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INVESTIGATION OF GENERALIZED SOLUTION OF ONE NONSELF-ADJOINT ONE-DIMENSIONAL MIXED PROBLEM FOR A CLASS OF SEMI-LINEAR PSEUDO-HYPERBOLIC EQUATIONS OF THE FOURTH ORDER

Abstract

At the paper one nonself-adjoint one dimensional mixed problem was investigated for one class of semi-linear pseudo-hyperbolic equations of the fourth order. The idea of generalized solution of considering mixed problem is introduced. By view of Bellman inequality the theorem on uniqueness on the whole of generalized solution was proved. Further, by the combination of generalized principle of condensed mappings by Shauder principle on fixed point the theorem of existence in a little of generalized solution was proved.

Besides by view of strong Shauder principle on fixed point the theorem on existence on the whole of generalized solution of considering mixed problem was proved.

The paper is devoted to the investigation of the existence and uniqueness of generalized solution of the following nonself-adjoint one-dimensional mixed problem for a class of semi-linear pseudo-hyperbolic equations of the fourth order

$$\begin{cases} u_{tt}(t,x)+u_{xxxx}(t,x)-\alpha\cdot u_{ttxx}(t,x)= \\ =F\left(t,x,u(t,x),u_t(t,x),u_x(t,x),u_{tx}(t,x),u_{xx}(t,x),u_{txx}(t,x),u_{xxx}(t,x)\right)\ (0\le t\le T, 0\le x\le 1), & (1)\\ u(0,x)=\varphi(x)\ (0\le x\le 1),\ u_t(0,x)=\psi(x)\ (0\le x\le 1), & (2)\\ u(t,0)=0,\ u_x(t,0)=u_x(t,1),\ u_{xx}(t,0)=0,\ u_{xxx}(t,0)=u_{xxx}(t,1)\ (0\le t\le T), & (3)\end{cases}$$

where $\alpha>0$ is a fixed number; $0<T<+\infty$; F, φ, ψ are given functions, and $u(t,x)$ is the desired function, moreover under the generalized solution of the problem (1)-(3) we understand the following.

Definition. *Under the generalized solution of the problem (1)-(3) we understand the function $u(t,x)$ having the properties:*

a) $u(t,x),\ u_x(t,x),\ u_{xx}(t,x), u_t(t,x), u_{tx}(t,x),\ u_{tt}(t,x)\in C([0,T]\times[0,1])$,
$u_{xxx}(t,x),\ u_{txx}(t,x),\ u_{ttx}(t,x)\in C([0,T];L_2(0,1))$;

b) both initial conditions (2) and the first three from the boundary conditions (3) are satisfied in usual meaning;

c) the integral identity

$$\int_0^T\int_0^1\left\{u_{tt}(t,x)V(t,x)-u_{xxx}(t,x)V_x(t,x)+\alpha\cdot u_{ttx}(t,x)V_x(t,x)-\mathsf{F}(u(t,x))\cdot V(t,x)\right\}dxdt=0 \quad (4)$$

is fulfilled for any function $V(t,x)$ having the properties

$$V(t,x)\in C([0,T]\times[0,1]),\ V_x(t,x)\in L([0,T];L_2(0,1)),\ V(t,0)=V(t,1)\ \forall t\in[0,T], \quad (5)$$

where

$$\mathsf{F}(u(t,x))\equiv F\left(t,x,u(t,x),u_t(t,x),u_x(t,x),u_{tx}(t,x),u_{xx}(t,x),u_{txx}(t,x),u_{xxx}(t,x)\right). \quad (6)$$

First of all let's denote the following works in some sense connected with the given paper. At the paper [1] the problem

$$\begin{cases} u_{tt}(t,x)+u_{xxxx}(t,x)-\alpha\cdot u_{ttxx}(t,x)=F(t,x,u(t,x))\ (0\le t\le T,\ 0\le x\le\pi), & (7)\\ u(0,x)=\varphi(x)\ (0\le x\le\pi),\ u_t(0,x)=\psi(x)\ (0\le x\le\pi), & (8)\\ u(t,0)=u(t,\pi)=u_{xx}(t,0)=u_{xx}(t,\pi)=0\ (0\le t\le T) & (9)\end{cases}$$

is considered, where $\alpha>0$; the theorem on the existence and uniqueness of a generalized solution of the problem (7)-(9) is constructed and the continuous dependence (in a certain sense) of the generalized solution of the problem (7)-(9) from the parameter α is studied.

And at paper [2] the existence of ω periodic solution of the problem

$$\begin{cases} u_{tt}(t,x)+F(u_{xx}(t,x))_{xx}-a\cdot u_{ttxx}(t,x)=g(t,x), & (10)\\ u(t,0)=u(t,\pi)=u_{xx}(t,0)=u_{xx}(t,\pi)=0 & (11)\end{cases}$$

is proved, where $\alpha\ge 0$ is a given constant; g is a given function, ω is periodic by t; F is Ishlinski operator. Let's denote also that the equation (10) describes the forced transverse vibration of an elastico-plastic pivot.

§1. Auxiliary facts.

Let's cite some known facts to investigate the problem (1)-(3).

1. The next lemma is true.

Lemma 1 (see [3], p.297). *The sequences of the functions*

$$X_0(x)=x,\ ...,\ X_{2k-1}(x)=x\cos 2\pi kx,\ X_{2k}(x)=\sin 2\pi kx,\ ..., \tag{12}$$

and

$$Y_0(x)=2,\ ...,\ Y_{2k-1}(x)=4\cos 2\pi kx,\ Y_{2k}(x)=4(1-x)\sin 2\pi kx,\ ... \tag{13}$$

form in $L_2(0,1)$ *biorthogonal system of functions.*

At the given paper essentially the following is used.

Theorem 1 (see [3], p.298-299). *The sequence of the functions (12), i.e. the system*

$$x,\ x\cos 2\pi x,\ \sin 2\pi x,\ ...,\ x\cos 2\pi kx,\ \sin 2\pi kx,\ ... \tag{14}$$

forms the basis at the space $L_2(0,1)$ *and for any function* $L_2(0,1)$ *the estimations*

$$\frac{3}{4}\|f(x)\|^2_{L_2(0,1)}\le\sum_{k=0}^{\infty}f_k^2\le 16\|f\|^2_{L_2(0,1)}, \tag{15}$$

where

$$f_k\equiv\int_0^1 f(x)Y_k(x)dx\ \ (k=0,1,...) \tag{16}$$

are true.

2. Since by theorem 1 the system (12) forms a basis in $L_2(0,1)$ and by lemma 1 the system (12) and (13) form in $L_2(0,1)$ a biorthogonal system functions then it is evident that every generalized solution $u(t,x)$ of the problem (1)-(3) has the form:

$$u(t,x)=\sum_{k=0}^{\infty}u_k(t)X_k(x), \tag{17}$$

where

$$u_k(t)=\int_0^1 u(t,x)Y_k(x)dx\qquad (k=0,1,...). \tag{18}$$

After applying the formal scheme of Fourier method for finding the functions $u_k(t)$ $(k=0,1,...)$ is led to the solution of the following countable system of non-linear integro-differentiable equations:

$$u_0(t)=\varphi_0+\psi_0\cdot t+\int_0^t\int_0^1(t-\tau)\mathrm{F}(u(\tau,x))Y_0(x)dxd\tau\ (t\in[0,T]), \tag{19}$$

$$u_{2k-1}(t)=\varphi_{2k-1}\cos\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t+\psi_{2k-1}\frac{\sqrt{1+4\alpha\pi^2k^2}}{4\pi^2k^2}\cdot\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t+$$
$$+\frac{1}{4\pi^2k^2\sqrt{1+4\alpha\pi^2k^2}}\int_0^t\int_0^1\mathrm{F}(u(\tau,x))Y_{2k-1}(x)\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}(t-\tau)dxd\tau$$
$$(k=1,2,...;t\in[0,T]), \tag{20}$$

$$u_{2k}(t)=-\varphi_{2k-1}\frac{4\pi k\cdot\left(1+2\alpha\pi^2k^2\right)}{\left(1+4\alpha\pi^2k^2\right)^{\frac{3}{2}}}t\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t+\varphi_{2k}\cos\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t-$$
$$-\psi_{2k-1}\frac{1+2\alpha\pi^2k^2}{\pi k\left(1+4\alpha\pi^2k^2\right)}\left\{\frac{\sqrt{1+4\alpha\pi^2k^2}}{4\pi^2k^2}\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t-t\cdot\cos\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}\right\}+$$
$$+\psi_{2k}\cdot\frac{\sqrt{1+4\alpha\pi^2k^2}}{4\pi^2k^2}\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t-\frac{\alpha}{\pi k\left(1+4\alpha\pi^2k^2\right)^{\frac{3}{2}}}\cdot\int_0^t\int_0^1\mathrm{F}(u(\tau,x)Y_{2k-1}(x))\times$$
$$\times\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}(t-\tau)dxd\tau-\frac{2\left(1+2\alpha\pi^2k^2\right)}{\pi k\left(1+4\alpha\pi^2k^2\right)^2}\int_0^t\left[\int_0^\tau\int_0^1\mathrm{F}(u(\sigma,x))Y_{2k-1}(x)\times\right.$$
$$\left.\times\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}(\tau-\sigma)dxd\sigma\right]\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}(t-\tau)d\tau+\frac{1}{4\pi^2k^2\sqrt{1+4\alpha\pi^2k^2}}\times$$
$$\times\int_0^t\int_0^1\mathrm{F}(u(\tau,x))Y_{2k}(x)\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}(t-\tau)dxd\tau\quad(k=1,2,...;t\in[0,T]), \tag{21}$$

where

$$\varphi_k\equiv\int_0^1\varphi(x)Y_k(x)dx\,,\ \ \psi_k\equiv\int_0^1\psi(x)Y_k(x)dx\quad(k=0,1,...), \tag{22}$$

and the operator F is determined by the relation (6).

Further, starting from the determination of generalized solution of the problem (1)-(3), the following is proved.

Lemma 2. *If $u(t,x)=\sum_{k=0}^{\infty}u_k(t)X_k(x)$ is any generalized solution of the problem (1)-(3), then the function $u_k(t)=\int_0 u(t,x)Y_k(x)dx$ $(k=0,1,...)$ satisfy on $[0,T]$ the system (19)-(21).*

3. At suppositions

$$\varphi(x)\in\mathrm{C}^{(2)}([0,1]),\ \varphi'''(x)\in L_2(0,1),\ \varphi(0)=0\,,\ \varphi'(0)=\varphi'(1)\,,\ \varphi''(0)=0\,, \tag{23}$$

$$\psi(x)\in\mathrm{C}^{(1)}([0,1]),\ \psi''(x)\in L_2(0,1),\ \psi(0)=0\,,\ \psi'(0)=\psi'(1) \tag{24}$$

the validity of the estimations are proved

$$\sum_{k=1}^{\infty}\left(k^3\varphi_{2k-1}\right)^2 \le \frac{1}{8\pi^6}\left\|\varphi'''(x)\right\|^2_{L_2(0,1)}, \quad \sum_{k=1}^{\infty}\left(k^3\varphi_{2k}\right)^2 \le \frac{1}{8\pi^6}\left\|(1-x)\varphi'''(x)-3\varphi''(x)\right\|^2_{L_2(0,1)}; \tag{25}$$

$$\sum_{k=1}^{\infty}\left(k^2\psi_{2k-1}\right)^2 \le \frac{1}{2\pi^4}\left\|\psi''(x)\right\|^2_{L_2(0,1)}, \quad \sum_{k=1}^{\infty}\left(k^2\psi_{2k}\right)^2 \le \frac{1}{2\pi^4}\left\|(1-x)\psi''(x)-2\psi'(x)\right\|^2_{L_2(0,1)}, \tag{26}$$

where the numbers φ_k, ψ_k $(k=0,1,...)$ are determined by the relation (22).

4. Let's denote by $B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$ the totality of all functions $u(t,x)$ of the form

$$u(t,x)=\sum_{k=0}^{\infty}u_k(t)X_k(x) \tag{27}$$

considering on $[0,T]\times[0,1]$ for which all functions $u_k(t)\in C^{(l)}([0,T])$ and

$$J_T(u)\equiv\sum_{i=0}^{l}\left\{\left(\max_{0\le t\le T}\left|u_0^{(i)}(t)\right|\right)^{\beta_i}+\sum_{k=1}^{\infty}\left(k^{\alpha_i}\cdot\max_{0\le t\le T}\left|u_{2k-1}^{(i)}(t)\right|\right)^{\beta_i}+\sum_{k=1}^{\infty}\left(k^{\alpha_i}\cdot\max_{0\le t\le T}\left|u_{2k}^{(i)}(t)\right|\right)^{\beta_i}\right\}^{\frac{1}{\beta_i}}<+\infty, \tag{28}$$

where functions $X_k(x)\,(k=0,1,...)$ are determined by the relation (12), $l\ge 0$ is an integer, $\alpha_i\ge 0, 1\le\beta_i\le 2\ \left(i=\overline{0,l}\right)$. Let's determine the norm on this set so: $\|u\|=J_T(u)$. It is evident that all these spaces are Banach spaces.

In the further for the functions $u(t,x)\in B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$ we'll use the notations

$$\|u\|_{B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,t}}\equiv\sum_{i=0}^{l}\left\{\left(\max_{0\le\tau\le t}\left|u_0^{(i)}(\tau)\right|\right)^{\beta_i}+\sum_{k=1}^{\infty}\left(k^{a_i}\max_{0\le\tau\le t}\left|u_{2k-1}^{(i)}(\tau)\right|\right)^{\beta_i}+\sum_{k=1}^{\infty}\left(k^{a_i}\max_{0\le\tau\le t}\left|u_{2k}^{(i)}(\tau)\right|\right)^{\beta_i}\right\}^{\frac{1}{\beta_i}}$$

$$(0\le t\le T). \tag{29}$$

Further for the function $u(t,x)=\sum_{k=0}^{\infty}u_k(t)X_k(x)$ $u_k(t)$ we'll call the function its k-th component. Let M be any non-empty set from $B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$. We'll denote by M_k the totality of k-th components all functions from M. The following is true

Theorem 2. *For the compactness of the set $M\subset B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$ in $B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$ it is necessary and sufficient the following two conditions are to be fulfilled:*

a) $\forall k\ (k=0,1,...)$ *the set* M_k *is compact in* $C^{(l)}([0,T])$;

b) $\forall\varepsilon>0$ *exist* k_ε *the same for all* $u(t,x)=\sum_{k=0}^{i}u_k(t)X_k(x)$, *such that*

$$\sum_{i=0}^{l}\left\{\sum_{k=k_\varepsilon}^{\infty}\left(k^{\alpha_i}\cdot\max_{0\le t\le T}\left|u_{2k-1}^{(i)}(t)\right|\right)^{\beta_i}+\sum_{k=k_\varepsilon}^{\infty}\left(k^{\alpha_i}\cdot\max_{0\le t\le T}\left|u_{2k}^{(i)}(t)\right|\right)^{\beta_i}\right\}^{\frac{1}{\beta_i}}<\varepsilon \quad \forall u\in M.$$

Further, at the given paper we often use the estimation

$$\forall u\in B^{3,2}_{2,2,T}\quad \|u\|_{B^{2,1}_{1,1,t}}\le\frac{\pi}{\sqrt{2}}\cdot\|u\|_{B^{3,2}_{2,2,t}}\quad (0\le t\le T). \tag{30}$$

5. Let $f(x)\in L_2(0,1)$. Then

$$\|f(x)\|^2_{L_2(0,1)}\le\frac{4}{3}\cdot\sum_{k=0}f_k^2, \tag{31}$$

where the numbers f_k $(k=0,1,...)$ are determined by the relation (16).

Let $f(x)\in C([0,1]),\ f'(x)\in L_2(0,1)$ and $f_0^2+\sum_{k=1}^{\infty}(k\cdot f_{2k-1})^2+\sum_{k=1}^{\infty}(k\cdot f_{2k})^2<+\infty$.

Then

$$\|f'(x)\|^2_{L_2(0,1)}\le 2\left(1+3\pi^2\right)\left\{f_0^2+\sum_{k=1}^{\infty}(k\cdot f_{2k-1})^2+\sum_{k=1}^{\infty}(k\cdot f_{2k})^2\right\}. \qquad (32)$$

Let $f(x)\in C^{(1)}([0,1]),\ f''(x)\in L_2(0,1)$ and $f_0^2+\sum_{k=1}^{\infty}\left(k^2 f_{2k-1}\right)^2+\sum_{k=1}^{\infty}\left(k^2 f_{2k}\right)^2<+\infty$.

Then

$$\|f''(x)\|^2_{L_2(0,1)}\le 8\pi^2\left(2+3\pi^2\right)\left\{f_0^2+\sum_{k=1}^{\infty}\left(k^2\cdot f_{2k-1}\right)^2+\sum_{k=1}^{\infty}\left(k^2\cdot f_{2k}\right)^2\right\}. \qquad (33)$$

Let $f(x)\in C^{(2)}([0,1]),\ f'''(x)\in L_2(0,1)$ and $f_0^2+\sum_{k=1}^{\infty}\left(k^3 f_{2k-1}\right)^2+\sum_{k=1}^{\infty}\left(k^3 f_{2k}\right)^2<+\infty$.

Then

$$\|f'''(x)\|^2_{L_2(0,1)}\le 48\pi^4\left(3+2\pi^2\right)\left\{f_0^2+\sum_{k=1}^{\infty}\left(k^3\cdot f_{2k-1}\right)^2+\sum_{k=1}^{\infty}\left(k^3\cdot f_{2k}\right)^2\right\}. \qquad (34)$$

6. In conclusion of this paragraph let's agree to suppose everywhere all values at this paper as real, all functions- real- valued, and integral everywhere understand in Lebesgue meaning.

§2. The investigation of uniqueness of generalized solution of the problem (1)-(3).

By view of the Belman inequality the following theorem on uniqueness in the whole of generalized solution of the problem (1)-(3) is proved.

Theorem 3. *Let*

1. $F(t,x,u_1,...,u_7)\in C\left([0,T]\times[0,1]\times(-\infty,\infty)^7\right)$.

2. $\forall R>0$ *in* $[0,T]\times[0,1]\times[-R,R]^5\times(-\infty,\infty)^2$

$$\left|F(t,x,u_1,...,u_7)-F(t,x,\tilde{u}_1,...,\tilde{u}_7)\right|\le C_R\cdot\sum_{i=1}^{7}\left|u_i-\tilde{u}_i\right|, \qquad (35)$$

where $C_R>0$ *is a constant.*

Then the problem (1)-(3) can't have more than one generalized solution.

§3. The investigation of existence in small the generalized solution of the problem (1)-(3).

At this paragraph by the combination of generalized principle of condensed mappings by Shauder principle on a fixed point, the following theorem of existence in a little (i.e. true at the sufficiently little values T) generalized solution of the problem (1)-(3) is proved.

Theorem 4. *Let*

1. $\varphi(x)\in C^{(2)}([0,1])$, $\varphi'''(x)\in L_2(0,1)$ *and* $\varphi(0)=0,\ \varphi'(0)=\varphi'(1),\ \varphi''(0)=0$;

$\psi(x)\in C^{(1)}([0,1])$, $\psi''(x)\in L_2(0,1)$ *and* $\psi(0)=0,\ \psi'(0)=\psi'(1)$.

2. $F(t,x,u_1,...,u_7)\in C\left([0,T]\times[0,1]\times(-\infty,\infty)^7\right)$.

3. $\forall R>0$ *in* $[0,T]\times[0,1]\times[-R,R]^5\times(-\infty,\infty)^2$

$$\left|F(t,x,u_1,...,u_5,u_6,u_7)-F(t,x,u_1,...,u_5,\tilde{u}_6,\tilde{u}_7)\right|\le C_R\left(|u_6-\tilde{u}_6|+|u_7-\tilde{u}_7|\right), \quad (36)$$

where $C_R>0$ *is a constant.*

Then the generalized solution of the problem (1)-(3) exists in small.

Proof. Let's determine in $B^{3,2}_{2,2,T}$ the operator (with respect to V) P_u for every fixed $u\in B^{2,1}_{1,1,T}$

$$P_u(V(t,x))=\tilde{V}(t,x)\equiv\sum_{k=0}^{\infty}\tilde{V}_k(t)X_k(x), \quad (37)$$

where

$$\tilde{V}_0(t)=\varphi_0+\psi_0\cdot t+2\int_0^t\int_0^1(t-\tau)\Phi_u(V(\tau,x))dxd\tau \quad (t\in[0,T]), \quad (38)$$

$$\tilde{V}_{2k-1}(t)=\varphi_{2k-1}\cdot\cos\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t+\psi_{2k-1}\frac{\sqrt{1+4\alpha\pi^2k^2}}{4\pi^2k^2}\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t+$$
$$+\frac{1}{\pi^2k^2\sqrt{1+4\alpha\pi^2k^2}}\cdot\int_0^t\int_0^1\Phi_u(V(\tau,x))\cos 2\pi kx\cdot\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}(t-\tau)dxd\tau$$
$$(k=1,2,...;t\in[0,T]), \quad (39)$$

$$\tilde{V}_{2k}(t)=-\varphi_{2k-1}\frac{4\pi k(1+2\alpha\pi^2k^2)}{(1+4\alpha\pi^2k^2)^{\frac{3}{2}}}t\cdot\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t+\varphi_{2k}\cos\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t-$$
$$-\psi_{2k-1}\frac{1+2\alpha\pi^2k^2}{\pi k(1+4\alpha\pi^2k^2)}\left\{\frac{\sqrt{1+4\alpha\pi^2k^2}}{4\pi^2k^2}\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t-t\cdot\cos\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t\right\}+$$
$$+\psi_{2k}\frac{\sqrt{1+4\alpha\pi^2k^2}}{4\pi^2k^2}\cdot\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}t-\frac{4\alpha}{\pi k(1+4\alpha\pi^2k^2)^{\frac{3}{2}}}\times$$
$$\times\int_0^t\int_0^1\Phi_u(V(\tau,x))\cos 2\pi kx\cdot\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}(t-\tau)dxd\tau-\frac{8(1+2\alpha\pi^2k^2)}{\pi k(1+4\alpha\pi^2k^2)^2}\times$$
$$\times\int_0^t\left[\int_0^\tau\int_0^1\Phi_u(V(\sigma,x))\cos 2\pi kx\cdot\sin\frac{2\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}(\tau-\sigma)dxd\sigma\right]\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}(t-\tau)d\tau+$$
$$+\frac{1}{\pi^2k^2\sqrt{1+4\alpha\pi^2k^2}}\int_0^t\int_0^1(1-x)\Phi_u(V(\tau,x))\sin 2\pi kx\sin\frac{4\pi^2k^2}{\sqrt{1+4\alpha\pi^2k^2}}(t-\tau)dxd\tau$$
$$(k=1,2,...;t\in[0,T]), \quad (40)$$

the functions $X_k(x)$ $(k=0,1,...)$ are determined by the relation (12), the numbers φ_k, ψ_k $(k=0,1,...)$ are determined by the relation (22), the functions $Y_k(x)$ $(k=0,1,...)$ are determined by the relation (13) and

$$\Phi_u(V(t,x))\equiv F(t,x,u(t,x),u_t(t,x),u_x(t,x),u_t(t,x),u_{xx}(t,x),V_{txx}(t,x),V_{xxxx}(t,x)). \quad (41)$$

It is evident that

$$\forall u \in B_{2,2,T}^{3,2} \quad \Phi_u(u(t,x)) = \mathsf{F}(u(t,x)), \tag{42}$$

where the operator F is determined by the relation (6).

By view of structure of the space $B_{1,1,T}^{2,1}$ for any $u \in B_{1,1,T}^{2,1}$ there exists such $R = R_u > 0$ that $\forall t \in [0,T]$ and $x \in [0,1]$:

$$-R_u \le u(t,x), u_t(t,x), u_x(t,x),\ u_{tx}(t,x),\ u_{xx}(t,x) \le R_u . \tag{43}$$

Then using the estimation (36) for $R = R_u$ by the estimations

$$|\Phi_u(V(t,x))| \le |\Phi_u(V(t,x)) - \Phi_u(0)| + |\Phi_u(0)| \le \mathsf{C}_{R_u}\left(|V_{txx}(t,x)| + |V_{xxx}(t,x)|\right) + \mathsf{A}_{R_u} \tag{44}$$

and the estimations (33), (34) it is easy to get that at any fixed $u \in B_{1,1,T}^{2,1}$ $\forall V \in B_{2,2,T}^{3,2}$ and $t \in [0,T]$:

$$\|\mathsf{P}_u(V)\|^2_{B_{2,2,t}^{3,2}} \equiv \left\| \sum_{k=0}^{\infty} \tilde{V}_k(t) X_k(x) \right\|^2_{B_{2,2,t}^{3,2}} \le a_0 + b_0 \int_0^t \int_0^1 \{\Phi_u(V(\tau,x))\}^2 dx d\tau \le$$

$$\le a_0 + 3b_0 \mathsf{A}^2_{R_u} T + 3b_0 \mathsf{C}^2_{R_u} \left\{8\pi^2\left(2+3\pi^2\right) + 48\pi^4\left(3+2\pi^2\right)\right\} \cdot \|V\|^2_{B_{2,2,T}^{3,2}} \cdot T , \tag{45}$$

where

$$a_0 \equiv 6\varphi_0^2 + \frac{6}{\alpha^2}\left\{\alpha\left(1+8T^2\right) + 4\left[\alpha\pi^2 + 2\left(\sqrt{\alpha} + 2\pi T\right)^2\right]\right\} \sum_{k=1}^{\infty} \left(k^3 \varphi_{2k-1}\right)^2 +$$

$$+12\left(1 + \frac{4\pi^2}{\alpha}\right) \sum_{k=1}^{\infty} \left(k^3 \varphi_{2k}\right)^2 + 6T^2 \cdot \psi_0^2 + 6\left[\frac{1}{16\pi^4}\left(1+4\alpha\pi^2\right) + \frac{1}{2\pi^4}\left(4T\pi^2 + \sqrt{1+4\alpha\pi^2}\right)^2 +\right.$$

$$\left. + \frac{1}{\alpha}\left(\alpha + 8T^2\right)\right] \sum_{k=1}^{\infty} \left(k^2 \psi_{2k-1}\right)^2 + 6\left[\frac{1}{8\pi^4}\left(1+4\alpha\pi^2\right) + 2\right] \sum_{k=1}^{\infty} \left(k^2 \psi_{2k}\right)^2 , \tag{46}$$

$$b_0 \equiv \frac{T}{4\alpha^2\pi^8}\left\{32\alpha^2\pi^8 T^2 + 3\alpha\pi^2 + 6\left[\left(\sqrt{\alpha} + 4\pi T\right)^2 + \alpha\pi^2\right]\right\} +$$

$$+ \frac{T}{\alpha^3\pi^6}\left\{16\alpha^3\pi^6 + 3\alpha\pi^2 + 6\left[\left(\sqrt{\alpha} + 4\pi T\right)^2 + \alpha\pi^2\right]\right\}, \tag{47}$$

A_{R_u} is a maximum of the function $|F(t,x,u_1,...,u_5,0,0)|$ in a closed domain $0 \le t \le T$, $0 \le x \le 1,\ -R_u \le u_1,...,u_5 \le R_u$ moreover the finiteness a_0 follows from the estimations (25), (26).

From (45) it follows that for any fixed $u \in B_{1,1,T}^{2,1}$ the operator P_u acts in $B_{2,2,T}^{3,2}$ moreover boundedly.

Further, analogously (45) by the mathematical induction method we show that at any fixed $u \in B_{1,1,T}^{2,1}$ $\forall V_1, V_2 \in B_{2,2,T}^{3,2}$ and $t \in [0,T]$

$$\left\|\mathsf{P}_u^n(V_1) - \mathsf{P}_u^n(V_2)\right\|^2_{B_{2,2,t}^{3,2}} = \left\|\mathsf{P}_u\left(\mathsf{P}_u^{n-1}(V_1)\right) - \mathsf{P}_u\left(\mathsf{P}_u^{n-1}(V_2)\right)\right\|^2_{B_{2,2,t}^{3,2}} \le$$

$$\le \left\{16 b_0 \pi^2\left(2 + 21\pi^2 + 12\pi^4\right) \cdot \mathsf{C}^2_{R_u}\right\}^n \cdot \|V_1 - V_2\|^2_{B_{2,2,T}^{3,2}} \cdot \frac{t^n}{n!}, \tag{48}$$

where n is any natural numbers.

Thus at any fixed $u \in B_{1,1,T}^{2,1}$ $\forall V_1, V_2 \in B_{2,2,T}^{3,2}$:

$$\left\|\mathsf{P}_u^n(V_1) - \mathsf{P}_u^n(V_2)\right\|_{B_{2,2,T}^{3,2}} \le q_n(u) \cdot \|V_1 - V_2\|_{B_{2,2,T}^{3,2}} , \tag{49}$$

where

$$q_n(u) \equiv \frac{1}{\sqrt{n!}} \cdot \left\{16 b_0 \pi^2 \left(2 + 21\pi^2 + 12\pi^4\right) \cdot \mathrm{C}^2_{R_u} \cdot T\right\}^{\frac{n}{2}}. \qquad (50)$$

It is evident that for sufficient big $n = n_u : q(u) < 1$. For such n the operator P_u^n seems condensed at the space $B^{3,2}_{2,2,T}$. Then by view of generalized principle of considered mappings. Unique in $B^{3,2}_{2,2,T}$ the fixed point V of the operator P_u^n is unique in $B^{3,2}_{2,2,T}$ fixed point of the operator P_u too:

$$V = \mathrm{P}_u(V), \; V \in B^{3,2}_{2,2,T}. \qquad (51)$$

Associated to every $u \in B^{2,1}_{1,1,T}$ unique in $B^{3,2}_{2,2,T}$ fixed point V of the operator P_u we generate the operator H :

$$H(u) = V = \mathrm{P}_u(V) \qquad (52)$$

acting from $B^{2,1}_{1,1,T}$ in $B^{3,2}_{2,2,T}$.

Further, it is shown that the operator H acts from $B^{2,1}_{1,1,T}$ in $B^{3,2}_{2,2,T}$ continuously, and as it acts in $B^{2,1}_{1,1,T}$ continuously. Besides it is shown that for any closed sphere K_r of the space $B^{2,1}_{1,1,T}$ radius r and with the center in zero of the set $H(K_r)$ is bounded in $B^{3,2,1}_{2,2,2,T}$. From here using theorem 2 for the space $B^{2,1}_{1,1,T}$ we get that the set $H(K_r)$ is compact in $B^{2,1}_{1,1,T}$. Consequently the operator H acts to $B^{2,1}_{1,1,T}$ compactly. Thus the operator H acts at the space $B^{2,1}_{1,1,T}$ completely continuously.

Let r be a fixed number, satisfying the condition $r > \frac{\pi}{\sqrt{2}} \cdot a_0$ where the number $\frac{\Re}{\Re}$ is determined by the relation (46). It is shown that at sufficiently little values T the operator H transforms the sphere K_r to itself. Consequently, by view of the Shauder principle on a fixed point at sufficiently little values T the operator H has in $B^{2,1}_{1,1,T}$ even one fixed point u:

$$u = H(u). \qquad (53)$$

Since the operator H acts from $B^{2,1}_{1,1,T}$ in $B^{3,2,1}_{2,2,2,T}$ then

$$u(t,x) \in B^{3,2,1}_{2,2,2,T}. \qquad (54)$$

Further, since $u = H(u) = V = \mathrm{P}_u(V)$ then $u = V$ and consequently

$$u = H(u) = \mathrm{P}_u(u).$$

Then by view of (42), $\Phi_u(u) = \mathrm{F}(u)$ and consequently for finding fixed point $u = u(t,x) = \sum_{k=0}^{\infty} u_k(t) X_k(x)$ the functions $u_k(t)$ $(k = 0,1,...)$ by view of the notations (13) satisfy on $[0,T]$ the system (19)-(21). Using this and the relation (54) it is shown that the function $u(t,x)$ is a generalized solution of the problem (1)-(3). Theorem is proved.

Remark 1. It follows to note that the condition 1 of theorem 4, imposed on initial functions $\varphi(x)$ and $\psi(x)$ aren't only sufficient, but also necessary for the existence of a generalized solution of the problem (1)-(3).

Remark 2. Since theorem 3 is the theorem on uniqueness in whole, and the theorem 4 is the theorem on the existence in a small of generalized solution of the problem (1)-(3) then from the two theorems it follows the following.

Theorem 5. *Let*

1. *The conditions of the theorem 3 be fulfilled.*
2. *The condition one of theorem 4 be fulfilled.*

Then the unique in the whole the generalized solution of the problem (1)-(3) exits in small.

§4. The investigation on the existence in a small the generalized solution of the problem (1)-(3).

At this paragraph by view of strong Shauder principle on a fixed point the following theorem on the existence in the whole of generalized solution of the problem (1)-(3) is proved.

Theorem 6. *Let*

1. *The all conditions of theorem 5 be fulfilled.*
2. *In* $[0,T]\times[0,1]\times(-\infty,\infty)^7$

$$\left|F(t,x,u_1,...,u_7)\right|\leq \mathrm{C}\cdot\left(1+|u_1|+...+|u_7|\right), \tag{55}$$

where $\mathrm{C}>0$ *is a constant.*

Then there exists the generalized solution of the problem (1)-(3).

Proof. Let H be an operator, introduced at the proving process of theorem 4. As it was said at the proving process of theorem 4 the operator H act at the space $B^{2,1}_{1,1,T}$ completely continuously; besides it acts from $B^{2,1}_{1,1,T}$ in $B^{3,2,1}_{2,2,2,T}$ boundedly. By the definition of the operator H :

$$\forall u\in B^{2,1}_{1,1,T}\quad H(u)=V=\mathrm{P}_u(V), \tag{56}$$

where the operator P_u is determined by the relation (37)-(40).

Let's consider now in $B^{2,1}_{1,1,T}$ the equation

$$u=\lambda H(u),\ 0\leq\lambda\leq 1 \tag{57}$$

and a priori we'll estimate all their possible in $B^{2,1}_{1,1,T}$ solution u. Since

$$u=\lambda H(u)=\lambda V=\lambda\mathrm{P}_u(V), \tag{58}$$

then completely analogously to (45) we get that $\forall t\in[0,T]$:

$$\|u\|^2_{B^{3,2}_{2,2,t}}\equiv\|\lambda H(u)\|^2_{B^{3,2}_{2,2,t}}\equiv\|\lambda V\|^2_{B^{3,2}_{2,2,t}}\equiv\|\lambda\mathrm{P}_u(V)\|^2_{B^{3,2}_{2,2,t}}\leq$$

$$\leq a_0\cdot\lambda^2+b_0\cdot\lambda^2\cdot\int_0^t\int_0^1\{\Phi_u(V(\tau,x))\}^2dxd\tau\leq a_0+b_0\cdot\lambda^2\cdot\int_0^t\int_0^1\{\Phi_u(V(\tau,x))\}^2dxd\tau, \tag{59}$$

where the numbers a_0 and b_0 are determined by the relations (46), (47).

From (59) using the inequality (55) and the relation $\lambda V=u$ we get that

$$\|u\|^2_{B^{3,2}_{2,2,t}}\leq a_0+8b_0\lambda^2\mathrm{C}^2\int_0^t\left\{1+\int_0^1u^2(\tau,x)dx+\int_0^1u^2_\tau(\tau,x)dx+\int_0^1u^2_x(t,x)dx+\int_0^1u^2_{\tau x}(\tau,x)dx+\right.$$

$$+\int_0^1 u_{xx}^2(\tau,x)dx+\int_0^1 V_{\tau xx}^2(\tau,x)dx+\int_0^1 V_{xxx}^2(\tau,x)dx\Big\}d\tau\le a_0+8b_0C^2T+8b_0C^2\int_0^t\Big\{\int_0^1 u^2(\tau,x)dx+$$

$$+\int_0^1 u_\tau^2(\tau,x)+\int_0^1 u_x^2(\tau,x)dx+\int_0^1 u_{\tau x}^2(\tau,x)dx+\int_0^1 u_{xx}^2(\tau,x)dx+\int_0^1 \lambda^2 V_{\tau xx}^2(\tau,x)dx+$$

$$+\int_0^1 \lambda^2 V_{xxx}^2(\tau,x)dx\Big\}d\tau=a_0+8b_0TC^2+8b_0\cdot C^2\cdot\int_0^t\Big\{\int_0^1 u^2(t,x)dx+\int_0^1 u_\tau^2(\tau,x)dx+$$

$$+\int_0^1 u_x^2(\tau,x)dx+\int_0^1 u_{\tau x}^2(\tau,x)dx+\int_0^1 u_{xx}^2(t,x)dx+\int_0^1 u_{\tau xx}^2(\tau,x)dx+\int_0^1 u_{xxx}^2(t,x)dx\Big\}d\tau. \quad (60)$$

On the other hand by view of the estimations (31)-(34) $\forall\tau\in[0,T]$:

$$\int_0^1 u^2(\tau,x)dx\le\frac{4}{3}\|u\|^2_{B^0_{2,\tau}}\le\frac{4}{3}\|u\|^2_{B^3_{2,\tau}}\le\frac{4}{3}\|u\|^2_{B^{3,2}_{2,2,\tau}}, \quad (61)$$

$$\int_0^1 u_\tau^2(\tau,x)dx\le\frac{4}{3}\|u_\tau\|^2_{B^0_{2,\tau}}\le\frac{4}{3}\|u_\tau\|^2_{B^2_{2,\tau}}\le\frac{4}{3}\|u\|^2_{B^{3,2}_{2,2,\tau}}, \quad (62)$$

$$\int_0^1 u_x^2(\tau,x)dx\le 2(1+3\pi^2)\|u\|^2_{B^1_{2,\tau}}\le 2(1+3\pi^2)\|u\|^2_{B^3_{2,\tau}}\le 2(1+3\pi^2)\|u\|^2_{B^{3,2}_{2,2,\tau}}, \quad (63)$$

$$\int_0^1 u_{\tau x}^2(\tau,x)dx\le 2(1+3\pi^2)\|u_\tau\|^2_{B^1_{2,\tau}}\le 2(1+3\pi^2)\|u_\tau\|^2_{B^2_{2,\tau}}\le 2(1+3\pi^2)\|u\|^2_{B^{3,2}_{2,2,\tau}}, \quad (64)$$

$$\int_0^1 u_{xx}^2(\tau,x)dx\le 8\pi^2(2+3\pi^2)\|u\|^2_{B^2_{2,\tau}}\le 8\pi^2(2+3\pi^2)\|u\|^2_{B^3_{2,\tau}}\le 8\pi^2(2+3\pi^2)\|u\|^2_{B^{3,2}_{2,2,\tau}}, \quad (65)$$

$$\int_0^1 u_{\tau xx}^2(\tau,x)dx\le 8\pi^2(2+3\pi^2)\|u_\tau\|^2_{B^2_{2,\tau}}\le 8\pi^2(2+3\pi^2)\|u\|^2_{B^{3,2}_{2,2,\tau}}, \quad (66)$$

$$\int_0^1 u_{xxx}^2(\tau,x)dx\le 48\pi^4(3+2\pi^2)\|u\|^2_{B^3_{2,\tau}}\le 48\pi^4(3+2\pi^2)\|u\|^2_{B^{3,2}_{2,2,\tau}}. \quad (67)$$

Now using the estimation (61)-(67) from (60) we'll get that $\forall t\in[0,T]$:

$$\|u\|^2_{B^{3,2}_{2,2,t}}\le a_0+8b_0T\cdot C^2+8b_0\cdot C^2\times$$

$$\times\left\{\frac{8}{3}+4(1+3\pi^2)+16\pi^2(2+3\pi^2)+48\pi^4(3+2\pi^2)\right\}\int_0^t\|u\|^2_{B^{3,2}_{2,2,t}}d\tau. \quad (68)$$

From (68) applying the Bellman inequality we get:

$$\|u\|^2_{B^{3,2}_{2,2,T}}\le\{a_0+8b_0T\cdot C^2\}\times$$

$$\times\exp\left\{8b_0C^2\left[\frac{8}{3}+4(1+3\pi^2)+16\pi^2(2+3\pi^2)+48\pi^4(3+2\pi^2)\right]\cdot T\right\}\equiv C_0^2. \quad (69)$$

Thus, all possible in $B^{2,1}_{1,1,T}$ solutions of the equations (57) a priori bounded in $B^{2,1}_{2,2,T}$ and as in $B^{2,1}_{1,1,T}$ because by view of (30), $\|u\|^2_{B^{2,1}_{1,1,T}}\le\frac{\pi}{\sqrt{2}}\|u\|_{B^{3,2}_{2,2,T}}$. Then by view of strong Shauder principle on a fixed point or principle of trivial rotation the operator H in

$B^{2,1}_{1,1,T}$ has a fixed point u, which as it was said at the proving process of theorem 4 is a generalized solution of the problem (1)-(3). Theorem is proved.

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