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WEIGHTED COMPOSITION OPERATORS ON $C_0(X,E)$

Abstract

We characterize the mapping inducing the weighted composition operators on $C_0(X,E)$- the space of E-valued continuous functions on a locally compact space X that vanish at infinity and equipped with the supremum norm. A few properties of the multiplication and weighted composition operators are discussed and some examples of them are presented to illustrate the theory.

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1. Introduction and Preliminaries.

Let X be a locally compact Hausdorff space and E a real or complex Banach space. The space of continuous E-valued function on X will be denoted by $C(X,E)$. We will consider a subspace $C_0(X,E)$ of $C(X,E)$ which is the set of all continuous E-valued functions on X that "vanish at infinity", i.e. $C_0(X.E)$ denote the space of all continuous functions $f: X \to E$ such that for all $\varepsilon > 0$, $\{x \in X : \|f(x)\|_E \geq \varepsilon\}$ is compact (where $\|\cdot\|_E$ denote the norm of E) equipped with the sup-norm $\|f\| = \sup\{\|f(x)\|_E : x \in X\}$. In case $E \in \{\mathbb{R}, \mathbb{C}\}$, we shall omit E from our notation and write $C_0(X)$ in place of $C_0(X,E)$. We recall that a weighted composition transformation $T = wC_\varphi$ on $C_0(X,E)$ induced by some selfmap φ of X and some map w from X into $B(E)$-the space of bounded linear operators on E is defined as $Tf(x) = wC_\varphi f(x) = w(x)f(\varphi(x))$, $f \in \ \in C_0(X,E)$. In case wC_φ is a bounded linear operator with range in $C_0(X,E)$ we call it a weighted composition operator. In the sequel, we write T in place of wC_φ. For a weighted composition operator T on $C_0(X,E)$, we have that $\sup\{\|w(x)\| : x \in X\} < +\infty$, and the map $w: X \to B(E)$ is continuous in the strong operator topology but not necessarily continuous in the uniform operator topology (see [5]).

If $w(x) = I$ the identity transformation on E for every $x \in X$, and φ is some selfmap of X, then we write $T = wC_\varphi$ as C_φ and call it the composition operator on $C_0(X,E)$ induced by φ. In case $\varphi(x) = x$ for every $x \in X$ and w is some map from X into $B(E)$, then we write $T = wC_\varphi$ as M_w and call it the multiplication operator on $C_0(X,E)$ induced by w, i.e. $M_w f(x) = w(x)f(x)$, $f \in C_0(X,E)$.

Weighted composition operators or composition operators on $C(X,E)$ (or on its certain subspaces) were studied in [1], [3], [5], and the case of $E = \mathbb{C}$ was considered in [2], [4], [6], when X is a compact Hausdorff space. In this paper we have characterized $T = wC_\varphi$ on $C_0(X,E)$ when X is a locally compact Hausdorff space and w is a continuous $B(E)$- valued function on X, and some examples are given to illustrate the results.

2. Functions inducing weighted composition operators.

Let $f \in C_0(X,E)$, then for any $\varepsilon > 0, \|f(x)\| < \varepsilon$ outside some compact subset K of X. Now if $\varphi^{-1}(K)$ is compact subset of X, where φ is continuous selfmap of X, then $\|f(\varphi(x))\| < \varepsilon$ outside $\varphi^{-1}(K)$ indicates that $f \circ \varphi$ vanishes at infinity on X.

Definition 2.1. *Let X be a locally compact Hausdorff space. A selfmap φ of X is said to be proper if $\varphi^{-1}(K)$ is compact subset of X for all compact $K \subseteq X$.*

Theorem 2.2. *Let X be a locally compact Hausdorff space, E a Banach space and the map $w: X \to B(E)$ be continuous.*

(a) If φ is continuous map from X into X, proper and there is a positive number M such that $\|w(x)\| \le M$, $\forall x \in X$, then T is weighted composition operator.

(b) If T is a weighted composition operator and there is a positive number m such that, for all $e \in E$, $m\|e\| \le \|w(x)e\|$, $\forall x \in X$, then φ is continuous and proper.

Proof. (a) We shall show that T is a bounded linear operator on $C_0(X,E)$. First of all, we show that T is an into map. Let $f \in C_0(X,E)$ than for any $\varepsilon > 0, \|f(x)\| < \varepsilon/M$ outside some compact subset K of X. Since φ is proper, $\varphi^{-1}(K)$ is compact in X. Now the fact that

$$\|w(x)f(\varphi(x))\| \le \|w(x)\|\|f(\varphi(x))\| \le M\|f(\varphi(x))\| < \varepsilon$$

outside $\varphi^{-1}(K)$ indicates that $w(\cdot)f(\varphi) \in C_0(X,E)$. This implies that $Tf \in C_0(X,E)$. Clearly T is linear on $C_0(X,E)$, it is enough to show that T is continuous at the origin. For this, suppose $\{f_\alpha\}$ is a net in $C_0(X,E)$ such that $\|f_\alpha\| \to 0$ we have

$$\|Tf_\alpha\| = \sup_{x\in X}\|Tf_\alpha(x)\| = \sup_{x\in X}\|w(x)f_\alpha(\varphi(x))\| \le$$

$$\le \sup_{x\in X}\|w(x)\|\|f_\alpha(\varphi(x))\| \le M\sup_{x\in X}\|f_\alpha(\varphi(x))\| \le M\|f_\alpha\| \to 0 .$$

Hence T is continuous on $C_0(X,E)$.

(b) Suppose $x_\alpha \to x$ in X. We want to show that $\varphi(x_\alpha) \to \varphi(x)$ in X. Suppose not, by passing to a subnet if necessary, we can assume that $\varphi(x_\alpha)$ either converges to some $y \ne \varphi(x)$ in X or ∞. Then for all f in $C_0(X,E)$, we have

$$w(x)f(y) = \lim_\alpha w(x_\alpha)f(\varphi(x_\alpha)) = \lim_\alpha Tf(x_\alpha) = Tf(x) = w(x)f(\varphi(x)).$$

Hence $w(x)(f(y) - f(\varphi(x))) = 0$, $\forall f \in C_0(X,E)$. As $w(x)$ is one-to-one, $f(y) = f(\varphi(x))$ for all $f \in C_0(X,E)$. Therefore, we obtain a contradiction $\varphi(x) = y$. If $\varphi(x_\alpha) \to \infty$ then a similar argument gives

$$\|w(x)f(\varphi(x))\| = \|T\, f(x)\| = \lim_\alpha\|w(x_\alpha)f(\varphi(x_\alpha))\| \le$$

$$\le \lim_\alpha\|w(x_\alpha)\|\|f(\varphi(x_\alpha))\| = \|w(x)\|\lim_\alpha\|f(\varphi(x_\alpha))\| = 0$$

for all $f\in C_0(X,E)$. Hence $f(\varphi(x))=0, \forall f\in C_0(X,E)$ and therefore $\varphi(x)=\infty$, a contradiction with $\varphi:X\to X$. Finally, let K be a compact subset of X and we are going to see that $\varphi^{-1}(K)$ is compact in X, or equivalently, closed in X_∞, the one-point compactification of X.

To see this, suppose $x_\alpha\to x$ in X_∞ and $x_\alpha\in\varphi^{-1}(K)$. Since K is compact, then without loss of generality, we can assume that $\varphi(x_\alpha)\to z$ for some $z\in K$. Now, by hypothesis of the theorem, we obtain

$$\|Tf(x)\|=\lim_\alpha\|Tf(x_\alpha)\|=\lim_\alpha\|w(x_\alpha)f(\varphi(x_\alpha))\|\geq$$
$$\geq m\lim_\alpha\|f(\varphi(x_\alpha))\|=m\|f(z)\|,\quad f\in C_0(X,E).$$

This implies that $x\neq\infty$, and continuity of φ gives $z=\varphi(x)$. Hence $x\in\varphi^{-1}(K)$.

Corollary 2.3. *Let u be a continuous scalar-valued function defined on X. Then:*

(a) Suppose that there are bounds $M, m>0$ such that $m\leq|u(x)|\leq M, \forall x\in X$. Then the operator $uC_\varphi: C_0(X)\to C_0(X)$ is a weighted composition operator if and only if φ is continuous and proper.

(b) If there is $M>0$ such that $|u(x)|\leq M$, $\forall x\in X$, then $M_u(f)=uf$ is a multiplication operator on $C_0(X)$.

Proof. Follows by taking $w(x)=u(x)I$ and $\varphi(x)=x$ respectively in the proof of the theorem (2.2).

Example 2.4. Let $X=(0,+\infty)$, $E=\mathbb{R}$ and define $w(x)=\sin x+2$ and

$$\varphi(x)=\begin{cases}\frac{1}{x}, & \text{if}\quad x\in(0,1),\\ 1, & \text{if}\quad x\in[1,+\infty).\end{cases}$$

Since $\varphi^{-1}(1)=[1,+\infty)$ is not compact, therefore φ is not proper and hence by corollary 2.3, T is not a weighted composition operator. In fact T is not even an into map. For, take $f(x)=\begin{cases}x, & \text{if}\quad x\in(0,1),\\ \frac{1}{x}, & \text{if}\quad x\in[1,+\infty).\end{cases}$

Then obviously $f\in C_0(X,E)$, but

$$Tf(x)=u(x)f(\varphi(x))==(\sin x+2)\cdot\begin{cases}\frac{1}{x}, & \text{if}\quad x\in(0,1),\\ 1, & \text{if}\quad x\in[1,+\infty)\end{cases}$$

and it is clear that $Tf\notin C_0(X,E)$.

Proposition 2.5. *Let X be a locally compact Hausdorff space. If φ is a continuous map from X onto X, proper, and w a $B(E)$- valued function defined on X with bounds $M,m>0$ such that $m\|e\|\leq\|w(x)e\|\leq M\|e\|$ for all $x\in X$ and $e\in E$, then T is weighted composition operator, one-to-one and has closed range.*

Proof. By hypothesis of the theorem and theorem (2.2) it is enough to show that for every $f \in C_0(X,E)$, there exists $c>0$ such that $\|Tf\| \ge c\|f\|$. If $f \in C_0(X,E)$, then we have

$$\|Tf\| = \sup_{x\in X}\|w(x)f(\varphi(x))\| \ge m\sup_{x\in X}\|f(\varphi(x))\| = m\sup_{x\in X}\|f(x)\| = m\|f\|.$$

This completes the proof of the theorem.

3. Functions inducing multiplication operators.

In this section we shall take X to be a completely regular Hausdorff space and give a characterization of multiplication operators on $C_0(X,E)$ induced by vector-valued (scalar-valued) functions.

Theorem 3.1. *Let X be a completely regular Hausdorff space, E a Banach space and map $w: X \to B(E)$ is continuous. Then $M_w : C_0(X,E) \to C_0(X,E)$ is a multiplication operator if and only if there is a positive number M such that for every $e \in E$, $\|w(x)e\| \le M\|e\|$, $\forall x \in X$.*

Proof. The sufficiency follows easily from theorem (2.2). For the necessity, let U be the closed unit ball in $C_0(X,E)$. Since M_w is continuous at the origin, there exists $r>0$ such that $T(rU) \subseteq U$. Take $x_0 \in X$, a nonzero $e_0 \in E$ and N is an open neighborhood of x_0 such that $\overline{N}$ is compact. Thus there exists $f \in C_0(X)$ such that $0 \le f \le 1$, $f(x_0)=1$ and $f(X \setminus N)=0$.

Define $g(x)= f(x)e_0$ for every $x \in X$. Then $g \in C_0(X,E)$, $0 \le \|g\| \le \|e_0\|$ and $\|g(x_0)\| = \|e_0\|$. If $h = r\|e_0\|^{-1} g$, then $h \in rU$ and hence $\|Th\| \le 1$. From this it follows that $\|w(x)h(x)\| \le 1$ for every $x \in X$. This implies that

$$\|w(x)f(x)e_0\| \le r^{-1}\|e_0\|, \quad \text{for every} \quad x \in X.$$

Thus $\|w(x_0)e_0\| \le r^{-1}\|e_0\|$. This completes the proof of the theorem.

Corollary 3.2. *Let E be a Banach algebra with unit e and let $w: X \to E$ be a continuous function. Then*

$$M_w : C_0(X,E) \to C_0(X,E)$$

is a multiplication operator if and only if there exists $M>0$ such that $\|w(x)\| \le M$, for every $x \in X$.

Example 3.3. Let $X = N$ be set of natural numbers with discrete topology and $E == l^2$. Define $w: N \to B(l^2)$ as $w(n) = C_\varphi^n$, where $C_\varphi : l^2 \to l^2$ is the composition operator induced by a map $\varphi: N \to N$. Then it can be seen that $\|w(n)f\| = \|C_\varphi^n f\| = \|f \circ \varphi^n\| \le \|f\|$, for every $n \in N$ and $f \in l^2$, and hence by theorem (3.1), M_w is a multiplication operator on $C_0(N,l^2)$.

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