and suppose that $b_i^+\neq 0$. We'll call conditions $b_i^+\neq 0, i=\overline{1,p}$ the conditions of sewing regularity (SR) at the points a_i, $i=\overline{1,p}$. Below in p.5 the contrary instance is given, showing the essentiality of SR condition during the investigation of the boundary value problem 1.

Lemma 3. *At* $|\rho|x\geq 1,\ x\in(0,T],\ j=1,2; m=0,1$ *the following asymptotic formulas hold:*

$$\varphi_j^{(m)}(x,\lambda)=d_j^0\rho^{-\mu_j}\left((i\rho)^m\exp(-i\pi\mu_j)\exp(i\rho x)[1]_0+(-i\rho)^m\exp(-i\rho x)[1]_0\right),\ x<a_1, \tag{30}$$

$$\varphi_j^{(m)}(x,\lambda)=d_j^0\rho^{-\mu_j}\Bigg\{(i\rho)^m\Bigg[B_N\exp(-i\pi\mu_j)+\sum_{l=1}^{N}\sum_{1\leq s_1<s_2<\ldots<s_l\leq N}B_{Nl}\exp\Bigg(\sum_{k=1}^{l}\left((-1)^{l-k+1}2i\rho a_{s_k}-\right.$$

$$\left.-i\pi\mu_i\delta_{s_k}\right)\Bigg)\Bigg]\exp(i\rho x)[1]_0+(-i\rho)^m\Bigg[B_N+\sum_{l=1}^{N}\sum_{1\leq s_1<s_2<\ldots<s_l\leq N}B_{Nl}\exp\Bigg(\sum_{k=1}^{l}\left((-1)^{l-k}2i\rho a_{s_k}-\right.$$

$$\left.-i\pi\mu_j(1-\delta_{s_k})\right)\Bigg)\Bigg]\exp(-i\rho x)[1]_0\Bigg\},\ a_N<x<a_{N+1},\ N=\overline{1,p}, \tag{31}$$

where

$$\delta_{s_k}=\begin{cases}1, & at\ k\ is\ even\\ 0, & at\ k\ is\ odd\end{cases}, B_{Nl}=B_{Nl}(s_1,\ldots,s_j)=\prod_{k=1}^{l}b_{s_k}^{-}\cdot\prod_{k=l+1}^{N}b_{s_k}^{+},$$

$$B_N=\sum_{k=1}^{N}b_k^+,\quad B_0:\equiv 1.$$

Here $\{s_k\}_{k=\overline{1,N}}$ *is permutation of numbers* $1,\ldots,N$, *numbers* $\{s_k\}_{k=\overline{l+1,N}}$ *are complement of* $\{s_k\}_{k=\overline{1,l}}$ *to the numbers* $1,\ldots,N$.

Proof. Formula (30) follows from (22), therefore we'll prove only formula (31). We expand $\varphi_j(x,\lambda)$ by the fundamental system of solutions $\{Y_k(x,\rho)\}_{k=1,2}$ at $a_{i-1} < x < a_i$, $i=\overline{1,p+1}$

$$\varphi_j(x,\lambda)=\sum_{k=1}^{2} A_{jki}(\rho)Y_k(x,\rho), \quad a_{i-1} < x < a_i . \tag{32}$$

By virtue of (23) and (27)

$$A_{jk0}(\rho)=d_{jk}(\rho). \tag{33}$$

In order to calculate $A_{jki}(\rho)$ we'll use the sewing conditions (28). At first we'll calculate A_{jk1}. For this substituting (32) into (28), we obtain

$$\begin{bmatrix} Y_1(a_1,\rho) & Y_2(a_1,\rho) \\ Y_1'(a_1,\rho) & Y_2'(a_1,\rho) \end{bmatrix}\begin{bmatrix} A_{j11}(\rho) \\ A_{j21}(\rho) \end{bmatrix}=\begin{bmatrix} a_{11}^{(1)} & 0 \\ a_{21}^{(1)} & a_{22}^{(1)} \end{bmatrix}\begin{bmatrix} Y_1(a_1,\rho) & Y_2(a_1,\rho) \\ Y_1'(a_1,\rho) & Y_2'(a_1,\rho) \end{bmatrix}\begin{bmatrix} A_{j10}(\rho) \\ A_{j20}(\rho) \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} A_{j11}(\rho) \\ A_{j21}(\rho) \end{bmatrix}=\begin{bmatrix} \eta_{11}^{(1)}(\rho) & \eta_{12}^{(1)}(\rho) \\ \eta_{21}^{(1)}(\rho) & \eta_{22}^{(1)}(\rho) \end{bmatrix}\begin{bmatrix} A_{j10}(\rho) \\ A_{j20}(\rho) \end{bmatrix}, \tag{34}$$

where

$$\eta_{1k}^{(1)}(\rho)=\frac{1}{w(\rho)}\left(a_{11}^{(1)}Y_k(a_1,\rho)Y_2'(a_1,\rho)-a_{21}^{(1)}Y_k(a_1,\rho)Y_2(a_1,\rho)-a_{22}Y_k'(a_1,\rho)Y_2(a_1,\rho)\right),$$

$$\eta_{2k}^{(1)}(\rho)=-\frac{1}{w(\rho)}\left(a_{11}^{(1)}Y_k(a_1,\rho)Y_1'(a_1,\rho)-a_{21}^{(1)}Y_k(a_1,\rho)Y_1(a_1,\rho)-a_{22}Y_k'(a_1,\rho)Y_1(a_1,\rho)\right), \tag{35}$$

$$w(\rho)=\det\left[Y_k^{(m-1)}(a,\rho)\right]_{k,m=1,2}.$$

Let for definiteness $\rho\in\overline{S}_0$. Then $R_k=\varepsilon_k$. Using (35) and (28), we find

$$\begin{bmatrix} \eta_{11}^{(1)}(\rho) & \eta_{12}^{(1)}(\rho) \\ \eta_{21}^{(1)}(\rho) & \eta_{22}^{(1)}(\rho) \end{bmatrix}=\begin{bmatrix} [b_1^+] & [b_1^-]\exp(-2i\rho a_1) \\ [b_1^-]\exp(-2i\rho a_1) & [b_1^+] \end{bmatrix}. \tag{36}$$

Substituting now (36) into (34) and using (32), (34) and (26) we obtain

$$A_{j11}(\rho)=d_j^0\rho^{-\mu_j}\left(b_1^+\exp(-i\pi\mu_j)+b_1^-\exp(-2i\rho a_1)\right)[1],$$

$$A_{j21}(\rho)=d_j^0\rho^{-\mu_j}\left(b_1^-\exp(-i\pi\mu_j)\exp(2i\rho a_1)+b_1^+\right)[1],$$

which with (32) and (25) gives (31).

In the case $N=2$ analogously we obtain, that

$$\begin{bmatrix} A_{j12}(\rho) \\ A_{j22}(\rho) \end{bmatrix}=\begin{bmatrix} \eta_{11}^{(2)}(\rho) & \eta_{12}^{(2)}(\rho) \\ \eta_{21}^{(2)}(\rho) & \eta_{22}^{(2)}(\rho) \end{bmatrix}\begin{bmatrix} A_{j11}(\rho) \\ A_{j21}(\rho) \end{bmatrix}.$$

Then for A_{jk2}, $k=1,2$ we obtain the following formulas:

$$A_{j12}(\rho)=d_j^0\rho^{-\mu_j}\left\{b_1^+b_2^+\exp(-i\pi\mu_j)+b_1^-b_2^+\exp(-2i\rho a_1)+\right.$$
$$\left.+b_1^-b_2^-\exp(-i\pi\mu_j)\exp(-2i\rho(a_2-a_1))+b_1^+b_2^-\exp(-2i\rho a_2)\right\}[1],$$

$$A_{j22}(\rho)=d_j^0\rho^{-\mu_j}\left\{b_1^+b_2^++b_1^-b_2^+\exp(-i\pi\mu_j)\exp(2ia_1)+\right.$$
$$\left.+b_1^+b_2^-\exp(-i\pi\mu_j)\exp(-2i\rho a_2)+b_1^-b_2^-\exp(-2i\rho(a_2-a_1))\right\}[1].$$

By the similar way any $N\le p$ for A_{jkN}, $i,k=1,2$ we obtain the following formulas:

$$A_{j1N}(\rho)=d_j^0\rho^{-\mu_j}(i\rho)^m\left[B_N\exp(-i\pi\mu_j)+\sum_{l=1}^{N}\sum_{1\le s_1<s_2<\ldots<s_l\le N}B_{Nl}\exp\left(\sum_{k=1}^{l}(-1)^{l-k+1}2i\rho a_{s_k}-\right.\right.$$

$$\left.\left.-i\pi\mu_j\delta_{s_k}\right)\right]\exp(i\rho x)[1]_0,$$

$$A_{j2N}(\rho)=d_j^0\rho^{-\mu_j}(-i\rho)^m\left[B_N+\sum_{l=1}^{N}\sum_{1\le s_1<s_2<\ldots<s_l\le N}B_{Nl}\exp\left(\sum_{k=1}^{l}(-1)^{l-k}2i\rho a_{s_k}-\right.\right.$$

$$\left.\left.-i\pi\mu_j(1-\delta_{s_k})\right)\right]\exp(-i\rho x)[1]_0,$$

which with (32) and (25) gives (31).

Denote $\Delta(\lambda)=\varphi_2(T,\lambda)$. Function $\Delta(\lambda)$ is integer analytical by λ function and its zeros $\{\lambda_n\}_{n\ge 1}$ coincide with the eigen values of the boundary value problem (1)-(4). At that if λ_n is zero multiplicity χ_n, then functions

$$\varphi_{ns}(x)=\frac{\partial^s}{\partial\lambda^s}\varphi_2(x,\lambda)\Big|_{\lambda=\lambda_n},\quad s=\overline{0,\chi_n-1}$$

form the chain of eigen and adjoint functions for the eigen value λ_n. Function $\Delta(\lambda)$ is called a characteristic function of the problem L. By virtue of (31) the following asymptotic formula hold:

$$\Delta(\lambda)=\Delta_0(\rho)\left(1+O\left(\frac{1}{\rho}\right)\right),\quad |\rho|\to\infty, \tag{37}$$

where

$$\Delta_0(\rho)=d_2^0\rho^{-\mu_2}\left\{\left[B_p\exp(-i\pi\mu_2)+\sum_{l=1}^{p}\sum_{1\le s_1<s_2<\ldots<s_l\le p}B_{pl}\exp\left(\sum_{k=1}^{l}\left((-1)^{l-k+1}2i\rho a_{s_k}-i\pi\mu_2\delta_{s_k}\right)\right)\right]\times\right.$$

$$\left.\times\exp(i\rho T)[1]_0+\left[B_p+\sum_{l=1}^{p}\sum_{1\le s_1<s_2<\ldots<s_l\le p}B_{pl}\exp\left(\sum_{k=1}^{l}\left((-1)^{l-k}2i\rho a_{s_k}-i\pi\mu_2(1-\delta_{s_k})\right)\right)\right]\exp(-i\rho T)[1]_0\right\}$$

Consider the case when $\operatorname{Im}\rho\ge 0$. Then taking out the multiplier $\exp(-i\rho T)$ from the expressions for $\Delta_0(\rho)$, we obtain

$$\Delta_0(\rho)=d_2^0\rho^{-\mu_2}e^{-i\rho T}\cdot B_p(1+\aleph_1(\rho)), \tag{38}$$

where

$$\aleph_1(\rho)=\exp(-i\pi\mu_2+2i\rho T)+\sum_{l=1}^{p}\sum_{1\le s_1<\ldots<s_l\le N}B'_{Nl}(s_1,\ldots,s_j)\exp\left(\sum_{k=1}^{l}\left((-1)^{l-k+1}2i\rho a_{s_k}-i\pi\mu_2\delta_{s_k}\right)\right)\times$$

$$\times\exp(2i\rho T)+\sum_{l=1}^{p}\sum_{1\le s_1<s_2<\ldots s_l\le N}B'_{Nl}(s_1,\ldots,s_j)\exp\left(\sum_{k=1}^{l}\left((-1)^{l-k}2i\rho a_{s_k}-i\pi\mu_2(1-\delta_{s_k})\right)\right),$$

where $B'_{Nl}=B_{Nl}/B_p$.

Since included in expression $\aleph_1(\rho)$ terms $\sum_{k=1}^{l}\left((-1)^{l-k+1}a_{s_k}+T\right)$ and $\sum_{k=1}^{l}(-1)^{l-k}a_{s_k}$ are positive, then there exists such constant C, that $|\aleph_1(\rho)|\le C$. Analogously, in the case $\operatorname{Im}\rho\le 0$ we obtain, that

$$\Delta_0(\rho)=d_2^0\rho^{-\mu_2}e^{i\rho T}\cdot B_p\left(1+\aleph_2(\rho)\right),$$

where

$$\aleph_2(\rho)=\exp(-i\pi\mu_2)+\sum_{l=1}^{p}\sum_{1\le s_1<\ldots s_l\le N}B'_{pl}(s_1,\ldots,s_j)\exp\left(\sum_{k=1}^{l}\left((-1)^{l-k+1}2i\rho a_{s_k}-i\pi\mu_2\delta_{s_k}\right)\right)\times$$

$$+\left[\sum_{l=1}^{p}\sum B'_{pl}(s_1,\ldots,s_j)\exp\left(\sum_{k=1}^{l}\left((-1)^{l-k}2i\rho a_{s_k}-i\pi\mu_2\left(1-\delta_{s_k}\right)\right)\right)\right]\exp(-2i\rho T).$$

Since included in the expression $\aleph_2(\rho)$ terms $\sum_{k=1}^{l}(-1)^{l-k+1}a_{s_k}$ and $\sum_{k=1}^{l}(-1)^{l-k}\left(a_{s_k}-T\right)$ are negative, then there exists such constant C, that $|\aleph_2(\rho)|<C$.

By virtue of the SR condition $B_N\ne 0$. Then using known methods (see, for example, [16]) it can be stated, that characteristic function and its zeros have the following properties:

1) at $|\rho|\to\infty$
$$\Delta(\lambda)=O\left(|\rho|^{-\operatorname{Re}\nu-1/2}\exp\left(|\operatorname{Im}\rho|T\right)\right);$$
2) there exist $h>0$ and $C_h>0$ such, that
$$|\Delta(\lambda)|\ge C_h|\rho|^{-\operatorname{Re}\nu-1/2}\exp\left(|\operatorname{Im}\rho|T\right)$$
at $|\operatorname{Im}\rho|\ge h$; consequently, all the eigen values $\lambda_n=\rho_n^2$ of the boundary value problem L lie in the strip $|\operatorname{Im}\rho|<h$;
3) number N_ξ of zeros $\Delta(\lambda)$ in the rectangular $\Pi_\xi:=\left\{|\operatorname{Im}\rho|<h,\operatorname{Re}\rho\in[\xi,\xi+1]\right\}$ is bounded on ξ;
4) denote $G_\delta=\left\{\rho:|\rho-\rho_n|\ge\delta,\ \forall n\right\}$, then
$$|\Delta(\lambda)|\ge G_\delta\cdot|\rho|^{-\operatorname{Re}\nu-1/2}\exp\left(|\operatorname{Im}\rho|T\right),\ \rho\in G_\delta;\qquad(39)$$
5) there exist numbers $R_N\to\infty$ such that at sufficiently small $\delta>0$ circles $|\rho|=R_N$ lie in G_δ at all N;
6) let $\left\{\rho_n^0\right\}$ be zeros of the function $\Delta_0(\rho)$ of the form (38), then at $n\to\infty$
$$\rho_n=\rho_n^0+0(1).$$

4. Weyl solution. Weyl function. Let function $\Phi(x,\lambda)$ be the solution of equation (1) and satisfy the conditions $\Phi(x,\lambda)\sim c_{10}\cdot x^{\mu_1}$, $x\to 0,\Phi(T,\lambda)=0$ and also sewing conditions (4). We'll call the function $\Phi(x,\lambda)$ the Weyl solution for the boundary value problem L (by analogy with the Weyl solution for the classical Sturm-Liouville problem). Denote

$$M(\lambda) = -\frac{\delta(\lambda)}{\Delta(\lambda)}, \qquad (40)$$

where $\delta(\lambda) := \varphi_1(T,\lambda)$. It's clear, that

$$\Phi(x,\lambda) = \varphi_1(x,\lambda) + M(\lambda)\varphi_2(x,\lambda). \qquad (41)$$

We'll call the function $M(\lambda)$ Weyl function for L. Weyl solution and Weyl function all meromorphic by λ functions with poles on spectrum of the problem L. From (40) and (41) it follows that

$$\Phi(x,\lambda) = -\frac{\Psi(x,\lambda)}{\Delta(\lambda)}, \qquad (42)$$

where

$$\Psi(x,\lambda) = \varphi_1(T,\lambda)\varphi_2(x,\lambda) - \varphi_2(T,\lambda)\varphi_1(x,\lambda). \qquad (43)$$

Function $\Psi(x,\lambda)$ is integer by λ solution of equation (1), satisfying the conditions $\Psi(T,\lambda)=0$, $\Psi'(T,\lambda)=\det A_N$ and also splice conditions (4). Note, that by virtue of (29), (41) and (42)

$$<\Phi(x,\lambda),\varphi_2(x,\lambda)> \equiv \begin{cases} 1, & x < a_1 \\ \prod_{k=1}^{N-1}\det A_k, & a_{N-1} < x < a_N,\ N = \overline{2, p+1}, \end{cases}$$

$$<\varphi_2(x,\lambda),\Psi(x,\lambda)> \equiv \Delta(\lambda)\cdot \begin{cases} 1, & x < a_1 \\ \prod_{k=1}^{N}\det A_k, & a_{N-1} < x < a_N,\ N = \overline{2, p+1}, \end{cases} \qquad (44)$$

where $< y, z > := yz' - y'z$.

Lemma 4. *At* $|\rho| \to \infty$, $|\rho|x \ge 1$, $x \in (0,T)$, $j \in (0,T)$, $j = 1,2$, $m = 0,1$ *the following asymptotic formulas hold:*

$$\Psi^{(m)}(x,\rho) = \frac{\det A_N}{2i\rho}\left(\exp(-i\rho(T-x))[1]_0 - \exp(i\rho(T-x))[1]_0\right),\ x > a_p, \qquad (45)$$

$$\Psi^{(m)}(x,\rho) = \frac{1}{\prod_{k=1}^{N-1}\det A_k}\frac{B_N}{2i\rho}\left(\exp(-i\rho(T-x))[1]_0 - \exp(i\rho(T-x))[1]_0\right) +$$

$$+\frac{1}{\prod_{k=1}^{N-1}\det A_k}\cdot\frac{1}{2i\rho}\sum_{l=1}^{N}\sum_{1\le s_1<s_2<\ldots<s_l\le N} B_{Nl}\left[\exp\left(\sum_{k=1}^{l}(-1)^{l-k}\left(T-2a_{s_k}\right) - i\rho x\right)[1]_0 - \right.$$

$$\left. - \exp\left(\sum_{k=1}^{l}(-1)^{l-k}\left(-2a_{s_k} - T\right) - i\rho x\right)[1]_0\right],\ a_N < x < a_{N+1}. \qquad (46)$$

Proof. We'll expand $\Psi(x,\lambda)$ by the fundamental system $\{Y_k(x,\rho)\}_{k=1,2}$ separately at $a_{N-1} < x < a_N$, $a_{p+1} = T$

$$\Psi(x,\rho) = \sum_{k=1}^{2} A_{kN-1}(\rho) Y_k(x,\rho), \qquad a_{N-1} < x < a_{N+1}. \qquad (47)$$

Using initial conditions $\Psi(T,\lambda)=0$, $\Psi'(T,\lambda)=\det A_N$, we calculate A_{kN}:

$$A_{1N}(\rho)=-\frac{\det A_N}{w(\rho)}Y_2(T,\rho),\qquad A_{2N}(\rho)=-\frac{\det A_N}{w(\rho)}Y_1(T,\rho),$$
$$w(\rho)=\det\left[Y_k^{(m-1)}(T,\rho)\right]_{k,m=1,2}. \tag{48}$$

Further, using sewing condition (4), as in the proof of lemma 3, we obtain

$$\begin{bmatrix}A_{1N}(\rho)\\A_{2N}(\rho)\end{bmatrix}=\begin{bmatrix}\eta_{11}^{(N)}(\rho) & \eta_{12}^{(N)}(\rho)\\ \eta_{21}^{(N)}(\rho) & \eta_{22}^{(N)}(\rho)\end{bmatrix}\begin{bmatrix}A_{1N-1}\\A_{2N-2}\end{bmatrix}, \tag{49}$$

where functions $\eta_{s,k}^{(N)}(\rho)$ are calculated by formula (35). Since $\det\left[\eta_{sk}^{(N)}(\rho)\right]_{s,k=1,2}\equiv$ $\equiv\det A_N$ then from (49) it follows

$$\begin{bmatrix}A_{1N-1}(\rho)\\A_{2N-1}(\rho)\end{bmatrix}=\frac{1}{\det A_N}\begin{bmatrix}\eta_{22}^{(N)}(\rho) & -\eta_{12}^{(N)}(\rho)\\ -\eta_{21}^{(N)}(\rho) & \eta_{11}^{(N)}(\rho)\end{bmatrix}\begin{bmatrix}A_{1N}(\rho)\\A_{2N}(\rho)\end{bmatrix}. \tag{50}$$

Let for definiteness $\rho\in\overline{S}_0$. Then $R_k=\varepsilon_k$. Substituting asymptotic formulas (25) and (36) into (48) and (50), we obtain

$$A_{1N}(\rho)=\frac{\det A_N}{2i\rho}\exp(-i\rho T)[1],\quad A_{2N}(\rho)=-\frac{\det A_N}{2i\rho}\exp(-i\rho T)[1],$$
$$A_{1N-1}(\rho)=\frac{1}{2i\rho}\left(b_N^+\exp(-i\rho T)+b_N^-\exp(i\rho(T-2a_N))\right)[1],$$
$$A_{2N-1}(\rho)=-\frac{1}{2i\rho}\left(b_N^-\exp(-i\rho(2a_N-T))+b_N^+\exp(i\rho T)\right)[1].$$

In the case $a_{N-2}<x<a_{N-1}$ for $A_{k,N-2}$, $k=1,2$ analogously we obtain that

$$A_{1N-2}(\rho)=\frac{1}{\det A_{N-1}}\cdot\frac{1}{2i\rho}\left\{b_N^+b_{N-1}^+\exp(-i\rho T)+b_N^-b_{N-1}^+\exp(i\rho(T-2a_N))+\right.$$
$$\left.+b_N^-b_{N-1}^-\exp(-i\rho(2a_N-2a_{N-1}-T))+b_N^+b_{N-1}^-\exp(i\rho(T-2a_{N-1}))\right\}[1],$$
$$A_{2N-2}(\rho)=\frac{-1}{\det A_{N-1}}\cdot\frac{1}{2i\rho}\left\{b_N^+b_{N-1}^+\exp(-i\rho T)+b_N^-b_{N-1}^+\exp(i\rho(2a_N-T))+\right.$$
$$\left.+b_N^-b_{N-1}^-\exp(-i\rho(2a_N-2a_{N-1}-T))+b_N^+b_{N-1}^-\exp(i\rho(2a_{N-1}-T))\right\}[1].$$

Using the mathematical induction method we obtain that

$$A_{11}(\rho)=\frac{1}{\prod\limits_{k=1}^{N-1}\det A_k}\cdot\frac{1}{2i\rho}\left\{B_N\exp(-i\rho T)+\sum_{l=1}^{N}\sum_{1\le s_1<s_2<\ldots<s_l\le N}B_{Nl}\exp\left(\sum_{k=1}^{l}(-1)^{l-k}\left(2a_{s_k}-T\right)\right)\right\}[1]$$

$$A_{21}(\rho)=\frac{-1}{\prod\limits_{k=1}^{N-1}\det A_k}\cdot\frac{1}{2i\rho}\left\{B_N\exp(-i\rho T)+\sum_{l=1}^{N}\sum_{1\le s_1<s_2<\ldots<s_l\le N}B_{Nl}\exp\left(\sum_{k=1}^{l}(-1)^{l-k}\left(T-2a_{s_k}\right)\right)\right\}[1]$$

which together with (47) reduces to (45) and (46).

Corollary 1. *At* $\rho\in G_\delta$ *the estimations*

$$\left|\Phi^{(m)}(x,\lambda)\right|\le C_\delta|\rho|^{m+\operatorname{Re}\nu-1/2}\exp(-|\operatorname{Im}\rho|x),\quad m=0,1,\ |\rho|x\ge1, \tag{51}$$

$$|M(\lambda)|\le C_\delta\cdot|\rho|^{2\nu} \tag{52}$$

hold.

Really, estimation (51) follows from (42), (39) and lemma 4, but estimation (52)- from (40), (39) and lemma 3.

5. Completion theorem. Let α be real number and $1 \le \rho < \infty$. Consider Banach spaces $\Phi_{\alpha,p} = \{f(x) : f(x)x^{-\alpha} \in L_p(0,T)\}$ with norm $\|f\|_{\alpha,p} = \|f(x)\cdot x^{-\alpha}\|_p$, where $\|\cdot\|$ is a norm in space $L_p(0,T)$. By $\Phi^*_{\alpha,p}$ we denote the conjugate space. It's clear, that $\Phi^*_{\alpha,p} = \Phi_{-\alpha,q}$ $(p^{-1} + q^{-1} = 1,\ p > 1)$. We'll show, that

$$\Phi_{\alpha,p} \subseteq \Phi_{\beta,s}\ , \quad 1 \le s \le p < \infty, \quad \beta - \alpha < s^{-1} - p^{-1} \tag{53}$$

(symbol $\subseteq$ denotes the dense inclusion [17]). Really, at $\alpha \ge \beta,\ s \le p$ we have $\Phi_{\alpha,p} \subseteq \Phi_{\beta,p}$, $\Phi_{\beta,p} \subseteq \Phi_{\beta,s}$, and (53) is evident. Suppose now, that $\alpha < \beta,\ s < p$. Consider function $f(x) \in \Phi_{\alpha,p}$. Assume $r = p/s,\ r' = p/(p-s)$. Then $r^{-1} + (r')^{-1} = 1$. Since $\beta - \alpha < s^{-1} - p^{-1}$, then $(\alpha - \beta)sr' > -1$. Using Hölder inequality, we obtain

$$\left\|f(x)x^{-\beta}\right\|_s \le \left\|f(x)x^{-\alpha}\right\|_{sr} \cdot \left\|x^{\alpha-\beta}\right\|_{sr'}$$

and, consequently, $\|f\|_{\beta,s} \le c \cdot \|f\|_{\alpha,p}$. Since $\Phi_{\alpha,p}$ is dense in $\Phi_{\beta,s}$, then we obtain (53). From (53), in particular, it follows that

$$\Phi_{\alpha,p} \subseteq L_s\,,\ 1 \le s \le p < \infty\,, \quad \alpha > p^{-1} - s^{-1}\,.$$

Denote $\omega = \operatorname{Re}\nu + \frac{1}{2}$. The following completion theorem is valid.

Theorem 1. *System of eigen and adjoint functions of the boundary value problem L is complete in spaces $\Phi_{\beta,s}$ at $1 \le s < \infty,\ \beta < \omega + \frac{1}{s}$.*

Proof. For brevity we confine ourselves to the case of simple spectrum, i.e. the case, when characteristic function $\Delta(\lambda)$ has only simple roots. The general case is considered analogously. Since eigen values $\{\lambda_n\}_{n\ge1}$ of the problem L are zeros of the characteristic function $\Delta(\lambda)$, then by virtue of (44) and sewing conditions (4) we have

$$\Psi(x,\lambda_n) = \beta_n \varphi_2(x,\lambda_n)\,, \qquad \beta_n \ne 0\,. \tag{54}$$

Functions $\varphi_2(x,\lambda_n)$ and $\Psi(x,\lambda_n)$ are eigen functions of the problem L for eigen values λ_n.

Let function $f(x),\ x \in (0,T)$ be such that

$$f(x)x^{\omega} \in L(0,T),\ \int_0^T \varphi_2(x,\lambda_n) f(x)dx = 0\,, \quad n \ge 1\,. \tag{55}$$

Consider the function

$$Y(x,\lambda) = \frac{1}{\Delta(\lambda)}\left(\Psi^*(x,\lambda)\int_0^x \varphi_2(t,\lambda)f(t)dt + \varphi_2^*(x,\lambda)\int_x^T \Psi(t,\lambda)f(t)dt\right), \tag{56}$$

where

$$\Psi^*(x,\lambda) = \frac{\Psi(x,\lambda)}{\eta(x)},\quad \varphi_2^*(x,\lambda) = \frac{\varphi_2(x,\lambda)}{\eta(x)},\quad \eta(x) = \begin{cases} 1, & x < a_1\,; \\ \prod_{k=1}^{N} \det A_k\,, & a_N < x < a_{N+1}\,. \end{cases} \tag{57}$$

Since by virtue of (44)

$$\varphi_2(x,\lambda)\Psi^{*'}(x,\lambda)-\varphi_2^{*'}(x,\lambda)\Psi(x,\lambda)\equiv\Delta(\lambda),$$

then by direct calculation we are convinced, that function $Y(x,\lambda)$ satisfies the differential equation

$$lY(x,\lambda)-\lambda Y(x,\lambda)=f(x) \tag{58}$$

separately at $x<a_1$ and at each $a_{N-1}<x<a_N$, $N=\overline{2,n}$. Using (54)-(57), we calculate the residue of function $Y(x,\lambda)$ at the points of spectrum $\lambda=\lambda_n$:

$$\operatorname*{Res}_{\lambda=\lambda_n}Y(x,\lambda)=\frac{1}{\dot\Delta(\lambda_n)}\left(\Psi^*(x,\lambda_n)\int_0^x\varphi_2(t,\lambda_n)f(t)dt+\varphi_2^*(x,\lambda_n)\int_x^T\Psi(t,\lambda_n)f(t)dt\right)=$$

$$=\frac{\beta_n}{\dot\Delta(\lambda_n)}\varphi_2^*(x,\lambda_n)\int_0^T\varphi_2(t,\lambda_n)f(t)dt,$$

where $\dot\Delta(\lambda)=\frac{d}{d\lambda}\Delta(\lambda)$. According to (55) we have

$$\operatorname*{Res}_{\lambda=\lambda_n}Y(x,\lambda)=0.$$

So, at each fixed $x\in(0,T)$ function $Y(x,\lambda)$ is an integer analytical by λ.

On the other hand, using results of p.3, 4, we estimate function $Y(x,\lambda)$ in domain G_δ. We fix $x\in(0,T)$. Then $|\rho|x\ge1$ at sufficiently large ρ. By virtue of (30), (31) and (45), (46) we have

$$\left|\varphi_2^{(m)}(x,\lambda)\right|\le C|\rho|^{m-\omega}\exp(|\operatorname{Im}\rho|x),\qquad |\rho|x\ge1,$$

$$\left|\Psi^{(m)}(x,\lambda)\right|\le C|\rho|^{m-1}\exp(|\operatorname{Im}\rho|(T-x)),\qquad |\rho|x\ge1. \tag{59}$$

Consequently, subject to (39) and (57) we obtain the estimate

$$|Y(x,\lambda)|\le C_\delta\left(|\rho|^{\omega-1}\exp(-|\operatorname{Im}\rho|x)\int_0^x|\varphi_2(t,\lambda)f(t)|dt+\right.$$

$$\left.+\exp(-|\operatorname{Im}\rho|(T-x))\int_x^T|\Psi(t,\lambda)f(t)|dt\right). \tag{60}$$

Further, by virtue (17) and (27)

$$|\varphi_2(t,\lambda)|\le Ct^\omega,\qquad t\le\frac{1}{|\rho|}, \tag{61}$$

and consequently,

$$\int_0^{1/|\rho|}|\varphi_2(t,\lambda)f(t)|dt\le C\int_0^{1/|\rho|}t^\omega|f(t)|dt.$$

Using (59), we have

$$\int_{1/|\rho|}^x|\varphi_2(t,\lambda)f(t)|\le C|\rho|^{-\omega}\int_{1/|\rho|}^x\exp(|\operatorname{Im}\rho|f(t))dt\le$$

$$\le C|\rho|^{-\omega}\exp(|\operatorname{Im}\rho|x)\int_{1/|\rho|}^x|f(t)|dt\le C\exp(|\operatorname{Im}\rho|x)\int_{1/|\rho|}^x t^\omega(f(t))dt,$$

and also

$$\int_x^T |\Psi(t,\lambda) f(t)| \le C|\rho|^{-1}\int_x^T \exp(|\operatorname{Im}\rho|(T-t))|f(t)|dt \le$$
$$\le C\exp(|\operatorname{Im}\rho|(T-x))|\rho|^{\omega-1}\int_x^T t^{\omega}|f(t)|dt .$$

Substituting into (60), we get estimation

$$|Y(x,\lambda)| \le C_\delta |\rho|^{\omega-1} , \qquad \rho \in G_\delta .$$

Since at fixed x function $Y(x,\lambda)$ is an integer by λ, then from the last estimation it follows, that $Y(x,\lambda)$ is a polynomial by λ, which together with (58) gives $Y(x,\lambda)\equiv 0$ and $f(x)=0$ a.e. on $(0,T)$.

So, we proved that at each p $(1 \le p < \infty)$ system of functions $\{\varphi_2(x,\lambda_n)\}_{n\ge 1}$ is complete in $\Phi_{\omega,p}$. Since by condition $\beta-\omega < 1/s$, then $\beta - \omega < 1/s - 1/p$ at sufficiently large p, and according to (53) $\Phi_{\omega,p} \subseteq \Phi_{\beta,s}$. Consequently, system of functions $\{\varphi_2(x,\lambda_n)\}_{n\ge 1}$ is complete in $\Phi_{\beta,s}$. Theorem 1 is proved.

Corollary 2. *System of eigen and adjoint functions of the problem L is complete in $L_s(0,T)$ at $1 \le s < \infty$.*

We'll give the contrary instance showing the essentiality of the SR condition $b^+ \ne 0$. Consider the boundary value L at $\nu_0 = 0, q(x) \equiv 0, T = \pi, a_1 = 3\pi/4$ (in case $n = 1$), $a_{11} = -a_{22} = 1, a_{12} = a_{21} = 0$, i.e. consider the problem

$$-y'' = \lambda y , \ 0 < x < \pi ,$$
$$y(0) = y(\pi) = 0 ,$$
$$y^{(m)}(a+0) = (-1)^m y^{(m)}(a-0), \ \ m = 0,1 , \quad a_1 = \frac{3}{4}\pi . \tag{62}$$

For this problem $b^+ = 0$, i.e. SR condition is violated. Characteristic function of the problem (62) has the form

$$\Delta(\lambda) = \frac{\sin\rho(2a-\pi)}{\rho} .$$

Eigen values $\lambda_n = \rho_n^2$ are $\rho_n = 2n, n \ge 1$, and eigen functions have the form

$$y_n(x) = \begin{cases} \sin 2nx, & x \le \dfrac{3\pi}{4}; \\ (-1)^{n-1}\sin 2nx, & x \ge \dfrac{3}{4}\pi . \end{cases}$$

System of functions $\{y_n(x)\}_{n\ge 1}$ is incomplete in $\Phi_{\beta,s}$ at $1 \le s < \infty$, $\beta < 1 + \dfrac{1}{s}$.

6. Inverse problem. In this point we study the inverse problem of restoration of the boundary value problem of the form (1)-(4) by the data of its spectral characteristics. We'll consider three statements of the inverse problems of the boundary value problem L by Weyl function and discrete spectral data. These inverse problems are generalization of the known inverse problems for Sturm-Liouville operator (see [3,4]).

Let's formulate the uniqueness theorem of solution of the inverse problem by Weyl function. For this along with L consider the boundary value problem $\tilde{L}$ of the same form, but with other potential $\tilde{v}_0/x^2+\tilde{q}(x)$. We'll stipulate, that if some symbol α denotes the object related to the problem L, then $\tilde{\alpha}$ will denote the object, related to the problem $\tilde{L}$.

Theorem 2. *If* $M(\lambda)=\tilde{M}(\lambda)$, *then* $L=\tilde{L}$. *So, representation of Weyl function uniquely defines the boundary problem* L.

Proof. Consider functions

$$P_m(x,\lambda)=\Phi(x,\lambda)\tilde{\varphi}_2^{(m)}(x,\lambda)-\varphi_2(x,\lambda)\tilde{\Phi}^{(m)}(x,\lambda),\quad m=0,1. \tag{63}$$

At each fixed $x\in(0,T]$ functions $P_m(x,\lambda)$ are memorphic by λ with poles at points $\lambda=\lambda_n$ and $\lambda=\tilde{\lambda}_n$. We'll fix $x\in(0,T]$. Then $|\rho|x\geq 1$ at sufficiently large ρ. Denote $G_\delta^0=G_\delta\cap\tilde{G}_\delta$. By virtue of (51) and (59) we have

$$|P_0(x,\lambda)|\leq\frac{C_\delta}{|\rho|},\quad |P_1(x,\lambda)|\leq C_\delta,\quad \rho\in G_\delta. \tag{64}$$

Substituting (41) into (63), we calculate

$$P_m(x,\lambda)=\varphi_1(x,\lambda)\tilde{\varphi}_2^{(m)}(x,\lambda)-\varphi_2(x,\lambda)\tilde{\varphi}_1^{(m)}(x,\lambda)+\left(M(\lambda)-\tilde{M}(\lambda)\right)\varphi_2(x,\lambda)\tilde{\varphi}_2^{(m)}(x,\lambda). \tag{65}$$

Since by condition $M(\lambda)=\tilde{M}(\lambda)$, then from (65) it follows, that at each fixed $x\in(0,T]$ functions $P_m(x,\lambda)$ are integer analytical by λ. Together with (64) it gives

$$P_0(x,\lambda)\equiv 0,\ P_1(x,\lambda)\equiv P(x).$$

But then

$$\Phi(x,\lambda)\tilde{\varphi}_2(x,\lambda)=\varphi_2(x,\lambda)\tilde{\Phi}(x,\lambda),$$

$$P(x)\tilde{\varphi}_2(x,\lambda)=\left(\Phi(x,\lambda)\tilde{\varphi}_2'(x,\lambda)-\varphi_2(x,\lambda)\tilde{\Phi}'(x,\lambda)\right)\tilde{\varphi}_2(x,\lambda)=$$
$$=\left(\tilde{\Phi}(x,\lambda)\tilde{\varphi}_2'(x,\lambda)-\tilde{\varphi}_2(x,\lambda)\tilde{\Phi}'(x,\lambda)\right)\varphi_2(x,\lambda)=\tilde{\eta}(x)\varphi_2(x,\lambda).$$

Analogously

$$P(x)\tilde{\Phi}(x,\lambda)=\tilde{\eta}(x)\Phi(x,\lambda).$$

So,

$$\frac{\varphi_2(x,\lambda)}{\tilde{\varphi}_2(x,\lambda)}=\frac{\Phi(x,\lambda)}{\tilde{\Phi}(x,\lambda)}=\frac{P(x)}{\tilde{\eta}(x)}. \tag{66}$$

Further from (30), (31) it follows that at $|\rho|\to\infty$, $\arg\rho\in[\varepsilon,\pi-\varepsilon]$, $\varepsilon>0$,

$$\varphi_2(x,\lambda)=d_2^0\rho^{-\omega}B_N\exp(-i\rho x)[1],\qquad a_N<x<a_{N+1},\ N=\overline{1,p}. \tag{67}$$

Analogously, using (42), (45), (46) and (37), (38) we obtain that at $|\rho|\to\infty$, $\arg\rho\in[\varepsilon,\pi-\varepsilon]$, $\varepsilon>0$

$$\Phi(x,\lambda)=\frac{\prod_{k=1}^{N}\det A_k\cdot\rho^{\omega-1}}{2id_2^0B_N}\exp(i\rho x)[1],\qquad a_N<x<a_{N+1},\ N=\overline{0,p}. \tag{68}$$

Substituting now (67) and (68) separately on each interval (a_N,a_{N+1}) into (66), we obtain $\omega=\tilde{\omega}$,

$$\frac{P(x)}{\det\tilde{A}_N}\equiv P_{N*}=const,$$

i.e.

$$\varphi_2(x,\lambda) \equiv P_{N*}\tilde{\varphi}_2(x,\lambda),\ \Phi(x,\lambda) \equiv P_{N*}\tilde{\Phi}(x,\lambda),\ a_N < x < a_{N+1}.$$

Consequently, $q(x) = \tilde{q}(x)$ a.e. at each $x \in (a_N, a_{N+1})$, i.e. a.e. at $(0,T)$. Theorem 2 is proved.

Consider now the inverse problem of restoration of L by discrete spectral characteristics. For brevity we confine ourselves to the case of simple spectrum. Denote

$$\alpha_n = \int_0^T \varphi_2(x,\lambda_n)\varphi_2^*(x,\lambda_n)dx.$$

We'll call set of numbers $\{\lambda_n, \alpha_n\}_{n\ge 1}$ the spectral data of the boundary problem L. Inverse problem is stated in the following form: by given spectral data to construct the problem L. We'll prove the uniqueness theorem of restoration of L by the spectral data.

Theorem 3. *If $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$ at all $n \ge 1$, then $L = \tilde{L}$. So, representation of spectral data uniquely defines the problem L.*

Proof. We'll show that the following correlation is valid

$$\alpha_n = \frac{\dot{\Delta}(\lambda_n)}{\beta_n}, \tag{69}$$

where numbers β_n are defined from correlation (54). Really, from correlations

$$-\varphi_2''(x,\lambda) + \left(\frac{\nu_0}{x^2} + q(x)\right)\varphi_2(x,\lambda) = \lambda\,\varphi_2(x,\lambda),$$

$$-\varphi_2^{*''}(x,\lambda) + \left(\frac{\nu_0}{x^2} + q(x)\right)\varphi_2^*(x,\lambda_n) = \lambda_n\,\varphi_2^*(x,\lambda_n)$$

we have

$$\frac{d}{dx} < \varphi_2(x,\lambda), \varphi_2^*(x,\lambda_n) >= (\lambda - \lambda_n)\varphi_2(x,\lambda)\varphi_2^*(x,\lambda_n)$$

separately on each interval, $a_{N-1} < x < a_N$. Consequently,

$$(\lambda - \lambda_n)\int_0^T \varphi_2(x,\lambda)\varphi_2^*(x,\lambda_n)dx = \sum_{k=0}^{n} < \varphi_2(x,\lambda), \varphi_2^*(x,\lambda_n) > \Big|_{a_k}^{a_{k+1}},$$

where $a_0 = 0$, a_{n+1}.

Using (61) and conditions (4), we obtain

$$(\lambda - \lambda_n)\int_0^T \varphi_2(x,\lambda)\varphi_2^*(x,\lambda_n)dx = \varphi_2(T,\lambda), \varphi_2^{*'}(T,\lambda_n)$$

and, consequently,

$$\int_0^T \varphi_2(x,\lambda)\varphi_2^*(x,\lambda_n)dx = \dot{\Delta}(\lambda_n)\varphi_2^{*'}(T,\lambda_n). \tag{70}$$

According to (54) and (57) $\Psi^{*'}(T,\lambda_n) = \beta_n\varphi_2^{*'}(T,\lambda_n)$, i.e. $\varphi_2^{*'}(T,\lambda_n) = \dfrac{1}{\beta_n}$ which together with (70) gives (69).

Further, by virtue (40)

$$\operatorname*{Res}_{\lambda\text{-}\lambda_n} M(\lambda) = -\frac{\delta(\lambda_n)}{\dot{\Delta}(\lambda_n)} = -\frac{\varphi_1(T,\lambda_n)}{\dot{\Delta}(\lambda_n)}.$$

According to (43) $\Psi(x,\lambda_n)=\varphi_1(T,\lambda_n)\varphi_2(x,\lambda_n)$. Comparing with (54), we obtain $\beta_n=\varphi_1(T,\lambda_n)$ and consequently,

$$\operatorname*{Res}_{\lambda\text{-}\lambda_n} M(\lambda)=-\frac{\beta_n}{\dot{\Delta}(\lambda_n)}=-\frac{1}{\alpha_n}. \tag{71}$$

Now let $\lambda=\tilde{\lambda}_n$, $\alpha=\tilde{\alpha}_n$ at all $n\geq 1$. We define functions $P_m(x,\lambda)$ by the formula (63). By virtue of (65) and (71) and functions $P_m(x,\lambda)$ are integer by λ at each fixed $x\in(0,T]$. Repeating now reasoning given at the proof of theorem 2, we obtain $L=\tilde{L}$.

Theorem 4. *If* $\lambda_{n1}=\tilde{\lambda}_{n1}$, $\lambda_{n2}=\tilde{\lambda}_{n2}$ *at all* $n\geq 0$, *then* $L=\tilde{L}$. *So, the problem* L *is uniquely defined by given two spectrums.*

Proof. Since $\Delta(\lambda)$ and $\tilde{\Delta}(\lambda)$ are integer analytical functions of half order, then accurate to constant multipliers they are defined by their zeros. Consequently,

$$\Delta(\lambda)=C_1\prod_{k=0}^{\infty}\left(1-\frac{\lambda}{\lambda_{k_1}}\right),\quad \tilde{\Delta}(\lambda)=\tilde{C}_1\prod_{k=0}^{\infty}\left(1-\frac{\lambda}{\tilde{\lambda}_k}\right),$$

where C_1, $\tilde{C}_1$ are constant values.

From condition $\lambda_{n1}=\tilde{\lambda}_{n1}$ it follows that $\Delta(\lambda)\equiv c\cdot\tilde{\Delta}(\lambda)$. On the other hand, from formula (37) for $\Delta(\lambda)$ we obtain that

$$\frac{\Delta(\lambda)}{\tilde{\Delta}(\lambda)}=\frac{d_2^0\cdot\rho^{-\mu_2}}{\tilde{d}_2^0\cdot\rho^{-\tilde{\mu}_2}}\cdot\left(1+O\left(\frac{1}{\rho}\right)\right)=\frac{d_2^0}{\tilde{d}_2^0}\cdot\rho^{\tilde{\mu}_2-\mu_2}\left(1+O\left(\frac{1}{\rho}\right)\right)\equiv C,\ \operatorname{Im}\rho\geq 0,\ |\rho|\to\infty.$$

From here we drive, that $\mu_2=\tilde{\mu}_2$.

By analogous reasonings, using expression $\delta(\lambda)$ we obtain, that $\mu_1=\tilde{\mu}_1$. Since

$$c_{10}\cdot c_{20}=\frac{1}{2\nu}=\frac{1}{\mu_2-\mu_1}=\frac{1}{\tilde{\mu}_2-\tilde{\mu}_1}=\frac{1}{2\tilde{\nu}},$$

and one of numbers c_{10}, c_{20} can be arbitrarily chosen, then we obtain that one of numbers β_{kj}^0 can be arbitrarily chosen. Without losing of generality assume $\beta_{11}^0=\tilde{\beta}_{11}^0=1$. Then

$$\beta_{21}^0=\beta_{11}^0\exp(i\pi\mu_1)=\tilde{\beta}_{11}^0\exp(i\pi\tilde{\mu}_1)=\tilde{\beta}_{21}^0,$$

$$\beta_{12}^0=\frac{1}{\beta_{11}^0}\cdot\frac{i}{\sin\pi\nu}=\frac{1}{\tilde{\beta}_{11}^0}\cdot\frac{i}{\sin\pi\tilde{\nu}}=\tilde{\beta}_{12}^0,$$

$$\beta_{22}^0=\beta_{12}^0\exp(i\pi\mu_2)=\tilde{\beta}_{12}^0\exp(i\pi\tilde{\mu}_2)=\tilde{\beta}_{22}^0.$$

On the other hand

$$\frac{\delta(\lambda)}{\Delta(\lambda)}=\frac{d_1^0}{d_2^0}\cdot\rho^{\mu_2-\mu_1}=\frac{\beta_{22}^0}{\beta_{21}^0}\cdot e^{-i\pi(\mu_2-\mu_1)}=\frac{\tilde{\beta}_{22}^0}{\tilde{\beta}_{21}^0}\cdot e^{-i\pi(\tilde{\mu}_2-\tilde{\mu}_1)}=\frac{\tilde{\delta}(\lambda)}{\tilde{\Delta}(\lambda)},$$

i.e. $M(\lambda)=\tilde{M}(\lambda)$. Then from theorem 2 we obtain that $L=\tilde{L}$. Consequently $q(x)=\tilde{q}(x)$, a.e. at $(0,T)$.

References

[1]. Litvinenko O.I., Soshnikov V.I. *Theory of non-homogeneous transmission lines and its applications in radio electronics.* M., "Radio", 1964, 536p. (in Russian)

[2]. Lapwood F.R., Usami T. *Free oscillations of the Earth.* Cambridge: Cambridge Univ. Press., 1981, 231p.
[3]. Levitan B.M., Sargsyan Ch.S. *Sturm-Liouville and Dirac operators.* M., “Nauka”, 1988, 432p. (in Russian)
[4]. Marchenko V.A. *Sturm-Liouville’s operators and their applications.* Kiev, “Naukova Dumka”, 1991, 232p. (in Russian)
[5]. Berezansky Yu.M. *The uniqueness theorem in the inverse spectral problem for Shrödinger equation.* Pr. Mosc. Math. Soc., 1958(7), p.3-51. (in Russian)
[6]. Nijnik L.P. *The inverse scattering problems for hyperbolic equations.* Kiev, “Naukova Dumka”, 1977, 329p. (in Russian)
[7]. Hald O.H. *Discontinuous inverse eigen-value problems.* Comm.Pure and Appl. Math., 1984(37), p.539-577.
[8]. Shepelsky D. *The inverse problem of reconstruction of the medium’s conductivity in a class of discontinuous and increasing functions.* Spectral operator theory and related topics: Adv. in Sov. Math., Providence; Amer. Math.Soc., 1994(19), p.209-232.
[9]. Kobayashi M. *A uniqueness for discontinuous inverse Sturm-Liouville problems with symmetric potentials.* Inverse Problems, 1985(5), p.767-781.
[10]. Gasymov M.G., Amirov R.Kh. *Direct and inverse problems for the second order differential operators in Coulomb singularity.* DAN AzSSR, 1985, v.41, №8. (in Russian)
[11]. Gasymov M.G. *The inverse problem by two spectrums for Sturm-Liouville equation with singularity.* DAN SSSR, 1965, v.161, №2. (in Russian)
[12]. Yurko V.A. *Integral transforms connected with differential operators having singularities inside the interval.* Integral Transforms and Special Functions, 1977, v.5, №3, p.309-322.
[13]. Yurko V.A. *On higher-order differential operators with a singular point.* Inverse Problems, 1993(9), p.495-502.
[14]. YurkoV.A. *Restoration of differential Sturm-Liouville operators with singularities inside the interval.* “Mat.zametki”, 1998, v.64, №1, p.143-156. (in Russian)
[15]. Yurko V.A. *The inverse problem for differential equations with singularity.* Diff.eq., 1992, v.28, №8, p.1355-1362. (in Russian)
[16]. Bellman R., Cooke K. *Differential-difference equations.* New-York, Acad.Press, 1963, 462p.
[17]. Kreyn S.G., Petunin Yu.I., Semenov E.M. *the interpolation of linear operators.* M., “Nauka”, 1982, 400p. (in Russian)
[18]. Amirov R.Kh., Yurko V.A. *On differential operators with singularity and discontinuity conditions inside the interval.* Ukr.math.jour., 2001, v.53, №11, p.1443-1458. (in Russian)

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