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THE FIRST BOUNDARY VALUE PROBLEM FOR NON-DIVERGENT LINEAR SECOND ORDER ELLIPTIC EQUATIONS OF CORDES TYPE

Abstract

In the paper the Dirichlet problem for non-divergent linear second order elliptic equations with, generally speaking, discontinuous coefficients, satisfying the Cordes condition is considered. The unique strong (almost everywhere) solvability of this problem is proved in the space $W_p^2(D)$, where p belongs to some segment containing point 2 .

Introduction. Let $\mathbf{E}_n$ be an Euclidian space of points $x=(x_1,...,x_n), n\geq 2$, $D\subset \mathbf{E}_n$ be a bounded domain with the boundary $\partial D\in C^2$. Consider in D the first boundary value problem

$$\mathrm{L}u=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i(x)\frac{\partial u}{\partial x_i}+c(x)u=f(x);\ x\in D\,, \tag{1}$$

$$u|_{\partial D}=0 \tag{2}$$

under the assumption that $\|a_{ij}(x)\|$ is a symmetric matrix. Moreover, for all $x\in D$ and $\xi\in \mathbf{E}_n$ the following condition is satisfied

$$\gamma|\xi|^2\leq \sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leq \gamma^{-1}|\xi|^2;\ \gamma\in(0,1]\text{-const} \tag{3}$$

Besides this we assume that all the coefficients of the operator L are real and measurable in D functions.

The aim of the present paper is finding the conditions on the coefficients of the equation (1), at fulfillment of which the first boundary value problem (1)-(2) is uniquely strongly (almost everywhere) solvable in space $\dot{W}_p^2(D)$ for any $f(x)\in L_p(D)$, $p\in[p_1,p_2]$, $p_1\in(1,2)$, $p_2\in(2,\infty)$. Note that if equation (1) doesn't contain minor terms,

$$\sum_{i=1}^{n}a_{ii}(x)=1;\ ess\sup_{x\in D}\sum_{i,j=1}^{n}a_{ij}^2(x)<\frac{1}{n-1}, \tag{4}$$

and boundary ∂D represents the surface of mean non-positive curvature, then analogous result at $p=2$ was established in [1]. If $a_{ij}\in C(\overline{D})$; $i,j=1,...,n$; then solvability of the problem (1)-(2) holds for any $p\in(1,\infty)$ (see [2-3]). We point also to the works [4-8], in which at $p=2$ under conditions of the type (4) the corresponding result was obtained for non-divergent second order parabolic equations. As was shown in [9], only condition (3) doesn't provide the solvability of the first boundary value problem in space $\dot{W}_p^2(D)$ for any $p\in(1,\infty)$. In connection with the questions of solvability of the boundary value problems for non-divergent elliptic equations we note papers [10-11].

1. Some auxiliary statements. At first we shall agree on some denotations and definitions. By u_i and u_{ij} we shall denote derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ respectively; $i, j = 1,...,n$; by $B_R(x^0)$ and $S_R(x^0)$ - the ball $\{x : |x - x^0| < R\}$ and the sphere $\{x : |x - x^0| = R\}$ respectively; $R > 0$, $x^0 \in \mathbf{E}_n$. Let $W_p^2(D), W_p^1(D)$ be Banach spaces of functions $u(x)$ given on D with finite norms

$$\|u\|_{W_p^1(D)} = \left(\int_D \left[|u|^p + \sum_{i=1}^n |u_i|^p \right] dx \right)^{\frac{1}{p}}$$

and

$$\|u\|_{W_p^2(D)} = \left(\int_D \left[|u|^2 + \sum_{i=1}^n |u_i|^p + \sum_{i,j=1}^n |u_{ij}|^p \right] dx \right)^{\frac{1}{p}}$$

respectively; $p \in (1, \infty)$. Denote by $\overset{\circ}{W}{}_p^1(D)$ the subspace of $W_p^1(D)$ containing as a dense set all infinitely differentiable functions from $C_0^\infty(D)$, and let $\dot{W}_p^2(D) = W_p^2(D) \cap \overset{\circ}{W}{}_p^1(D)$.

Function $u(x) \in \dot{W}_p^2(D)$ is called the strong solution of the first boundary value problem (1)-(2), if it satisfies equation (1) almost everywhere in D.

Everywhere below the notation $C(...)$ means that the positive constant C depends only on contents of brackets. Denote by $B = B_R(x^0)$ the ball such that $\overline{B} \in D$.

Lemma 1. *If* $u(x) \in C_0^\infty(B)$, *then*

$$\int_B \sum_{i,j=1}^n u_{ij}^2 dx = \int_B (\Delta u)^2 dx .$$

Proof. We have

$$\int_B (\Delta u)^2 dx = \int_B \sum_{i,j=1}^n u_{ii} u_{jj} dx = -\sum_{i,j=1}^n \int_B u_i u_{jji} dx = \int_B \sum_{i,j=1}^n u_{ij}^2 dx ,$$

and the lemma is proved.

Lemma 2. *If* $u(x) \in C_0^\infty(B)$ *and* $p \in (1, \infty)$ *then*

$$\int_B \sum_{i,j=1}^n |u_{ij}|^p dx \le C_1(p,n) \int_B |\Delta u|^p dx .$$

Proof. We'll consider only the case $n \ge 3$. Let $G(t) = |t|^{2-n}$, $b = \frac{1}{n\omega_n (2-n)}$, where ω_n is the volume of n- dimensional unit ball. We have

$$u(x) = b \int_B G(x-y) \Delta u(y) dy = b \int_B G(y-x) F(y) dy = b \int_{\mathbf{E}_n} G(\vartheta) F(\vartheta + x) d\vartheta ,$$

where $F(y) = \Delta u(y)$. So, for $i, j = 1,...,n$

$$u_i(x) = b \int_{\mathbf{E}_n} G(\vartheta) F_i(\vartheta + x) d\vartheta ,$$

$$u_{ij}(x)=b\int_{\mathbf{E}_n} G(\vartheta)F_{ij}(\vartheta+x)d\vartheta=b\int_B G(x-y)F_{ij}(y)dy=$$
$$=b\lim_{\rho\to 0}\int_{B^\rho} G(x-y)F_{ij}(y)dy==b\lim_{\rho\to 0} J_{ij}(\rho), \tag{5}$$

where $B^\rho = B\setminus \overline{B}_\rho(x)$.

On the other hand,

$$J_{ij}(\rho)=-\int_{B^\rho} G_j(y-x)F_i(y)dy+\int_{S\rho(x)} G_j(x-y)F_i(y)\cos(\overline{n},y_j)dS_y=$$
$$=k^1_{ij}(\rho)+k^2_{ij}(\rho);\ i,j=1,...,n; \tag{6}$$

where $\overline{n}$ is a unit vector of internal normal line to $\partial B_\rho(x)=S_\rho(x)$ (i.e. external with respect to $\partial B_\rho(x)$).

Further,

$$k^2_{ij}(\rho)=\rho^{2-n}\int_{S\rho(x)} F_i(y)\cos(\overline{n},y_j)dS_y,\ i,j=1,...,n.$$

Taking into account that for $i=1,...,n$ $|F_i(y)|\le M<\infty$, we obtain

$$\left|k^2_{ij}(\rho)\right|\le M\rho^{2-n} mes\, S_\rho(x)=Mn\omega_n\rho;\ i,j=1,...,n.$$

Thus, from (6) we conclude, that for $i,j=1,...,n$

$$\lim_{\rho\to 0} J_{ij}=\lim_{\rho\to 0} k^1_{ij}(\rho). \tag{7}$$

Analogously we derive for $i,j=1,...,n$

$$k^1_{ij}(\rho)=\int_{B^\rho} G_{ij}(x-y)F(y)dy-\int_{S_\rho(x)} G_j(y-x)F(y)\cos(\overline{n},y_i)dS_y=k^3_{ij}(\rho)+k^4_{ij}(\rho), \tag{8}$$

and further

$$bk^4_{ij}(\rho)=-\frac{1}{n\omega_n\rho^n}\int_{S_\rho(x)} F(y)\cos(\overline{n},y_i)(y_j-x_i)dS_y=$$
$$=-\frac{1}{n\omega_n\rho^n}\int_{S_\rho(x)} F(y)\cos(\overline{n},y_i)\cos(\overline{n},y_j)dS_y=$$
$$=-\frac{F(x)}{n\omega_n\rho^{n-1}}\int_{S_\rho(x)}\cos(\overline{n},y_i)\cos(\overline{n},y_j)dS_y-$$
$$-\frac{1}{n\omega_n\rho^{n-1}}\int_{S_\rho(x)}[F(y)-F(x)]\cos(\overline{n},y_i)\cos(\overline{n},y_j)dS_y=k^5_{ij}(\rho)+k^6_{ij}(\rho). \tag{9}$$

It's easy to see that if $i\ne j$, then $k^5_{ij}(\rho)=0$. Besides,

$$\left|k^5_{ii}(\rho)\right|=-\frac{F(x)}{n^2\omega_n\rho^{n-1}} mes S_\rho(x)=-\frac{F(x)}{n};\ i=1,...,n;$$

and by virtue of continuity of the function $F(y)$

$$\lim_{\rho\to 0} k^6_{ij}(\rho)=0;\ i,j=1,...,n. \tag{10}$$

Now from (5) and (7)-(9) it follows, that for $i,j=1,...,n$

$$u_{ij}(x)=-\frac{\delta_{ij}}{n}F(x)+K^{ij}*F\ ,$$

where δ_{ij} is Cronecker symbol, $K^{ij}*F$ is a singular integral with the kernel in G_{ij}.
By Calderon-Zygmund theorem [12] for $p\in(1,\infty)$, $i,j=1,...,n$

$$\left\|K^{ij}*F\right\|_{L_p(\mathbf{E}_n)}\le C_{ij}(p,n)\|F\|_{L_p(\mathbf{E}_n)}=C_{ij}(p,n)\|F\|_{L_p(B)}. \qquad (11)$$

The validity of the statement of the lemma from (10)-(11) follows.

Denote now by $\overset{\circ}{W}{}_p^2(B)$ and $\overset{\circ}{V}{}_p^2(B)$ the closures of $C_0^\infty(B)$ by norms

$$\|u\|_{\overset{\circ}{W}{}_p^2(B)}=\left(\int_B\sum_{i,j=1}^n\left|u_{ij}\right|^p dx\right)^{\frac{1}{p}}$$

and

$$\|u\|_{\overset{\circ}{V}{}_p^2(B)}=\left(\int_B|\Delta u|^p dx\right)^{\frac{1}{p}}$$

respectively, $p\in(1,\infty)$. According to Friedrichs inequality [13] and lemma 2 functionals defined above are really norms. Denote by $T(p)$ an operator mapping each function $u(x)\in\overset{\circ}{V}{}_p^2(B)$ to the function itself as an element of the space $\overset{\circ}{W}{}_p^2(B)$. According to lemma 2 operator $T(p)$ is bounded. Denote by $K(p)$ its norm. By lemma 1 $K(2)=1$. Let p_0 be arbitrary fixed number from the interval $(1,2)$. By Riesz-Thorin theorem on convexity [14] for any $p\in[p_0,2]$

$$K(p)\le(K(p_0))^{1-\theta}(K(2))^{\theta}=(K(p_0))^{1-\theta},$$

where $\theta=\dfrac{2(p-p_0)}{p(2-p_0)}$. So,

$$K(p)\le(K(p_0))^{\frac{p_0(2-p)}{p(2-p_0)}}.$$

We'll fix $p_0=\dfrac{3}{2}$ and denote $a=\max\left\{\dfrac{3}{2},\left(K\left(\dfrac{3}{2}\right)\right)^2\right\}$.

Since for $p\in\left[\dfrac{3}{2},2\right]$ $\dfrac{p_0(2-p)}{p(2-p_0)}\le\dfrac{2-p}{2-p_0}=2(2-p)$, then, finally, we obtain

$$K(p)\le a^{2-p}.$$

Thus, we proved the following statement.

Lemma 3. *If* $u(x)\in\overset{\circ}{W}{}_p^2(B)$, *then for any* $p\in\left[\dfrac{3}{2},2\right]$ *the following inequality holds*

$$\|u\|_{\overset{\circ}{W}{}_p^2(B)}\le a^{2-p}\|u\|_{\overset{\circ}{V}{}_p^2(B)}.$$

Note that here constant a>1 depends only on n.

Denote for $p\in\left[\frac{3}{2},2\right]$ $\sup\limits_D\left(\sum\limits_{i,j=1}^n\left|a_{ij}(x)-\delta_{ij}\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$ by δ_p (for brevity we write sup instead of ess sup), and let $\delta_2=\delta$, $h=\max\left\{\frac{1-\gamma^2}{\gamma},1\right\}$

Lemma 4. *For any* $p\in\left[\frac{3}{2},2\right]$ *the following estimation holds*

$$\delta_p\le h^{\frac{2-p}{p}}\delta^{\frac{2(p-1)}{p}}.$$

Proof. From condition (3) it follows that for $i=1,...,n$

$$\gamma-1\le a_{ii}(x)-1\le\gamma^{-1}-1\ ,$$

and since $\gamma-1\ge1-\gamma^{-1}$, then

$$\left|a_{ii}(x)-1\right|\le\frac{1-\gamma}{\gamma}. \tag{12}$$

If $i\ne j$, then

$$2\gamma\le a_{ii}(x)+a_{jj}(x)+2a_{ij}(x)\le2\gamma^{-1}.$$

Therefore,

$$\left|a_{ij}(x)\right|\le\frac{1-\gamma^2}{\gamma}. \tag{13}$$

From (12)-(13) we conclude, that for $i,j=1,...,n$

$$\left|a_{ij}(x)-\delta_{ij}\right|\le h.$$

On the other hand, subject to (14)

$$\delta_p\le\sup_D\left(\sum_{i,j=1}^n\left(a_{ij}(x)-\delta_{ij}\right)^2\left|a_{ij}(x)-\delta_{ij}\right|^{\frac{2-p}{p-1}}\right)^{\frac{p-1}{p}}=h^{\frac{2-p}{p}}\delta^{\frac{2(p-1)}{p}},$$

and the lemma is proved.

Lemma 5. *Let* $\delta<1$. *Then there exists* $p_1(\gamma,\delta,n)\in\left[\frac{3}{2},2\right)$, *such that for all* $p\in[p_1,2]$

$$a^{2-p}\delta_p\le\delta^{\frac{1}{3}}.$$

Proof. According to the previous lemma

$$a^{2-p}\delta_p\le a^{2-p}h^{\frac{2-p}{p}}\delta^{\frac{2(p-1)}{p}}.$$

But $h^{\frac{1}{p}}\le h^{\frac{2}{3}}=h_1$, $\frac{p-1}{p}\ge\frac{1}{3}$. Therefore,

$$a^{2-p}\delta_p\le(ah_1)^{2-p}\delta^{\frac{2}{3}}. \tag{14}$$

Let now $p_1=\max\left\{\frac{3}{2},\ 2-\frac{\ln\frac{1}{\delta}}{3\ln(ah_1)}\right\}$. Then for $p\in[p_1,2]$ $(ah_1)^{2-p}\le\delta^{-\frac{1}{3}}$, and the statement of the lemma follows from (14).

2. Interior a priori estimation. Consider the operator

$$\mathrm{L}_0=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}.$$

Lemma 6. *If with respect to coefficients of the operator* L_0 *condition (3) be fulfilled and* $\delta<1$ *,then for all* $p\in[p_1,2]$ *and any function* $u(x)\in\overset{\circ}{W}{}_p^2(B)$ *the following estimation is valid*

$$\|u\|_{\overset{\circ}{W}{}_p^2(B)}\le C_2(\gamma,\delta,n)\|\mathrm{L}_0u\|_{L_p(B)}.$$

Proof. According to lemma 3

$$\|u\|_{\overset{\circ}{W}{}_p^2(B)}\le a^{2-p}\|\Delta u\|_{L_p(B)}\le a^{2-p}\|\mathrm{L}_0u\|_{L_p(B)}+a^{2-p}\|(\mathrm{L}_0-\Delta)u\|_{L_p(B)}\le$$

$$\le a^{\frac{1}{2}}\|\mathrm{L}_0u\|_{L_p(B)}+a^{2-p}\|(\mathrm{L}_0-\Delta)u\|_{L_p(B)}. \tag{15}$$

But on the other hand,

$$\|(\mathrm{L}_0-\Delta)u\|_{L_p(B)}\le\left(\int_B\sum_{i,j=1}^{n}|u_{ij}|^p\left(\sum_{i,j=1}^{n}|a_{ij}(x)-\delta_{ij}|^{\frac{p}{p-1}}\right)^{p-1}dx\right)^{\frac{1}{p}}\le\delta_p\|u\|_{\overset{\circ}{W}{}_p^2(B)}.$$

Therefore, from (15) and lemma 5 we conclude

$$\|u\|_{\overset{\circ}{W}{}_p^2(B)}\le a^{\frac{1}{2}}\|\mathrm{L}_0u\|_{L_p(B)}+a^{2-p}\delta_p\|u\|_{\overset{\circ}{W}{}_p^2(B)}\le a^{\frac{1}{2}}\|\mathrm{L}_0u\|_{L_p(B)}+\delta^{\frac{1}{3}}\|u\|_{\overset{\circ}{W}{}_p^2(B)},$$

and statement of the lemma is proved.

Everywhere later on without separately mentioning it we shall assume that radius R of the ball B doesn't exceed 1.

Lemma 7. *If conditions of the previous lemma are fulfilled, then for all* $p\in[p_1,2]$ *and any function* $u(x)\in C_0^\infty(B)$ *the following inequality is valid*

$$\|u\|_{W_p^2(B)}\le C_3(\gamma,\delta,n)\|\mathrm{L}_0u\|_{L_p(B)}.$$

Proof. For proving it is sufficient to apply the Friedrichs inequality and lemma 6. We'll impose the following Cordes conditions on the leading coefficients of the operator L.

$$\sigma=\sup_D\frac{\sum_{i,j=1}^{n}a_{ij}^2(x)}{\left[\sum_{i=1}^{n}a_{ii}(x)\right]^2}<\frac{1}{n-1}. \tag{16}$$

At that we'll assume that condition (16) is fulfilled accurate up to equivalence and nonsingular linear transformation, i.e. equation (1) can be replaced by the equivalent to it equation $\mathrm{L}_1u=f_1(x)$, and domain D can be covered by a finite number of subdomains $D_1,...,D_m$, such that in each D_i there exists nonsingular linear transformation A_i, under which image of the operator L_1 satisfies condition (16) in the image of subdomain D_i, $i=1,...,m$.

Lemma 8. *Condition $\delta<1$ accurate up to equivalence and nonsingular linear transformation coincides with condition (16).*

Proof. It's easy to see that equation (1) can be replaced by equivalent to it equation, whose matrix of principle part has a unit trace. Therefore, without loss of generality, it can be assumed that $\sum_{i=1}^{n} a_{ii}(x)=1$. We'll make the following transformation

$y_i = kx_i; i=1,...,n$; where $k=\left(\sup\limits_{D}\sum_{i,j=1}^{n} a_{ij}^2(x)\right)^{\frac{1}{2}}$. Then if $\|A_{ij}(y)\|$ is the matrix of main part of image of the operator L, then $A_{ij}(y)=k^2 a_{ij}\left(\frac{y}{k}\right); i,j=1,...,n$. Condition $\delta<1$ in new variables will take the form

$$\sup_{D}\sum_{i,j=1}^{n} A_{ij}(y)-2k^2+n<1, \tag{17}$$

where $\widetilde{D}$ is the image of domain D. It's clear that (17) is equivalent to condition

$$\sup_{D}\sum_{i,j=1}^{n} a_{ij}^2(x)<\frac{1}{n-1}.$$

We'll impose now the following conditions on the minor coefficients of the operator L. For $p\in[p_1,2]$

$$b_i(x)\in L_r(D);\, i=1,...,n;\ r=\begin{cases} n; & if\ p<2, n\ge 2\, and\ p=2, n>2 \\ 2+\nu_1; & if\ p=n=2, \end{cases} \tag{18}$$

$$c(x)\in L_m(D), m=\begin{cases} \max\left\{p,\frac{n}{2}\right\}; & if\ p<2, n\ge 2\, and\ p=2, n\ne 4 \\ 2+\nu_2; & if\ p=2, n=4, \end{cases} \tag{19}$$

where ν_1,ν_2 are some positive constants.

Lemma 9. *Let with respect to coefficients of the operator L conditions (3), (16), (18) and (19) be fulfilled. Then there exist constants $C_4(\gamma,\sigma,n)$ and $R_0(\gamma,\sigma,n,\mathrm{B},c)$, such that if radius R of the ball B doesn't exceed R_0, then for any function $u(x)\in C_0^\infty(B)$ and any $p\in[p_1,2]$ the following estimation is valid*

$$\|u\|_{W_p^2(B)}\le C_4\|\mathrm{L}u\|_{L_p(B)}.$$

Here $\mathrm{B}=(b_1(x),...,b_n(x))$.

Proof. We restrict ourselves with the case $n>4$. We'll use the following imbedding theorem ([15]): let $q\in(1,n)$; then for any function $u(x)\in \overset{\circ}{W}{}_q^1(B)$ the following inequality holds

$$\|u\|_{L_{\frac{nq}{n-q}}(B)}\le C_5(q,n)\sum_{i=1}^{n}\|u_i\|_{L_q(B)}. \tag{20}$$

According to lemma 7

$$\|u\|_{W_p^2(B)} \le C_3\|Lu\|_{L_p(B)} + C_3\sum_{i=1}^{n}\|b_i(x)u_i\|_{L_p(B)} + C_3\|c(x)u\|_{L_p(B)}. \quad (21)$$

We'll fix arbitrary $i, 1 \le i \le n$ and assume $q = p$ in (20) and instead of function $u(x)$ - the function $u_i(x)$. We obtain

$$\|b_i u_i\|_{L_p(B)} \le \|b_i\|_{L_n(B)}\|u_i\|_{L_{\frac{np}{n-p}}(B)} \le C_5\|b_i\|_{L_n(B)}\sum_{i,j=1}^{n}\|u_{ij}\|_{L_p(B)}.$$

So,

$$\sum_{i=1}^{n}\|b_i u_i\|_{L_p(B)} \le C_5\sum_{i=1}^{n}\|b_i\|_{L_n(B)}\|u\|_{W_p^2(B)} = C_6(n)\Gamma_1(R)\|u\|_{W_p^2(B)}, \quad (22)$$

where $\Gamma_1(R) \to 0$ as $R \to 0$ by virtue of condition (18) and $C_6 = \sup\limits_{p\in[p_1,2]} C_5(p,n)$.

Analogously we have

$$\|cu\|_{L_p(B)} \le \|c\|_{L_{\frac{n}{2}}(B)}\|u\|_{L_{\frac{pn}{n-2p}}(B)}. \quad (23)$$

We put $q = \dfrac{pn}{n-2p}$ in (20). Then

$$\|u\|_{L_{\frac{pn}{n-2p}}(B)} \le C_7(p,n)\sum_{i=1}^{n}\|u_i\|_{L_{\frac{pn}{n-p}}(B)}.$$

Denoting by $C_8 = \sup\limits_{p\in[p_1,2]} C_7(p,n)$, we conclude

$$\|u\|_{L_{\frac{pn}{n-2p}}(B)} \le C_8(p,n)\sum_{i=1}^{n}\|u_i\|_{L_{\frac{pn}{n-p}}(B)}.$$

Again using (20) from (23) we obtain

$$\|cu\|_{L_p(B)} \le C_8C_5(p,n)\|c\|_{L_{\frac{n}{2}}(B)}\sum_{i,j=1}^{n}\|u_{ij}\|_{L_p(B)} \le C_8C_6\Gamma_2(R)\|u\|_{W_p^2(B)}, \quad (24)$$

where $\Gamma_2(R) \to 0$ as $R \to 0$ by virtue of condition (19).

Taking into account (22) and (24) in (21) we arrive at the estimation

$$\|u\|_{W_p^2(B)} \le C_3\|Lu\|_{L_p(B)} + C_3C_6(\Gamma_1(R) + C_8\Gamma_2(R))\|u\|_{W_p^2(B)}.$$

Now it is sufficient to choose R_0 so small that

$$\Gamma_1(R_0) + C_8\Gamma_2(R_0) \le \frac{1}{2C_3C_6},$$

and lemma is proved.

Remark. Note, that number R_0 doesn't depend on choice of point x^0 - the center of the ball B. It follows from absolute continuity of Lebesgue integral.

Everywhere later on not mentioning this especially we'll assume that $R \le R_0$. Denote for $R_1 < R$ $B_{R_1}(x^0)$ by B_1.

Lemma 10. *Let with respect to coefficients of the operator* L *conditions (3), (16), (18) and (19) be fulfilled. Then there exists a constant* $C_9(\gamma,\sigma,n,B)$ *such that for any function* $u(x) \in C^\infty(\overline{B})$ *and any* $p \in [p_1, 2]$ *the following estimation holds.*

$$\|u\|_{W_p^2(B_1)} \le C_4 \|\mathcal{L}u\|_{L_p(B)} + \frac{C_9}{(R-R_1)^2}\|u\|_{W_p^1(B)}.$$

Proof. Let the function $\eta(x)\in C_0^\infty(B_R)$ be such that $\eta(x)=1$ in B_1, $0\le\eta(x)\le 1$, moreover,

$$|\eta_i(x)| \le \frac{C_{10}}{R-R_1}, \quad |\eta_{ij}(x)| \le \frac{C_{10}}{(R-R_1)^2}; \; i,j=1,\dots,n, \tag{25}$$

where $C_{10}=C_{10}(n)$. Applying lemma 9 to the function $u\eta$, we obtain

$$\|u\|_{W_p^2(B_1)} \le C_4 \|\mathcal{L}(u\eta)\|_{L_p(B)}. \tag{26}$$

But on the other hand,

$$|\mathcal{L}(u\eta)| \le |\mathcal{L}u| + |u|\left(\left|\sum_{i,j=1}^n a_{ij}(x)\eta_{ij}\right| + \left|\sum_{i=1}^n b_i(x)\eta_i\right|\right) + 2\left|\sum_{i,j=1}^n a_{ij}(x)u_i\eta_j\right|, \tag{27}$$

and further subject to (25)

$$\left|\sum_{i,j=1}^n a_{ij}(x)\eta_{ij}\right| \le \frac{C_{11}(\gamma,n)}{(R-R_1)^2},$$

$$\left|\sum_{i=1}^n b_i(x)\eta_i\right| \le \frac{C_{10}}{R-R_1}\sum_{i=1}^n |b_i(x)|,$$

$$2\left|\sum_{i,j=1}^n a_{ij}(x)u_i\eta_j\right| \le 2\left(\sum_{i,j=1}^n a_{ij}(x)u_iu_j\right)^{\frac12}\left(\sum_{i,j=1}^n a_{ij}(x)\eta_i\eta_j\right)^{\frac12} \le 2\gamma^{-1}\left(\sum_{i=1}^n u_i^2\right)^{\frac12}\left(\sum_{i=1}^n \eta_i^2\right)^{\frac12} \le$$

$$\le 2\gamma^{-1}\sum_{i=1}^n |u_i|\sum_{i=1}^n |\eta_i| \le \frac{2n\gamma^{-1}C_{10}}{R-R_1}\sum_{i=1}^n |u_i|.$$

So, from (27) we conclude

$$\|\mathcal{L}(u\eta)\|_{L_{p(B)}} \le \|\mathcal{L}u\|_{L_{p(B)}} + \frac{C_{11}}{(R-R_1)^2}\|u\|_{L_{p(B)}} + \frac{C_{10}}{R-R_1}\left\|u\sum_{i=1}^n b_i\right\|_{L_{p(B)}} + \frac{C_{12}(\gamma,n)}{R-R_1}\sum_{i=1}^n \|u_i\|_{L_p(B)}. \tag{28}$$

Now we'll use the following imbedding theorem [15]: let $q\in(1,n)$; then for any function $u(x)\in W_q^1(B)$ the following inequality holds

$$\|u\|_{L_{\frac{nq}{n-q}}(B)} \le C_{13}(q,n)\|u\|_{W_q^1(B)}. \tag{29}$$

Putting $q=p$ in (29), we have

$$\left\|u\sum_{i=1}^n b_i\right\|_{L_{p(B)}} \le n\|u\|_{L_{\frac{nq}{n-q}}(B)}\sum_{i=1}^n \|b_i\|_{L_n(B)} \le nC_{14}(\mathcal{B})C_{13}\|u\|_{W_q^1(B)} \le nC_{14}C_{15}(n)\|u\|_{W_q^1(B)}, \tag{30}$$

where $C_{15}=\sup\limits_{p\in[p_1,2]} C_{13}(p,n)$. Taking into account (30) in (28) and remembering that $R\le 1$, we conclude

$$\|\mathcal{L}(u\eta)\|_{L_{p(B)}} \le \|\mathcal{L}u\|_{L_{p(B)}} + \frac{C_{16}(\gamma,n,\mathcal{B})}{(R-R_1)^2}\|u\|_{W_q^1(B)}.$$

Now the statement of the lemma follows from (26). Denote $B_{\frac{R}{2}(x^0)}$ by B_2.

Lemma 11. *Let with respect to coefficients of the operator* L *conditions of the previous lemma be fulfilled. Then there exists constant* $C_{17}(\gamma,\sigma,n,\mathrm{B})$ *such that for any function* $u(x)\in W_p^2(B)$, *any* $\varepsilon>0$ *and* $p\in[p_1,2]$ *the following estimation holds*

$$\|u\|_{W_p^2(B_2)}\le C_4\|\mathrm{L}u\|_{L_p(B)}+\varepsilon\|u\|_{W_p^2(B)}+\frac{C_{17}}{\varepsilon R^4}\|u\|_{L_p(B)} .$$

Proof. We'll use the following interpolation inequality ([12]): let $p\in(1,\infty)$; then for any function $u(x)\in W_p^2(B)$ for any $\varepsilon>0$ the following estimation holds.

$$\|u\|_{W_p^2(B)}\le\varepsilon\|u\|_{W_p^2(B)}+\frac{C_{18}(p,n)}{\varepsilon}\|u\|_{L_p(B)}. \tag{31}$$

It's evident that it suffices to prove lemma for functions $u(x)\in C^\infty(\overline{B})$. We'll fix arbitrary $\varepsilon>0$ and let $\varepsilon_1>0$ be a number, which will be chosen later. According to lemma 10 and inequality (31)

$$\|u\|_{W_p^2(B_2)}\le C_4\|\mathrm{L}u\|_{L_p(B)}+\frac{4C_9}{R^2}\|u\|_{W_p^2(B)}\le C_4\|\mathrm{L}u\|_{L_p(B)}+\frac{4C_9\varepsilon_1}{R^2}\|u\|_{W_p^2(B)}+\frac{4C_9C_{19}}{\varepsilon_1 R^2}\|u\|_{L_p(B)},$$

where $C_{19}=\sup\limits_{p\in[p_1,2]}C_{18}(p,n)$

Then it's sufficient to choose $\varepsilon_1=\dfrac{\varepsilon R^2}{4C_9}$, and the lemma is proved.

Denote for any $\rho>0$ set $\{x:x\in D, dist(x,\partial D)>\rho\}$ by D_ρ.

Lemma 12. *Let conditions (3), (16), (18) and (19) be satisfied with respect to the coefficients of the operator* L. *Then for any* $u(x)\in W_p^2(B)$, *any* $\varepsilon>0$, $\rho>0$ *and* $p\in[p_1,2]$ *the following estimation holds*

$$\|u\|_{W_p^2(D_\rho)}\le C_{20}(\gamma,\sigma,n,\rho,\mathrm{B},D)\|\mathrm{L}u\|_{L_p(D)}+\varepsilon\|u\|_{W_p^2(D)}+\frac{C_{21}(\gamma,\sigma,n,\rho,\mathrm{B},D)}{\varepsilon}\|u\|_{L_p(D)}.$$

Proof. We restrict ourselves with the case $u(x)\in C^\infty(\overline{B})$. Besides this, without loss of generality, we'll assume, that $\rho\le R_0$. We'll fix arbitrary $\varepsilon>0$ and let $\varepsilon_2>0$ be a number, which will be chosen later. We'll cover $\overline{D}_\rho$ by the system of balls $\left\{B_{\frac{\rho}{2}}(x_\nu)\right\}$ and choose from this cover a finite subcover $B^1,...,B^N$. It's evident that number N depends only on ρ,n and $diam\,D$. Applying for each $i=1,...,n$ lemma 11, we obtain

$$\|u\|^p_{W_p^2(B^i)}\le 3^{p-1}\left(C_4^p\|\mathrm{L}u\|^p_{L_p(D)}+\varepsilon_2^p\|u\|^p_{W_p^2(D)}+\frac{C_{17}^p}{\varepsilon_2^p\rho^{4p}}\|u\|^p_{L_p(D)}\right). \tag{32}$$

Summing inequalities (32) over i from 1 to N, we conclude

$$\|u\|^p_{W_p^2(D_\rho)}\le 3^{p-1}N\left(C_4^p\|\mathrm{L}u\|^p_{L_p(D)}+\varepsilon_2^p\|u\|^p_{W_p^2(D)}+\frac{C_{17}^p}{\varepsilon_2^p\rho^{4p}}\|u\|^p_{L_p(D)}\right).$$

Now it's sufficient to choose $\varepsilon_2=\dfrac{\varepsilon}{3N}$, and the lemma is proved.

3.The basic coercive estimation. The statement of lemma 12 is true without any requirements respective to the boundary ∂D. All following statements of the present paper holds under condition $\partial D \in C^2$, which we'll always assume to be satisfied.

Lemma 13. *Let with respect to the coefficients of the operator* L *conditions (3), (16), (18) and (19) be fulfilled. Then there exist positive constants* ρ_1, C_{22} *and* C_{23}, *depending only on* $\gamma, \sigma, n, \mathrm{B}, c$ *and domain D such that for any* $u(x) \in \dot{W}_p^2(D)$, *any* $\varepsilon > 0$ *and* $p \in [p_1, 2]$ *the following estimation holds*

$$\|u\|_{W_p^2(D/D_\rho)} \le C_{22} \|\mathrm{L}u\|_{L_p(D)} + \varepsilon \|u\|_{W_p^2(D)} + \frac{C_{23}}{\varepsilon} \|u\|_{L_p(D)}.$$

Proof. It's sufficient to prove lemma for functions $u(x) \in C^\infty(\overline{D})$, $u|_{\partial D} = 0$. Besides this, without loss of generality, we'll assume that coefficients of the operator L are infinitely differentiable in $\overline{D}$. We'll fix arbitrary $\varepsilon > 0$ and point $x^0 \in \partial D$. Let's make the orthogonal coordinate transformation $x \to y$ such that tangent hyperplane to $\partial\widetilde{D}$ at the point y^0 be perpendicular to Oy_n axis. Here $\widetilde{D}$ and y^0 are images of domain D and point x^0 at such transformation, respectively. Denote by the $\widetilde{u}(y)$ image of the function $u(x)$. For simplicity we'll assume that equation of $\partial\widetilde{D}$ in intersection of $\partial\widetilde{D}$ with some neighborhood O_h of the point y^0 is given by equation $y_n = \varphi(y_1, ..., y_{n-1})$ with twice continuously differentiable function φ, and a part of $\widetilde{D}$ adjacent to $\partial\widetilde{D} \cap O_h$ belongs to the set $\{y : y_n > \varphi(y_1, ..., y_{n-1})\}$. Let $A(x) = \|a_{ij}(x)\|$ be the matrix of leading coefficients of the operator L, $\widetilde{A}(y) = \|\widetilde{a}_{ij}(y)\|$, where $\widetilde{a}_{ij}(y)$ are leading coefficients of image $\widetilde{\mathrm{L}}$ of the operator L at our transformation, $i, j = 1, ..., n$. We'll show now that eigenvalues of matrices A and $\widetilde{A}$ coincide. Really, we'll fix arbitrary point $x \in D$,and let λ be an arbitrary eigenvalue of the matrix A, and x^λ is corresponding eigenvector. By virtue of orthogonality of our transformation there exists a non-degenerated matrix T such that $\widetilde{A} = T^{-1}AT$. Denote $T^{-1}x^\lambda$ by y^λ. We have

$$\widetilde{A}y^\lambda = T^{-1}Ax^\lambda = \lambda T^{-1}x^\lambda = \lambda y^\lambda.$$

On the other hand, condition (16) can be written in the following form

$$\sigma = \sup_D \frac{\sum_{i=1}^n \lambda_i^2(x)}{\left[\sum_{i=1}^n \lambda_i(x)\right]^2} < \frac{1}{n-1},$$

where $\lambda_i(x)$ are eigenvalues of the matrix $A(x)$; $i = 1, ..., n$. So, condition (16) is satisfied also for the operator $\widetilde{\mathrm{L}}$ with the same constant σ. Analogously we can show that for the operator $\widetilde{\mathrm{L}}$ conditions (3) (with the same constant γ) and also (18)-(19) are satisfied. Let's make one more transformation $z_i = y_i; i = 1, ..., n-1$; $z_n = y_n - \varphi(y_1, ..., y_{n-1})$. Let L', D' and z^0 be images of the operator L, domain $\widetilde{D}$ and point y^0 at our transformation, respectively, and $a'_{ij}(z)$ be leading coefficients of the operator L'; $i,j=1,...,n$.

It's easy to see that

$$a'_{ij}(z)=\sum_{k,l=1}^{n}\tilde{a}_{kl}(y)\frac{\partial z_i}{\partial y_k}\frac{\partial z_j}{\partial y_l};\, i,j=1,...,n.$$

Therefore,

$$a'_{ij}(z)=\tilde{a}_{ij}(y);\text{ if } 1\le i,j\le n-1;$$

$$a'_{nj}(z)=-\sum_{k=1}^{n-1}\tilde{a}_{kj}(y)\frac{\partial\varphi}{\partial y_k}+\tilde{a}_{nj}(y);\text{ if } 1\le j\le n-1;$$

$$a'_{nn}(z)=\sum_{k,l=1}^{n-1}\tilde{a}_{kl}(y)\frac{\partial\varphi}{\partial y_k}\frac{\partial\varphi}{\partial y_l}-2\sum_{k=1}^{n-1}\tilde{a}_{nk}(y)\frac{\partial\varphi}{\partial y_k}+\tilde{a}_{nn}(y).$$

Since $\frac{\partial\varphi}{\partial y_i}\left(y^0\right)=0$ for $i=1,...,n-1$, then there exists $h_1\left(y^0,\varphi\right)$ such that for $h\le h_1$ in intersection $D'\cap B_h\left(z^0\right)$ condition (16) with the constant $\sigma'=\frac{\sigma+\frac{1}{n-1}}{2}$ is fulfilled. Besides this, for the operator L' in the stated intersection conditions (3) (with constant $\frac{\gamma}{2}$), and also (18)-(19) be fulfilled. At that, if $b'_i(x)$; $i=1,...,n$; and $c'(x)$ are the coefficients at first derivatives of solution and at solution itself of the operator L', then the value $\sum_{i=1}^{n}\|b'_i\|_{L_n\left(D'\cap B_{h_1}(z0)\right)}+\|c'\|_{L_{\frac{n}{2}}\left(D'\cap B_{h_1}(z0)\right)}$ is bounded from above by a constant, dependent only on $\sum_{i=1}^{n}\|b_i\|_{L_n(D)}+\|c\|_{L_{\frac{n}{2}}(D)}$ and function φ. Suppose $r=r\left(z^0\right)=\min\{h_1,R_0\}$, and let $u'(z)$ be an image of the function $\tilde{u}(y)$ under such a transformation. It's clear that in variables z the intersection $D'\cap B_r\left(z^0\right)$ a semiball $B_r^+=\left\{z:\left|z-z^0\right|<r,z_n>0\right\}$. We continue the function $u'(z)$ to get an odd function and coefficients of the operator $\tilde{L}$ -to get an even function through the hyperplane $z_n=0$ in $B_r\left(z^0\right)\setminus B_r^+$ and denote the obtained function and operator again by $u'(z)$ and $\tilde{L}$ respectively. Since $u'(z)\in W_p^2\left(B_r\left(z^0\right)\right)$, then according to lemma 11

$$\|u'\|_{W_p^2\left(B_{\frac{r}{2}}\left(z^0\right)\right)}\le C_4\|L'u'\|_{L_p\left(B_r\left(z^0\right)\right)}+\varepsilon\|u'\|_{W_p^2\left(B_r\left(z^0\right)\right)}+\frac{C_{17}}{\varepsilon_3 r^4}\|u'\|_{L_p\left(B_r\left(z^0\right)\right)}, \tag{33}$$

where $\varepsilon_3>0$ will be chosen later. But on the other hand, each of norms in the right hand side of (33) is a corresponding norm taken by the semiball B_r^+ and multiplied by $2^{\frac{1}{p}}$. Therefore, from (33) we conclude

$$\|u'\|_{W_p^2\left(B_{\frac{r}{2}}^+\right)}\le C_4\|L'u'\|_{L_p\left(B_r^+\right)}+\varepsilon\|u'\|_{W_p^2\left(B_r^+\right)}+\frac{C_{17}}{\varepsilon_3 r^4}\|u'\|_{L_p\left(B_r^+\right)}. \tag{34}$$

We'll cover $\partial D'$ by the system of balls $\left\{ B_{\frac{r(z^\nu)}{2}}(z^\nu) \right\}$ and choose from this cover a finite subcover $B^{(1)},...,B^{(M)}$. At that number M is defined only by values γ, σ, n, functions $b_i(x); i=1,...,n; c(x)$ and domain D. Coping out inequalities of the type (34) for each semiball $B^+_{r(z^\nu)}$; $\nu = 1,...,M$; raising the both parts of the obtained inequalities to a power p and summing by ν from 1 to M, we obtain

$$\|u'\|^p_{W^2_p(\mathrm{B})} \le 3^{p-1} M \left(C_4^p \|\mathrm{L}'u'\|^p_{L_p(D')} + \varepsilon_3^p \|u'\|_{W^2_p(D')} + \frac{C_{17}^p}{\varepsilon_3^p r_6^{4p}} \|u'\|_{L_p(D')} \right),$$

where $\mathrm{B} = \bigcup_{\nu=1}^{M} B^+_{\frac{r(z^\nu)}{2}}$ and $r_0 = \min\{r(z^1),...,r(z^M)\}$. Returning to the variables x and noting that image B contains the set $D \setminus D_{\rho_1}$ with some $\rho_1 = (\gamma, \sigma, n, \mathbf{B}, c, D)$, we conclude that

$$\|u\|_{W^2_p(D \setminus D_{\rho_1})} \le C_{24} \|\mathrm{L}u\|_{L_p(D)} + C_{25}\varepsilon_3 \|u\|_{W^2_p(D)} + \frac{C_{26}}{\varepsilon_3} \|u\|_{L_p(D)},$$

where constants C_{24}, C_{25} and C_{26} depend only on $\gamma, \sigma, n, \mathbf{B}, c$ and domain D. Now it's sufficient to choose $\varepsilon_3 = \frac{\varepsilon}{C_{25}}$, and the lemma is proved.

Later on the notation $C(\mathrm{L}, n, D)$ means that positive constant C depends on abovementioned parameters.

From lemmas 12 and 13 follows

Theorem 1. *Let with respect to coefficients of the operator* L *conditions (3), (16), (18) and (19) be fulfilled. Then for any function* $u(x) \in \dot{W}^2_p(D)$, *any* $p \in [p_1, 2]$ *the following estimation holds*

$$\|u\|_{W^2_p(D)} \le C_{27}(\mathrm{L}, n, D)\left(\|\mathrm{L}u\|_{L_p(D)} + \|u\|_{L_p(D)} \right).$$

We'll prove now the coercive estimation for the operator L in small measure domains.

Theorem 2. *If conditions of the theorem 1 are satisfied, then there exists a constant* $d(\mathrm{L}, n, D)$ *such that when* $mes\, D \le d$ *for any function* $u(x) \in \dot{W}^2_p(D)$ *and for any* $p \in [p_1, 2]$ *the following estimation holds*

$$\|u\|_{W^2_p(D)} \le C_{28} \|\mathrm{L}u\|_{L_p(D)}, \tag{35}$$

where $C_{28} = 2C_{27}$.

Proof. Let constant C_6 have the same meaning as in lemma 9. We'll use inequality (20) at $q = p$. We have

$$\|u\|_{L_p(D)} \le (mes\, D)^{\frac{1}{n}} \|u\|_{L_{\frac{nq}{n-q}}(D)} \le C_6 d^{\frac{1}{n}} \sum_{i=1}^{n} \|u_i\|_{L_p(D)} \le C_6 d^{\frac{1}{n}} \|u\|_{W^2_p(D)}.$$

Now it's sufficient to choose d from condition $C_6 C_{27} d^{\frac{1}{n}} = \frac{1}{2}$, and the required estimation (35) follows from theorem 1. The theorem is proved.

Theorem 3. *If conditions of the theorem 1 are satisfied, then there exists constant* $\mu_0(\mathrm{L}, n, D)$ *such that for any function* $u(x) \in \dot{W}_p^2(D)$, *any* $\mu \ge \mu_0$ *and* $p \in [p_1, 2]$ *the following estimation holds*

$$\|u\|_{W_p^2(D)} \le C_{29}(\mathrm{L}, n, D) \|\mathrm{L}u - \mu u\|_{L_p(D)}.$$

Proof. In $(n+1)$- dimensional Euclidean space $\mathbf{E}_{n+1}$ of points (x,t) consider a cylindrical domain $Q_{T_0} = D \times (0, T_0)$ and in it the operator $\mathrm{L}_1 = \mathrm{L} + \frac{\partial^2}{\partial t^2}$. We'll choose number T_0 from condition $mes_{n+1} Q_{T_0} = d$. At that the constant d of the previous theorem corresponds to the operator L_1, the dimension of space $n+1$ and domain Q_{T_0}. Denote $\frac{\pi^2}{T_0^2}$ by μ_0, and let for $\mu \ge \mu_0$ number $T \le T_0$ be such that $\frac{\pi^2}{T^2} = \mu$. If $u(x) \in \dot{W}_p^2(D)$, then function $\vartheta(x,t) = u(x) \sin \frac{\pi t}{T}$ is an element of the space $\dot{W}_p^2(Q_T)$. According to theorem 2

$$\|\vartheta\|_{W_p^2(Q_T)} \le C_{29} \|\mathrm{L}_1 \vartheta\|_{L_p(Q_T)},$$

where constant C_{29} coincides with constant C_{28} taken for the operator L_1, dimension of space $n+1$ and domain Q_{T_0}. But $\mathrm{L}_1 \vartheta = \sin \frac{\pi t}{T} (\mathrm{L}u - \mu u)$. Therefore,

$$\int_0^T \left| \sin \frac{\pi t}{T} \right|^p dt \int_D \left(|u|^p + \sum_{i=1}^n |u_i|^p + \sum_{i,j=1}^n |u_{ij}|^p \right) dx \le C_{29}^p \int_0^T \left| \sin \frac{\pi t}{T} \right|^p dt \int_D |\mathrm{L}u - \mu u|^p dx,$$

and statement of the theorem is proved.

4. Case $p > 2$. Let $p \in \left[2, \frac{5}{2}\right]$, and $K(p)$ have the same meaning as in lemma 3. By Riesz-Thorin theorem for any $p \in \left[2, \frac{5}{2}\right]$

$$K(p) \le (K(2))^{1-\theta} \left(K\left(\frac{5}{2}\right) \right)^\theta = \left(K\left(\frac{5}{2}\right) \right)^\theta,$$

where $\theta = \frac{2(p-2)}{p\left(\frac{5}{2} - 2\right)}$. Denoting $\max\left\{ \frac{5}{2}, \left(K\left(\frac{5}{2}\right) \right)^2 \right\}$ by $a_1(n)$, we obtain

$$K(p) \le a_1^{p-2}.$$

Thus, the following analogue of lemma 3 is valid.

Lemma 14. *If* $u(x) \in \overset{\circ}{W}{}_p^2(B)$, *then for any* $p \in \left[2, \frac{5}{2}\right]$

$$\|u\|_{\overset{\circ}{W}{}^2_p(B)} \le a_1^{p-2}\|u\|_{\overset{\circ}{V}{}^2_p(B)}.$$

The analogues of lemma 4 and 5 are proved quite similarly.

Lemma 15. *For any* $p\in\left[2,\frac{5}{2}\right]$ *the following estimation holds*

$$\delta_p \le n^{\frac{p-2}{p}}\delta .$$

Lemma 16. *Let* $\delta<1$. *Then there exists* $p_2(\gamma,\delta,n)\in\left(2,\frac{5}{2}\right]$ *such that for all* $p\in[2,p_2]$

$$a_1^{2-p}\delta_p \le \delta^{\frac{1}{3}}.$$

We'll impose the following restrictions on the minor coefficients of the operator L for $p\in[2,p_2]$

$$b_i(x)\in L_q(D);\, i=1,...,n\,;\ q=\begin{cases} n; & if\ p\ne n\\ 2+\nu_3; & if\ p=n=2,\end{cases} \tag{36}$$

$$c(x)\in L_l(D),\, l=\begin{cases} \max\left\{p,\frac{n}{2}\right\}; & if\ p\ne \frac{n}{2}\\ 2+\nu_4; & if\ p=2, n=4,\end{cases} \tag{37}$$

where ν_3 and ν_4 are some positive constants.

Using the scheme developed in lemmas 6-13 and taking into account lemmas 14-16 we make sure in validity of theorems 1-3 for $p\in[2,p_2]$ and $u(x)\in\dot W_p^2(D)$ if with respect to coefficients of the operator L conditions (3), (16), (36) and (37) are fulfilled. We'll combine conditions (18) and (36), assuming that $p\in[p_1,p_2]$ or, namely, we'll assume that minor coefficients $b_i(x)$ of the operator L satisfy the condition

$$b_i(x)\in L_\theta(D);\, i=1,...,n\,;\ \theta=\begin{cases} \max\{p,n\}; & if\ p\ne n\\ 2+\nu_3; & if\ p=n=2\end{cases} \tag{38}$$

Theorem 4. *Let with respect to coefficients of the operator* L *conditions (3), (16), (37) and (38) be fulfilled .Then there exist constants* $d_1(\mathsf{L},n,D)$, $\mu_1(\mathsf{L},n,D)$, $C_{30}(\mathsf{L},n,D)$ *and* $C_{31}(\mathsf{L},n,D)$ *such that for any function* $u(x)\in\dot W_p^2(D)$ *and for any* $p\in[p_1,p_2]$ *the following estimations are valid*

$$\|u\|_{W_p^2(D)} \le C_{30}\|\mathsf{L}u\|_{L_p(D)},$$

if $mesD\le d_1$, *and*

$$\|u\|_{W_p^2(D)} \le C_{31}\|\mathsf{L}u-\mu u\|_{L_p(D)},$$

if $\mu\ge\mu_1$.

5. Solvability of the first boundary value problem. Consider now the first boundary value problem (1)-(2) and also the problem

$$\mathsf{L}u-\mu u=f(x);\ x\in D, \tag{39}$$

$$u\big|_{\partial D}=0, \tag{40}$$

assuming that $\mu\ge\mu_1$.

Theorem 5. *Let in domain D the coefficients of the operator* L *be given, satisfying the conditions (3), (16), (37) and (38). Then if* $mesD\le d_1$ *(*$\mu\ge\mu_1$*), then first boundary value problem (1)-(2) ((39)-(40)) is uniquely strongly solvable in the space* $\dot W_p^2(D)$ *for any* $f(x)\in L_p(D)$, $p\in[p_1,p_2]$. *At that for solution* $u(x)$ *the following estimation is valid*

$$\|u\|_{W_p^2(D)}\le C_{30}(C_{31})\|f\|_{L_p(D)}. \tag{41}$$

Proof. We'll prove the theorem by the method of continuation by parameter, restricting ourselves with the case of the boundary value problem (1)-(2). We introduce for $t\in[0,1]$ the family of operators $\mathsf{L}_t=t\mathsf{L}+(1-t)\Delta$. It's easy to see that conditions (3) and (16) are fulfilled for the operators L_t with constants γ and σ respectively. We'll show this on example of condition (16). According to lemma 6 the last is equivalent to condition $\delta<1$. Let $a_{ij}^t(x)$ be leading coefficients of the operator L_t; $i,j=1,...,n$, and

$\delta^t=\sup\limits_D\left(\sum\limits_{i,j=1}^n\left[a_{ij}^t(x)-\delta_{ij}\right]^2\right)^{\frac{1}{2}}$. We have

$$\delta^t=\sup_D\left(\sum_{i,j=1}^n\left[ta_{ij}(x)+(1-t)\delta_{ij}-\delta_{ij}\right]^2\right)^{\frac{1}{2}}=t\sup_D\left(\sum_{i,j=1}^n\left[a_{ij}(x)-\delta_{ij}\right]^2\right)^{\frac{1}{2}}=t\delta\le\delta .$$

Besides this, if $b_i^t(x)$ and $c^t(x)$ ($i,j=1,...,n$) are minor coefficients of the operator L_t, then the value $\sum\limits_{i=1}^n\left\|b_i^t\right\|_{L_\theta(D)}+\left\|c^t\right\|_{L_l(D)}$ is estimated from above by constant dependent only on $\sum\limits_{i=1}^n\left\|b_i\right\|_{L_\theta(D)}+\left\|c\right\|_{L_l(D)}$. Hence, it follows that the statement of theorem 4 is valid for the operator L_t with the constant C_{30} independent on t. Denote by A a set of all points of the segment $[0,1]$, for which the problem

$$\mathsf{L}_t u=f(x);\ x\in D,\ u\in\dot W_p^2(D) \tag{42}$$

has a solution. At once note that by virtue of theorem 4 this solution is unique. We'll show now that the set A is non-empty and open and closed simultaneously with respect to $[0,1]$. Then A coincides with the segment $[0,1]$, and , in particular, problem (42) is uniquely solvable at $t=1$, when $\mathsf{L}_1=\mathsf{L}$. At that estimation (41) follows from theorem 4. Nonemptiness of the set A follows from the fact, that problem (42) is solvable at $t=0$, when $\mathsf{L}_0=\Delta$ (see [2]). We'll prove that set A is open with respect to $[0,1]$. Let $t^0\in\mathsf{A}$, $t\in[0,1]$ be such that $\left|t-t^0\right|<\alpha$, where $\alpha>0$ will be chosen later. We'll represent problem (42) in the following form

$$\mathsf{L}_{t^0}u=f(x)+\left(\mathsf{L}_{t^0}-\mathsf{L}_t\right)u;\ x\in D,\ u\in\dot W_p^2(D). \tag{43}$$

It's easy to see that $\mathsf{L}_{t^0}-\mathsf{L}_t=\left(t^0-t\right)(\mathsf{L}-\Delta)$. Consider auxiliary problem

$$\mathsf{L}_{t^0}u=f(x)+\left(t^0-t\right)(\mathsf{L}-\Delta)\vartheta;\ x\in D,\ u\in\dot W_p^2(D), \tag{44}$$

where $\vartheta(x)\in \dot{W}_p^2(D)$. Proceeding similarly as in lemma 9, it can be shown that

$$\|(\mathrm{L}-\Delta)\vartheta\|_{L_p(D)} \le C_{32}(\mathrm{L},n,\mathbf{B},c)\|\vartheta\|_{W_p^2(D)}.$$

Thus, the operator Υ mapping each function $\vartheta(x)\in \dot{W}_p^2(D)$ to the solution $u(x)$ of problem (44) (i.e. $u=\Upsilon\vartheta$) is defined.

We'll show that in corresponding way chosen α mapping Υ is contraction. Let $u^1=\Upsilon\vartheta^1$, $u^2=\Upsilon\vartheta^2$. We have

$$\mathrm{L}_{t^0}\left(u^1-u^2\right)=\left(t^0-t\right)(\mathrm{L}-\Delta)\left(\vartheta^1-\vartheta^2\right);\ u^1-u^2\in \dot{W}_p^2(D).$$

Then according to theorem 4

$$\left\|u^1-u^2\right\|_{W_p^2(D)}=C_{30}\alpha C_{32}\left\|\vartheta^1-\vartheta^2\right\|_{W_p^2(D)},$$

and it's sufficient to choose $\alpha=\dfrac{1}{2C_{30}C_{32}}$. Then operator Υ has a fixed point $u=\Upsilon u$. But for $u=\vartheta$ problem (44) coincides with problem (43), i.e. with (42). Openness of the set A is proved. We'll prove now it's closeness. Let $t^m\in \mathrm{A}$; $m=1,2,...;t^0=\lim\limits_{m\to\infty} t^m$. We'll show that $t^0\in \mathrm{A}$. Denote by $u^m(x)$ a solution of boundary value problem

$$\mathrm{L}_{t^m}u^m=f(x);\ x\in D,\ u^m\in \dot{W}_p^2(D).$$

Then according to theorem 4

$$\left\|u^m\right\|_{W_p^2(D)}\le C_{30}\|f\|_{L_p(D)}.$$

Thus, sequence $\{u^m(x)\}$ is bounded by the norm of $\dot{W}_p^2(D)$. Hence, it follows that it is weakly compact, i.e. there exist sequence $m_k\to\infty$ as $k\to\infty$ and the function $u(x)\in \dot{W}_p^2(D)$ such that $u(x)$ is a weak limit of the sequence $\{u^{m_k}(x)\}$ as $k\to\infty$. From here, in particular, it follows that for any function $\varphi(x)\in C^\infty(\overline{D})$

$$\left\langle \mathrm{L}_{t^0}u^{m_k},\varphi\right\rangle\to\left\langle \mathrm{L}_{t^0}u,\varphi\right\rangle\ (k\to\infty),$$

where $\langle u,\vartheta\rangle=\int\limits_D u\vartheta dx$. But

$$\left\langle \mathrm{L}_{t^0}u^{m_k},\varphi\right\rangle=\left\langle\left(\mathrm{L}_{t^0}-\mathrm{L}_{t^{m_k}}\right)u^{m_k},\varphi\right\rangle+\left\langle \mathrm{L}_{t^0}u^{m_k},\varphi\right\rangle=i_1+i_2.$$

We have

$$|i_1|\le\left|t^0-t^{m_k}\right|\left|\left\langle(\mathrm{L}-\Delta)u^{m_k},\varphi\right\rangle\right|\le\left|t^0-t^{m_k}\right|C_{33}(\varphi,p)C_{32}\left\|u^{m_k}\right\|_{W_p^2(D)}\le$$
$$\le C_{30}C_{32}C_{33}\left|t^0-t^{m_k}\right|\|f\|_{L_p(D)}.$$

Thus, $i_1\to 0$ as $k\to\infty$. On the other hand, $i_2=\langle f,\varphi\rangle$. Thus, for any function $\varphi(x)\in C^\infty(\overline{D})$

$$\left\langle \mathrm{L}_{t^0}u,\varphi\right\rangle=\langle f,\varphi\rangle.$$

This means that $\mathrm{L}_{t^0}u=f(x)$ a.e. in D, i.e. $t^0\in \mathrm{A}$. The theorem is proved.

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