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THE ASYMPTOTIC BEHAVIOR OF EIGENVALUES OF ONE BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL-OPERATOR EQUATION OF THE SECOND ORDER WITH DISCONTINUOUS COEFFICIENT

Abstract

At the paper the asymptotic formulas are got for eigenvalues of one boundary value problem for differential-operator equation of the second order with discontinuous coefficient.

Let H be a separable Hilbert space. Let's consider in H the equation

$$-a(x)u''(x)-Au(x)-\lambda u(x)=0, \quad x\in\Omega=[0,b)\cup(b,1] \tag{1}$$

with the boundary conditions

$$L_1u\equiv\alpha_1u(0)+\alpha_2u'(0)=0,$$
$$L_2u\equiv\beta_1u(1)+\beta_2u'(1)=0, \tag{2}$$

where A is a self-adjoint positive definite operator in $H\left(A=A^*\geq C^2I\right)$ with the determination domain $D(A)$, λ is a spectral parameter

$$a(x)=\begin{cases}a_1>0, & x\in[0,b),\ b\in(0,1),\\ a_2>0, & x\in(b,1],\ a_1\neq a_2,\end{cases}$$

α_i, β_i $(i=1,2)$ are real numbers, moreover $\alpha_2, \beta_1\neq 0$. Let's superpose at the point $x=b$ the additional condition (conjugation condition) on the function $u(x)$

$$L_3u\equiv\delta_1u(b-0)+\delta_2u(b+0)=0,$$
$$L_4u\equiv\gamma_1u'(b-0)+\gamma_2u'(b+0)=0, \tag{3}$$

where $u(b-0)$ and $u(b+0)$ are left and right limit values $u(x)$ at the point $x=b$, δ_i, γ_i $(i=1,2)$ are real numbers δ_1, $\gamma_2\neq 0$.

Let's denote by $L_2((0,1);H)$ a set of all vector-functions $x\to u(x):(0,1)\to H$ strongly measurable and such that $\int_0^1\|u(x)\|_H^2dx<\infty$. As known $L_2((0,1);H)$ is Hilbert space with respect to scalar product

$$(u,\vartheta)_{L_2((0,1);H)}=\int_0^1(u(x),\vartheta(x))_H\,dx.$$

Let $A=A^2\geq C^2I$ in H. Since A^{-1} is bounded in H, then

$$H(A)=\left\{u: u\in D(A), \|u\|_{H(A)}=\|Au\|_H\right\}$$

is a Hilbert space whose norm is equivalent to the norm of the graph of the operator A.

Let's associate to (1)-(3) the operator L defined by the equalities

$D(\mathsf{L})=W_2^2(\Omega;H(A),H,L_\nu u=0,\ \nu=1\div 4)=\{u(x):$ of almost all $x\in\Omega$ $u(x)\in D(A)$, $Au(x)$, $u''(x)\in L_2(\Omega;H)$ and satisfies the conditions (2) and (3) $\}$,

$$\mathsf{L}u\equiv -a(x)u''(x)+Au(x).$$

Let's agree to write in the form $\{u_1, u_2\}$ every function $u \in L_2(\Omega; H)$ whose contraction which on the $L_2((0,b); H)$ and $L_2((b,1); H)$ coincides with $u_1(x)$ and $u_2(x)$ respectively.

Then the equation (1) splits to the system of differential-operator equations of the second order at the direct sum $\mathsf{H} \equiv L_2((0,b); H) \oplus L_2((b,1); H)$

$$-a_1 u_1''(x) + A u_1(x) = \lambda u_1(x), \quad x \in [0,b),$$
$$-a_2 u_2''(x) + A u_2 = \lambda u_2, \quad x \in (b,1] \tag{4}$$

and the boundary conditions (2) and (3) correspondingly take the following form:

$$L_1 u \equiv \alpha_1 u_1(0) + \alpha_2 u_1'(0) = 0,$$
$$L_2 u \equiv \beta_1 u_2(1) + \beta_2 u_2'(1) = 0, \tag{5}$$
$$L_3 u \equiv \delta_1 u_1(b) + \delta_2 u_2(b) = 0,$$
$$L_4 u \equiv \gamma_1 u'(b) + \gamma_2 u'(b) = 0. \tag{6}$$

It is known that the direct sum $\mathsf{H} \equiv L_2((0,b); H) \oplus L_2((b,1); H)$ is a Hilbert space with the second scalar product

$$(\{u_1, \vartheta_1\}, \{u_2, \vartheta_2\})_{\mathsf{H}} = (u_1, u_2)_{L_2((0,b);H)} + (\vartheta_1, \vartheta_2)_{L_2((b,1);H)}.$$

The aim of the given paper is to study the asymptotic distribution of eigen-values of the operator L knowing the asymptotic distribution of eigen-numbers of the operator A. In case when $a(x) \equiv 1$ an asymptotics of eigen-values of some boundary value problems for the Stourm-Luiville equation on a finite segment was studied in the papers V.I. Gorbachuk, M.l. Gorbachuk [1], V.I. Gorbachuk [2], V.A. Mikhaylec [3], V.I. Gorbachuk, M.A. Ribak [4] and others.

In the paper by O.Sh. Mukhtarov's [5] the asymptotic behavior of eigen-values of the conjugation problem was investigated for ordinary-differential equation of the second order.

Theorem. *Let*

1. $A = A^* \ge C^2 I$ *in* H *and* A^{-1} *be completely continuous in* H;
2. $\alpha_i, \beta_i, \delta_i, \gamma_i$ $(i = 1,2)$ *be real numbers, moreover* $\alpha_2, \beta_1, \delta_1, \gamma_2 \neq 0$, $a_1, \gamma_2, \delta_2 = a_2 \delta_1 \gamma_1$;
3. $\left| \sqrt{a_1}\, \delta_1 \gamma_2 + \sqrt{a_2}\, \delta_2 \gamma_1 \right| \neq 0$.

Then the eigen-numbers of the problem (1)-(3) form two infinite sequences $\lambda_{n,k}^1, \lambda_{n,k}^2$ $(n,k = 1,2,...)$ *with the asymptotic*

$$\lambda_{n,k}^1 = \mu_k + \gamma_n, \quad \lambda_{n,k}^2 = \mu_k + \xi_n,$$

where

$$\gamma_n \sim \frac{a_1}{b^2} \pi^2 n^2, \quad \xi_n \sim \frac{a_2}{(1-b)^2} \pi^2 n^2,$$

and $\mu_k = \mu_k(A)$ *are eigen-values of the operator* A.

Proof. Let's consider the equation $(\mathsf{L} - \lambda I)u = 0$ in the space H. Since the equation $(\mathsf{L} - \lambda I)u = 0$ is reduced to the boundary value problem (4)-(6), then it is evident that it is sufficient to find eigen-numbers of the spectral problem (4)-(6) for finding the eigen-numbers of the operator L.

Let's denote by φ_k the eigen-elements of the operator A corresponding to the eigen-values $\mu_k(A)$. It is known that the $\{\varphi_k\}$ forms the orthonormalized basis. Then allowing for the spectral distribution for the coefficients $u_{1k}=(u_1,\varphi_k)$ and $u_{2k}=(u_2,\varphi k)$ we'll get the following problem:

$$-u''_{1k}(x)+b_1(\mu_k-\lambda)u_{1k}(x)=0, \quad x\in[0,b), \tag{7}$$

$$-u''_{2k}(x)+b_2(\mu_k-\lambda)u_{2k}(x)=0, \; x\in(b,1], \tag{8}$$

$$\begin{aligned}&\alpha_1u_{1k}(0)+\alpha_2u'_{1k}(0)=0,\\&\beta_1u_{2k}(1)+\beta_2u'_{2k}(1)=0,\end{aligned} \tag{9}$$

$$\begin{aligned}&\delta_1u_{1k}(b)+\delta_2u_{2k}(b)=0,\\&\gamma_1u'_{1k}(b)+\gamma_2u'_{2k}(b)=0,\end{aligned} \tag{10}$$

where $b_i=\frac{1}{a_i}\;(i=1,2)$.

The general solution of the ordinary differential equations (7)-(8) correspondingly has the following form:

$$u_{1k}(x)=c_1e^{-x\sqrt{b_1}\sqrt{\mu_k-\lambda}}+c_2e^{-(b-x)\sqrt{b_1}\sqrt{\mu_k-\lambda}}, \tag{11}$$

$$u_{2k}(x)=c_3e^{-(x-b)\sqrt{b_2}\sqrt{\mu_k-\lambda}}+c_4e^{-(1-x)\sqrt{b_2}\sqrt{\mu_k-\lambda}}, \tag{12}$$

where $c_i\;(i=1\div 4)$ is an arbitrary constant.

Substituting (11) and (12) in (9) and (10), we'll get the system with respect to the $c_i\;(i=1\div 4)$ whose determinants has the form

$$K(\lambda)=\begin{vmatrix}a_{11}&a_{12}&0&0\\0&0&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{vmatrix},$$

where $a_{11}=\alpha_1-\alpha_2\sqrt{b_1(\mu_k-\lambda)}$, $a_{12}=\left(\alpha_1+\alpha_2\sqrt{b_1(\mu_k-\lambda)}\right)e^{-b\sqrt{b_1(\mu_k-\lambda)}}$,

$a_{23}=\left(\beta_1-\beta_2\sqrt{b_2(\mu_k-\lambda)}\right)e^{-(1-b)\sqrt{b_2(\mu_k-\lambda)}}$, $a_{24}=\beta_2\sqrt{b_2(\mu_k-\lambda)}+\beta_1$,

$a_{31}=\delta_1e^{-b\sqrt{b_1(\mu_k-\lambda)}}$, $a_{32}=\delta_1$, $a_{33}=\delta_2$, $a_{34}=\delta_2e^{-(1-b)\sqrt{b_2(\mu_k-\lambda)}}$,

$a_{41}=-\gamma_1\sqrt{b_1(\mu_k-\lambda)}e^{-b\sqrt{b_1(\mu_k-\lambda)}}$, $a_{42}=\gamma_1\sqrt{b_1(\mu_k-\lambda)}$,

$a_{43}=-\gamma_2\sqrt{b_2(\mu_k-\lambda)}$, $a_{44}=\gamma_2\sqrt{b_2(\mu_k-\lambda)}\;e^{-(1-b)\sqrt{b_2}\sqrt{\mu_k-\lambda}}$.

Calculating $K(\lambda)$ we'll get

$$\begin{aligned}K(\lambda)=&\left[\left(\alpha_1-\alpha_2\sqrt{b_1(\mu_k-\lambda)}\right)-\left(\alpha_1+\alpha_2\sqrt{b_1(\mu_k-\lambda)}\right)e^{-2b\sqrt{b_1(\mu_k-\lambda)}}\right]\times\\&\times\Big\{-2\left(\beta_1-\beta_2\sqrt{b_2(\mu_k-\lambda)}\right)\delta_1\gamma_2\sqrt{b_2(\mu_k-\lambda)}e^{-2(1-b)\sqrt{b_2(\mu_k-\lambda)}}+\Big[\left(\beta_1+\beta_2\sqrt{b_2(\mu_k-\lambda)}\right)-\\&-\left(\beta_1-\beta_2\sqrt{b_2(\mu_k-\lambda)}\right)e^{-2(1-b)\sqrt{b_2(\mu_k-\lambda)}}\Big]\left(-\delta_1\gamma_2\sqrt{b_2}-\delta_2\gamma_1\sqrt{b_1}\right)\sqrt{\mu_k-\lambda}\Big\}-\\&-2\left[\left(\beta_1+\beta_2\sqrt{b_2(\mu_k-\lambda)}\right)-\left(\beta_1-\beta_2\sqrt{b_2(\mu_k-\lambda)}\right)e^{-2(1-b)\sqrt{b_2(\mu_k-\lambda)}}\right]\times\\&\times\left(\alpha_1+\alpha_2\sqrt{b_1(\mu_k-\lambda)}\right)\gamma_1\delta_2\sqrt{b_1(\mu_k-\lambda)}e^{-b\sqrt{b_1(\mu_k-\lambda)}}.\end{aligned}$$

Then it is evident that the eigen-values of the problem (4)-(6) (of the operator L) there are zeros of the following equations

$$\left(\alpha_1+\alpha_2\sqrt{b_1(\mu_k-\lambda)}\right)-\left(\alpha_1-\alpha_2\sqrt{b_1(\mu_k-\lambda)}\right)e^{2b\sqrt{b_1(\mu_k-\lambda)}}=0, \tag{13}$$

$$\left(\beta_1 - \beta_2\sqrt{b_2(\mu_k - \lambda)}\right) - \left(\beta_1 + \beta_2\sqrt{b_2(\mu_k - \lambda)}\right)e^{2(1-b)\sqrt{b_2(\mu_k - \lambda)}} = 0 . \qquad (14)$$

Thus, the spectrum of the operator L consists of such real $\lambda \neq \mu_k$ which if only at one k satisfy the equations (13), (14). Let's search the eigenvalues of the operator L less μ_k. Let's denote by $\sqrt{\mu_k - \lambda} = y$. The equation (13) in this case is equivalent with the equation

$$\frac{k_1 + y}{k_1 - y} = e^{k_2 y} \quad , \quad 0 < y < \sqrt{\mu_k} \ , \qquad (15)$$

where $k_1 = \dfrac{\alpha_1}{\alpha_2\sqrt{b_1}}$, $k_2 b\sqrt{b_1} > 0$.

Let's prove the absence of solutions of the equation (15) on the interval $\left(0, \sqrt{\mu_k}\right)$ by the methods of analysis. Let's denote by $f(y)$ the left hand side of the equation (15) and make its investigation: by $f(y) = \dfrac{k_1 + y}{k_1 - y}$, $y \neq k_1$, $f'(y) = \dfrac{2k_1}{(k_1 - y)^2}$, $sign\, f'(y) = = sign(\alpha_1, \alpha_2)$, then $f(y)$ increases when $\alpha_1 \cdot \alpha_2 > 0$ and decreases when $\alpha_1 \cdot \alpha_2 < 0$, $f(0) = 1$, $\lim\limits_{y \to k_1 - 0} f(y) = sign\, k_1 \cdot (+\infty)$, $\lim\limits_{y \to k_1 + 0} f(y) = sign\, k_1 \cdot (-\infty)$, $\lim\limits_{y \to \infty} f(y) = -1$.

So, in case $0 < \lambda < \mu_k$ the equation (13) has the solution when $\alpha_1 \cdot \alpha_2 < 0$. When $\alpha_1 \cdot \alpha_2 > 0$ the equation (13) may have only one zero on $(0, k_1)$.

Denoting by y_0 this zero and substituting $y_1 = y - y_0$ we'll get the equivalent equation which will not have the zeros.

In the second case, i.e. when $\lambda > \mu_k$ the equation (13) is equivalent to the equation

$$ctg\ b\sqrt{b_1}\, z = \frac{\alpha_1}{\alpha_2\sqrt{b_1}\, z},$$

where $z = \sqrt{\lambda - \mu_k}$, $0 < z < \infty$.

Let's consider the function

$$\vartheta_k(z) = ctg\ b\sqrt{b_1}\ z - \frac{\alpha_1}{\alpha_2\sqrt{b_1}\, z} = \frac{\varphi_k(z)}{\alpha_2\sqrt{b_1}\, z},$$

$$\varphi_k(z) = \alpha_2\sqrt{b_1}\, z\, ctg\ b\sqrt{b_1}\ z - \alpha_1 .$$

The zeros of the functions $\varphi_k(z)$ and $\vartheta_k(z)$ coincide. The function $\varphi_k(z)$ is determined on $(0, +\infty)$ except the points $b\sqrt{b_1}\ z = n\pi$ $(n = 1, 2, \ldots)$, i.e. $z \neq \dfrac{n\pi}{b\sqrt{b_1}}$. Since at every interval $\left(\dfrac{n\pi}{b\sqrt{b_1}}, \dfrac{(n+1)\pi}{b\sqrt{b_1}}\right)$ $\varphi_k(n)$ run from $-\infty$ to $+\infty$ and it's derivative

$$\varphi_k'(z) = \frac{\alpha_2\sqrt{b_1}\left(\sin 2b\sqrt{b_1}\, z - 2b\sqrt{b_1}\, z\right)}{\sin^2 b\sqrt{b_1}\ z}$$

is positive when $\alpha_2 < 0$ (negative when $\alpha_2 > 0$) then on it the $\varphi_k(z)$ has only one zero $z_{n,k}$:

$$\frac{n\pi}{b\sqrt{b_1}} < z_{n,k} < \frac{(n+1)\pi}{b\sqrt{b_1}} \qquad (n = 1,2,...).$$

Consequently, the first series of eigen0values we can represent in the following form:

$$\lambda^1_{n.k} = \mu_k + \gamma_n ,$$

where $\gamma_n \sim \frac{a_1}{b^2}\pi^2 n^2$.

Zeros of the equation (14), which corresponds to the second series of eigen-values are investigated, analogously.

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