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TO THE THEORY OF INVERSE SCATTERING PROBLEMS ON THE LINE

Abstract

The approach for the solution of the inverse scattering problem for the third-order differential equations, based on the Gelfand-Levitan-Marchenko's formalism is given in the article.

Introduction.

A number of papers [1-4] have been dedicated to the investigation of the inverse scattering problem on an axis for higher order differential operators by a Riemany problem method. But for such operators excluding the second order operator ([5]-[8]), the traditional approach of solutions of the inverse problem based on the Gelfand-Levitan-Marchenko formalism in general case wasn't considered. We note only papers [9], [10] in which the inverse problem is investigated for the higher order operators with analytical coefficients and [11], where a part of the data of the inverse problem is artificially becomes zero. The rigid constraints in these papers are connected with questions of existence of the triangular representation of Jost's solution.

It's shown that such representations exist, if the analytical functions are coefficients of the equations, but any smoothness of coefficients in general case isn't sufficient for their existence ([13], [14]).

In [15], [16] the integral representations are obtained for a Jost's solution of ordinary differential equations, whose coefficients maybe non-analytical functions, too. Using these representations in the present paper the solution of the inverse scattering problem for the equation

$$iy''' - i\{q(x)y' + (q(x)y)'\} + p(x)y = ik^3 y, \quad x \in \mathbb{R}, \tag{1}$$

where the functions $p(x)$ and $q(x)$ are assumed real and such that $q(x)$ is absolutely continuous and

$$\int_{-\infty}^{\infty} (1+|x|)|q(x)|dx < +\infty, \qquad \int_{-\infty}^{\infty} (1+x^2)\{|p(x)| + |q'(x)|\}dx < +\infty \tag{2}$$

is reduced to the solution of a system of the Gelfand-Levitan-Marchenko type integral equations. Note that the inverse scattering problem for the equation (1) is closely connected with integration of the Cauchy problem for the Boussinesq equation ([11], [12]).

§1. Direct scattering problem.

We introduce the following designations

$$Q_0(x) = q'(x) - i\,p(x), \quad Q_1(x) = -2q(x), \quad D_x = \frac{\partial}{\partial x},$$

$$\sigma^{\pm}(x) = \pm\sum_{j=0}^{1} \int_x^{\pm\infty} \frac{(\pm s \mp x)^{2-j}}{(2-j)!} |Q_j(s)| ds,$$

$$\Omega^{\pm} = \left\{ k \in \mathbb{C} : |\arg(\mp k)| < \frac{\pi}{3} \right\}, \quad \Delta^{\pm} = \left\{ k \in \mathbb{C} : \left|\arg k - \frac{2\pi \pm \pi}{2}\right| < \frac{\pi}{3} \right\}.$$

Under $\lambda^{3/2}$ we'll understand the basic branch of this function. Let I_t^α and D_t^α be the integration and differentiation operators of the functional order α by the variable t:

$$I_t^\alpha \varphi(x,t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(x,s)ds,$$

$$D_t^\alpha \varphi(x,t)=D_t I_t^{1-\alpha}\varphi(x,t).$$

When $k\in\overline{\Omega^\pm}$ we determine Jost's solution of the equation (1) assuming

$$\lim_{x\to\pm\infty} f_\pm(x,k)e^{-kx}=1.$$

These solutions are representability in the form of [16],

$$f_\pm(x,k)=e^{kx}\left(1+\int_0^{+\infty} \mathsf{F}_\pm(x,t)e^{-(\mp k)^{3/2}}dt\right), \tag{3}$$

where the kernels $\mathsf{F}_\pm(x,\cdot)\in L_1(0,\infty)$ and have the following properties:

1) $\int_0^{+\infty}|\mathsf{F}_\pm(x,t)|dt\le \exp\{C\sigma^\pm(x)\}-1$, where C is some positive constant ;

2) on the semi-axis $(0,+\infty)$ there exist the summable by t derivatives $\left(D_t^{2/3}\right)^\ell D_x^m \mathsf{F}_\pm(x,t)$, $1\le \ell+m\le 2$, $\ell, m=0,1,2$ and

$$\lim_{x\to\pm\infty}\int_0^{+\infty}\left|\left(D_t^{2/3}\right)^\ell D_x^m \mathsf{F}_\pm(x,t)\right|dt=0;$$

3) the functions $\mathsf{F}_\pm(x,t)$ satisfy the equation

$$D_x\left(D_x^2\mp 3D_t^{2/3}D_x+\left(D_t^{2/3}\right)^2\right)\mathsf{F}_\pm(x,t)-2q(x)\left(D_x^2\mp 3D_t^{2/3}\right)\mathsf{F}_\pm(x,t)-$$
$$-\left(q'(x)+i\,p(x)\right)\mathsf{F}_\pm(x,t)=0$$

and the conditions

$$I_t^{1/3}\mathsf{F}_\pm(x,t)\Big|_{t=0}=\pm\frac{2}{3}\int_x^{\pm\infty}q(s)ds,$$

$$I_t^{1/3}D_t^{2/3}\mathsf{F}_\pm(x,t)\Big|_{t=0}=\frac{1}{3}\int_x^{\pm\infty}\left(q'(s)-i\,p(s)\right)ds+\frac{4}{9}\int_x^{\pm\infty}q(s)\int_s^{\pm\infty}q(\xi)d\xi\,ds. \tag{4}$$

We also note that $\mathsf{F}_\pm(x,\cdot)\in L_1(0,\infty)$ ([15]).

The function

$$h(x,k)=\begin{cases}\left[\bar f_+\left(x,-\alpha^2\bar k\right),\ \bar f_-\left(x,-\alpha\bar k\right)\right], k\in\overline{\Delta^+},\\ \left[\bar f_+\left(x,-\alpha\bar k\right),\ \bar f_-\left(x,-\alpha^2\bar k\right)\right], k\in\overline{\Delta^-},\end{cases}$$

where $[f,g]=f'g-f\,g'$ is a solution of the equation (1) and the following relations [11] are satisfied

$$\frac{1}{T_\pm\left(\pm\alpha^2\sigma\right)}\cdot\frac{h(x,\pm\alpha\sigma)}{\sigma\left(\alpha^2-1\right)}=f_\pm(x,\pm\alpha\sigma)-\frac{R_\pm(\pm\alpha\sigma)}{T_\pm\left(\pm\alpha^2\sigma\right)}f_\pm\left(x,\pm\alpha^2\sigma\right),$$
$$\frac{1}{T_\pm(\pm\alpha\sigma)}\cdot\frac{h\left(x,\pm\alpha^2\sigma\right)}{\sigma(\alpha-1)}=f_\pm\left(x,\pm\alpha^2\sigma\right)-\frac{R_\pm\left(\pm\alpha^2\sigma\right)}{T_\pm(\pm\alpha\sigma)}f_\pm(x,\pm\alpha\sigma), \tag{5}$$

$$T_+(k)=\bar{T}_-(-\bar{k}),\ \ k\in\overline{\Omega^+}\setminus\{0\},$$
$$R_+(\alpha^j\sigma)=\alpha^j\bar{R}_-(-\alpha^j\sigma),\ \ j=1,2,$$
$$T_-(\sigma)=T_+(\alpha\sigma)T_+(\alpha^2\sigma)-R_+(\alpha\sigma)R_+(\alpha^2\sigma),\ \ \sigma>0. \tag{6}$$

We'll need the following easily proved lemma

Lemma. *Let $y(x,k_1)$ and $z(x,k_2)$ be solutions of the equation (1) for $k=k_1$ and $k=k_2$ respectively. If $k_1^3=-\bar{k}_2^3$ the expression*

$$\{y(x,k_1),\bar{z}(x,k_2)\}\equiv y''(x,k_1)\bar{z}(x,k_2)+y(x,k_1)\bar{z}''(x,k_2)-$$
$$-y'(x,k_1)\bar{z}'(x,k_2)-2q(x)y(x,k_1)\bar{z}(x,k_2)$$

is independent of x.

Using this lemma and asymptotic of the solutions $f_\pm$ when $|x|\to+\infty$, we have

$$T_+(k)=\frac{\{f_+(x,k),\bar{f}_-(x,-\bar{k})\}}{3k^2},\ \ k\in\overline{\Omega}^+\setminus\{0\},$$
$$R_+(\alpha^j\sigma)=\frac{\{f_+(x,\alpha^j\sigma),\bar{f}_-(x,-\alpha^j\sigma)\}}{3\alpha^j\sigma^2},\ \ \sigma>0. \tag{7}$$

Further, since Wronskian of three solutions of the equation (1) is independent of x, we have

$$W(h(x,k),f_+(x,\alpha k),f_-(x,\alpha^2 k))=9k^4T_+(\alpha k)\bar{T}_+(-\alpha\bar{k}),\ k\in\Delta^-,$$
$$W(h(x,k),f_+(x,\alpha^2 k),f_-(x,\alpha k))=9k^4T_+(\alpha^2 k)\bar{T}_+(-\alpha^2\bar{k}),\ k\in\Delta^+. \tag{8}$$

By the standard basing on the formulas (7), (8) and selfadjointe of the operator L generated in the space $L_2(\mathrm{R})$ on right hand side of the equation (1) it's established that the function $T_+(k)$ hasn't zeros in the sector $\frac{5\pi}{6}<\arg k<\frac{7\pi}{6}$ ([11]) and have zeros in the form of $k_{j,s}=i\alpha^s\mu_{j,s}$, where $(-1)^s\mu_{j,s}<0,\ s=1,2$ iff the solutions $f_+(x,k_{j,s})$ and $f_-(x,-\bar{k}_{j,s})$ are linear dependent:

$$f_+(x,k_{j,s})=c_{j,s}^+\,f_-(x,-\bar{k}_{j,s}), \tag{9}$$

in addition the numbers $\mu_{j,s}^3$ will be eigen-values of the operator L. Besides the function $T_+(k)$ can have infinite number in the sector $\frac{2\pi}{3}\le\arg k<\frac{5\pi}{6}$ and $\frac{7\pi}{6}\le\arg k<\frac{4\pi}{3}$. These zeros haven't spectral interpretation ([11], [4]). Later on we'll assume that such zeros absent. Further the operator L has finite number of eigen-values (see, for ex. [10]), consequently and number of zeros $k_{j,s}$ is finite.

Using the first formula from (7) for inverse quantities of norms of eigen-functions the equality

$$\left(m_{j,s}^+\right)^{-2}=\int_{-\infty}^{\infty}\left|f_+(x,k_{j,s})\right|^2dx=-\bar{c}_{j,s}^+\dot{T}_+(k_{j,s}) \tag{10}$$

is established, where the point means differentiation with respect to k. Therefore $\dot{T}_+(k_{j,s})\neq 0$, and consequently zeros of the function $T_+(k)$ are simple.

The combination of quantities

$$\left\{\frac{R_+(\alpha\sigma)}{T_+(\alpha^2\sigma)},\ \frac{R_+(\alpha^2\sigma)}{T_+(\alpha\sigma)},\ \mu_{j,s}\in\mathrm{R},\ m_{j,s}>0\right\} \tag{11}$$

we'll call scattering data of the equation (1). The "transmission" coefficient $T_+(k)$ although doesn't form scattering data but isn't determined by them unique ([11])

$$T_+(k)=\prod_{j,s}\frac{k-k_{j,s}}{k-\bar{k}_{j,s}}\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{\alpha^2}{\alpha^2\sigma-k}-\frac{\alpha}{\alpha\sigma-k}\right)\ln b(\sigma)d\sigma\right\}, \tag{12}$$

where

$$b(\sigma)=\begin{cases}1-\dfrac{R+(\alpha\sigma)R_+(\alpha^2\sigma)}{T_+(\alpha\sigma)T_+(\alpha^2\sigma)},\ \sigma>0,\\ 1-\overline{\left[\dfrac{R+(-\alpha\sigma)R_+(-\alpha^2\sigma)}{T_+(-\alpha\sigma)T_+(-\alpha^2\sigma)}\right]},\ \sigma<0\ .\end{cases}$$

§2. The inverse scattering problem.

The inverse scattering problem for the equation (1) is in restoration of the coefficients $p(x)$ and $q(x)$ by the scattering data (11). For the solution of the inverse problem we derive the basic Marchenko type integral equations. Using the inversion formula ([18]) from the representation (3) we have

$$\mathrm{F}_\pm(x,t)=\frac{3i}{4\pi}\int_{\Gamma_{-\frac{\pi}{3}}}\left[f_\pm(x,\mp k)e^{\pm kx}-1\right]k^{1/2}e^{tk^{3/2}}dk-$$

$$-\frac{3i}{4\pi}\int_{\Gamma_{\frac{\pi}{3}}}\left[f_\pm(x,\mp k)e^{\pm kx}-1\right]k^{1/2}e^{tk^{3/2}}dk\,.$$

Here and further by Γ_α we'll denote the ray $\arg k=\alpha$. From the relations (5) we find $f_\pm(x,\mp k)$ and putting them in the last qualities we obtain

$$\mathrm{F}_\pm(x,t)=\frac{3i}{4\pi}\int_{\Gamma_{-\frac{\pi}{3}}}\left[\frac{h(x,\mp k)e^{\pm kx}}{T_\pm(\mp\alpha k)k(\alpha^2-\alpha)}-1\right]k^{1/2}e^{tk^{3/2}}dk+$$

$$+\frac{3i}{4\pi}\int_{\Gamma_{-\frac{\pi}{3}}}\frac{R_\pm(\mp k)}{T_\pm(\mp\alpha k)}f_\pm(x,\mp\alpha k)e^{\pm kx}k^{1/2}e^{tk^{3/2}}dk-\int_{\Gamma_{\frac{\pi}{3}}}\left[\frac{h(x,\mp k)e^{\pm kx}}{T_\pm(\mp\alpha^2k)k(\alpha^2-\alpha)}-1\right]k^{1/2}e^{tk^{3/2}}dk-$$

$$-\frac{3i}{4\pi}\int_{\Gamma_{\frac{\pi}{3}}}\frac{R_\pm(\mp k)}{T_\pm(\mp\alpha^2k)}f_\pm(x,\mp\alpha^2k)e^{\pm kx}k^{1/2}e^{tk^{3/2}}dk\,. \tag{13}$$

Now at first in the first and third integrals we substitute the right hand side of (13) by the contours of integration respectively to $\Gamma_{\frac{4\pi}{3}}$ and $\Gamma_{\frac{2\pi}{3}}$ and we take into account that $h(x,i\mu_{j,s})=0$. Then in the obtained integrals instead of $h(x,\mp k)$ we put their expressions from the relations (5). Thus in all integrals in the right hand of (13) only the solutions $f_\pm$ will take part and if instead of $f_\pm$ we put their representation (3), then for the kernels $\mathsf{F}_\pm$ we obtain the following system of integral equations

$$\left(\mathsf{F}_+(x,t),\mathsf{F}_-(x,t)\right)+(1,1)\Phi(x,t,0)+\int_0^{+\infty}\left(\mathsf{F}_+(x,\xi),\mathsf{F}_-(x,\xi)\right)\Phi(x,t,\xi)ds=0\,, \qquad (14)$$

$$\Phi(x,t,\xi)=\begin{pmatrix} a^+(x,t,+\xi) & b^+(x,t+i\xi)+c^+(x,t-i\xi) \\ b^-(x,t+i\xi)+c^-(x,t-i\xi) & a^-(x,t+\xi)\end{pmatrix},$$

$$a^\pm(x,t)=-\frac{3i}{4\pi}\int_{\Gamma_{-\frac{\pi}{3}}}\frac{R_\pm(\mp k)}{T_\pm(\mp\alpha k)}e^{\pm(1-\alpha)kx}k^{1/2}e^{tk^{3/2}}dk+\frac{3i}{4\pi}\int_{\Gamma_{\frac{\pi}{3}}}\frac{R_\pm(\mp k)}{T_\pm(\mp\alpha^2 k)}e^{\pm(1-\alpha^2)kx}k^{1/2}e^{tk^{3/2}}dk\,,$$

$$b^\pm(x,t)=-\frac{3i}{4\pi}\int_{\Gamma_{-\frac{2\pi}{3}}}\frac{T_\pm(\pm\alpha^2 k)}{T_\mp(\pm\alpha k)}k^{1/2}e^{tk^{3/2}}dk-\frac{3i}{4\pi}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{R_\pm(\pm k)}{T_\mp(\pm\alpha^2 k)}e^{\pm(\alpha-1)kx}k^{1/2}e^{tk^{3/2}}dk\,,$$

$$c^\pm(x,t)=\frac{3i}{4\pi}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{T_\pm(\pm\alpha k)}{T_\mp(\pm\alpha^2 k)}k^{1/2}e^{tk^{3/2}}dk+\frac{3i}{4\pi}\int_{\Gamma_{-\frac{2\pi}{3}}}\frac{R_\pm(\pm k)}{T_\mp(\pm\alpha k)}e^{\pm(\alpha^2-1)kx}k^{1/2}e^{tk^{3/2}}dk\,. \qquad (15)$$

As we'll see later on the system of integral equation (14) in general, hasn't unique solution. However if in addition we require that the solution satisfies the conditions (9), i.e.

$$e^{k_{j,s}x}\left(1+\int_0^{+\infty}\mathsf{F}_+(x,t)e^{-(-k_{js})^{3/2}t}dt\right)=c^+_{j,s}e^{-\bar{k}_{j,s}x}\left(1+\int_0^{+\infty}\mathsf{F}_-(x,t)e^{-(-\bar{k}_{js})^{3/2}t}dt\right), \qquad (16)$$

where $\bar{c}^+_{j,s}=-\left[\dot{T}_+(k_{j,s})m^{+2}_{j,s}\right]^{-1}$, then the uniqueness of solution will be ensured. A system of the integral equations (14), (16) will be called basic integral equations of the inverse problem.

The following theorem shows that the solution of the inverse scattering problem for the equation (1) is unique.

Theorem. *Let* $f_\pm(\cdot)\in L_1(0,\infty)\cap L_2(0,\infty)$ *be a solution of a system of the equations*

$$\left(f_+(t),f_-(t)\right)+\int_0^{+\infty}\left(f_+(\xi),f_-(\xi)\right)\Phi(x,t,\xi)d\xi=0\,, \qquad (17)$$

$$\int_0^{+\infty}f_+(\xi)e^{-(-k_{js})^{3/2}\xi}d\xi=c^+_{j,s}e^{-2\operatorname{Re}k_{js}x}\int_0^{+\infty}f_-(\xi)e^{-(-\bar{k}_{js})^{3/2}\xi}d\xi\,. \qquad (18)$$

Then $f_+(t)=f_-(t)=0$.

Proof. We introduce the following denotations

$$R^x_\pm(\pm\alpha^j\sigma)=R_\pm(\pm\alpha^j\sigma)e^{\pm(-1)^j(\alpha-\alpha^2)\sigma x}\,,\quad j=1,2\,,$$

$$M_\pm(k)=\int_0^{+\infty}f_\pm(\xi)e^{-\xi(\mp k)^{3/2}}d\xi\,,\quad k\in\Omega^\pm\,.$$

Substituting in the equation (17) the representation (15) of the functions $a^\pm$, $b^\pm$, $c^\pm$ we have

$$f_\pm(t)+\frac{3}{4\pi i}\int_{\Gamma_{-\frac{\pi}{3}}}\frac{R^x_\pm(\mp k)}{T_\pm(\mp\alpha k)}M_\pm(\mp\alpha k)k^{1/2}e^{tk^{3/2}}dk-$$

$$\frac{3}{4\pi i}\int_{\Gamma_{\frac{\pi}{3}}}\frac{R_{\pm}^{x}(\mp k)}{T_{\pm}(\mp\alpha^2 k)}M_{\pm}(\mp\alpha^2 k)k^{1/2}e^{tk^{3/2}}dk=\varphi_{\pm}(t), \tag{19}$$

where

$$\varphi_{\pm}(t)=\frac{3}{4\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{T_{\mp}(\mp\alpha k)}{T_{\pm}(\mp\alpha^2 k)}\left[M_{\mp}(\mp k)-\frac{R_{\mp}^{x}(\mp k)}{T_{\mp}(\mp\alpha k)}M_{\mp}(\mp\alpha k)\right]k^{1/2}e^{tk^{3/2}}dk-$$

$$-\frac{3}{4\pi i}\int_{\Gamma_{-\frac{2\pi}{3}}}\frac{T_{\mp}(\mp\alpha^2 k)}{T_{\pm}(\mp\alpha k)}\left[M_{\mp}(\mp k)-\frac{R_{\mp}^{x}(\mp k)}{T_{\mp}(\mp\alpha^2 k)}M_{\mp}(\mp\alpha^2 k)\right]k^{1/2}e^{tk^{3/2}}dk .$$

On the other hand from definition of the function $M_{\pm}(k)$ by virtue of the inversion formula we have (see [18])

$$f_{\pm}(t)=\frac{3}{4\pi i}\int_{\Gamma_{\frac{\pi}{3}}}M_{\pm}(\mp k)k^{1/2}e^{tk^{3/2}}dk-\frac{3}{4\pi i}\int_{\Gamma_{-\frac{\pi}{3}}}M_{\pm}(\mp k)k^{1/2}e^{tk^{3/2}}dk .$$

Therefore (19) is transformed to the form

$$\frac{3}{4\pi i}\int_{\Gamma_{\frac{\pi}{3}}}\left[M_{\pm}(\mp k)-\frac{R_{\pm}^{x}(\mp k)}{T_{\pm}(\mp\alpha^2 k)}M_{\pm}(\mp\alpha^2 k)\right]k^{1/2}e^{tk^{3/2}}dk-$$

$$-\frac{3}{4\pi i}\int_{\Gamma_{-\frac{\pi}{3}}}\left[M_{\pm}(\mp k)-\frac{R_{\pm}^{x}(\mp k)}{T_{\pm}(\mp\alpha^2 k)}M_{\pm}(\mp\alpha k)\right]k^{1/2}e^{tk^{3/2}}dk=\varphi_{\pm}(t).$$

Hence denoting

$$\Phi_{\pm}(z)\equiv\int_{0}^{+\infty}\varphi_{\pm}(t)e^{-tz^{3/2}}dt=$$

$$=\frac{3}{4\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{T_{\mp}(\mp\alpha k)}{T_{\pm}(\mp\alpha^2 k)}\left[M_{\mp}(\mp k)-\frac{R_{\mp}^{x}(\mp k)}{T_{\mp}(\mp\alpha k)}M_{\mp}(\mp\alpha k)\right]\frac{k^{1/2}}{z^{3/2}-k^{3/2}}dk-$$

$$-\frac{3}{4\pi i}\int_{\Gamma_{-\frac{2\pi}{3}}}\frac{T_{\mp}(\mp\alpha^2 k)}{T_{\pm}(\mp\alpha k)}\left[M_{\mp}(\mp k)-\frac{R_{\mp}^{x}(\mp k)}{T_{\mp}(\mp\alpha^2 k)}M_{\mp}(\mp\alpha^2 k)\right]\frac{k^{1/2}}{z^{3/2}-k^{3/2}}dk ,$$

we obtain the following relations

$$M_{\pm}(\mp k)-\frac{R_{\pm}^{x}(\mp k)}{T_{\pm}(\mp\alpha^2 k)}M_{\pm}(\mp\alpha^2 k)=\Phi_{\pm}(k), \quad k\in\Gamma_{\pi/3} ,$$

$$M_{\pm}(\mp k)-\frac{R_{\pm}^{x}(\mp k)}{T_{\pm}(\mp\alpha k)}M_{\pm}(\mp\alpha k)=\Phi_{\pm}(k), \quad k\in\Gamma_{-\pi/3} . \tag{20}$$

Allowing for these equalities for the function $\Phi_{\pm}(z)$ we find

$$\Phi_{\pm}(z)=\frac{3}{4\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{T_{\mp}(\mp\alpha k)}{T_{\pm}(\mp\alpha^2 k)}\Phi_{\mp}(\pm k)\frac{k^{1/2}}{z^{3/2}-k^{3/2}}dk-$$
$$-\frac{3}{4\pi i}\int_{\Gamma_{-\frac{2\pi}{3}}}\frac{T_{\mp}(\mp\alpha^2 k)}{T_{\pm}(\mp\alpha k)}\Phi_{\mp}(\pm k)\frac{k^{1/2}}{z^{3/2}-k^{3/2}}dk$$

and since $\Phi_{\pm}(z)$ are regular in the sector $-\frac{2\pi}{3}<\arg z<\frac{2\pi}{3}$, and the integrands of the first integrals are regular in the sector Δ^{-}, the integrands of the second integrals- in the sector Δ^{+}, then changing the contour of integration from here we have

$$\Phi_{\pm}(z)=\frac{3}{4\pi i}\int_{\Gamma_{\frac{\pi}{3}}}\frac{T_{\mp}(\mp\alpha k)}{T_{\pm}(\mp\alpha^2 k)}\Phi_{\mp}(\pm k)\frac{k^{1/2}}{z^{3/2}-k^{3/2}}dk-$$
$$-\frac{3}{4\pi i}\int_{\Gamma_{-\frac{\pi}{3}}}\frac{T_{\mp}(\mp\alpha^2 k)}{T_{\pm}(\mp\alpha k)}\Phi_{\mp}(\pm k)\frac{k^{1/2}}{z^{3/2}-k^{3/2}}dk\,.$$

From these formulas we obtain the following relations connecting the functions Φ_{+} and Φ_{-}

$$\Phi_{\pm}(-\alpha\sigma)=\frac{T_{\mp}(\pm\sigma)}{T_{\pm}(\pm\alpha^2\sigma)}\Phi_{\mp}(\mp\alpha\sigma),$$
$$\Phi_{\pm}(-\alpha^2\sigma)=\frac{T_{\mp}(\pm\sigma)}{T_{\pm}(\pm\alpha\sigma)}\Phi_{\mp}(\mp\alpha^2\sigma)\,,\quad \sigma>0\,. \tag{21}$$

Consider now the regular in the sectors Δ^{+} and Δ^{-} functions

$$P(z)=\begin{cases}(\alpha^2-\alpha)\Phi_{-}(z)T_{-}(\alpha z), & z\in\Delta^{+}\,,\\ (\alpha-\alpha^2)\Phi_{-}(z)T_{+}(\alpha^2 z), & z\in\Delta^{-}\,,\end{cases}$$
$$\widetilde{P}(z)=\begin{cases}(\alpha^2-\alpha)\Phi_{+}(-z)T_{-}(\alpha^2 z), & z\in\Delta^{+}\,,\\ (\alpha-\alpha^2)\Phi_{+}(-z)T_{+}(\alpha z), & z\in\Delta^{-}\,.\end{cases}$$

The relations (21) show that

$$\widetilde{P}(-\alpha^2\sigma)=(\alpha-\alpha^2)\Phi_{+}(\alpha^2\sigma)T_{+}(-\sigma)=(\alpha-\alpha^2)\Phi_{-}(-\alpha^2\sigma)T_{-}(-\alpha\sigma)=P(-\alpha^2\sigma),$$
$$\widetilde{P}(-\alpha\sigma)=(\alpha-\alpha^2)\Phi_{+}(\alpha\sigma)T_{+}(-\sigma)=(\alpha-\alpha^2)\Phi_{-}(-\alpha\sigma)T_{-}(-\alpha^2\sigma)=P(-\alpha\sigma),$$

consequently $P(z)=\widetilde{P}(z)$.

Thus

$$\Phi_{\pm}(z)=\begin{cases}\pm\dfrac{P(\mp z)}{T_{\pm}(\mp\alpha z)(\alpha-\alpha^2)}, & z\in\Delta^{+}\,,\\ \pm\dfrac{P(\mp z)}{T_{\pm}(\mp\alpha^2 z)(\alpha^2-\alpha)}, & z\in\Delta^{-}.\end{cases}$$

Therefore we can write the relations (20) in the form of

$$M_{\pm}(\mp k)-\frac{R_{\pm}^{x}(\mp k)}{T_{\pm}(\mp\alpha^2 k)}M_{\pm}(\mp\alpha^2 k)=\frac{\pm P(\mp k)}{T_{\pm}(\mp\alpha^2 k)(\alpha^2-\alpha)}\,,\quad k\in\Gamma_{\pi/3}\,,$$

$$M_{\pm}(\mp k)-\frac{R_{\pm}^{x}(\mp k)}{T_{\pm}(\mp\alpha k)}M_{\pm}(\mp\alpha k)=\frac{\pm P(\mp k)}{T_{\pm}(\mp\alpha k)(\alpha-\alpha^2)}, \quad k\in\Gamma_{-\pi/3} \ . \qquad (22)$$

These relations coincide with relations from theorem 4 of article [11], p.586. but in our case the functions $T_{+}^{-1}(\alpha^2 k)$ and $T_{+}^{-1}(\alpha k)$ already have poles in the sectors Δ^{+} and Δ^{-1} respectively.

In the case of absence of poles in the indicated paper is proved that from relations (22) it follows $M_{+}(k)=M_{-}(k)=P(k)=0$. We show the correctness of these equalities in our case too. We'll use the condition (18)

$$M_{+}(i\alpha^s\mu_{j,s})=C_{j,s}^{*}M_{-}(i\alpha^s\mu_{j,s}), \qquad C_{j,s}^{*}=c_{j,s}^{+}e^{-2\operatorname{Re}k_{j,s}} .$$

On the other hand by the Cauchy theorem we have

$$\frac{1}{2\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{M_{+}(k)\overline{M}_{-}(-\bar{k})}{T_{+}(k)}dk-\frac{1}{2\pi i}\int_{\Gamma_{\frac{4\pi}{3}}}\frac{M_{+}(k)\overline{M}_{-}(-\bar{k})}{T_{+}(k)}dk=$$

$$=\sum\operatorname*{Re}_{k=k_{j,s}}s\frac{M_{+}(k)\overline{M}_{-}(-\bar{k})}{T_{+}(k)}=\sum\frac{M_{+}(k_{j,s})\overline{M}_{-}(-\bar{k}_{j,s})}{\dot{T}_{+}(k_{j,s})}=\sum\frac{|M_{+}(k_{j,s})|^2}{\overline{C}_{j,s}^{*}\overset{\circ}{T}_{+}(k_{j,s})} .$$

On other hand following as in [11] we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{M_{+}(k)\overline{M}_{-}(-\bar{k})}{T_{+}(k)}dk-\frac{1}{2\pi i}\int_{\Gamma_{\frac{4\pi}{3}}}\frac{M_{+}(k)\overline{M}_{-}(-\bar{k})}{T_{+}(k)}dk=$$

$$=-\frac{1}{6\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{P(k)\overline{P}(-\bar{k})}{T_{+}(\alpha k)\overline{T}_{+}(-\alpha\bar{k})}dk+\frac{1}{6\pi i}\int_{\Gamma_{\frac{4\pi}{3}}}\frac{P(k)\overline{P}(-\bar{k})}{T_{+}(\alpha^2 k)\overline{T}_{+}(-\alpha^2\bar{k})}dk=$$

$$=\frac{1}{6\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}|\alpha-\alpha^2|^2\,\overline{\Phi}_{+}(\bar{k})\Phi_{+}(-k)dk-\frac{1}{6\pi i}\int_{\Gamma_{\frac{4\pi}{3}}}|\alpha^2-\alpha|^2\,\Phi_{+}(-k)\overline{\Phi}_{+}(\bar{k})dk=$$

$$=\frac{1}{2\pi i}\int_{\Gamma_{\frac{\pi}{2}}}\Phi_{+}(-k)\overline{\Phi}_{+}(\bar{k})dk-\frac{1}{2\pi i}\int_{\Gamma_{\frac{\pi}{2}}}\Phi_{+}(-k)\overline{\Phi}_{+}(\bar{k})dk=$$

$$=\frac{1}{2\pi}\int_0^{+\infty}|\Phi_{+}(-i\sigma)|^2d\sigma+\frac{1}{2\pi}\int_0^{+\infty}|\Phi_{+}(i\sigma)|^2d\sigma .$$

Comparing the last relations we have

$$\sum\frac{|M_{+}(k_{j,s})|^2}{-\overline{C}_{j,s}^{*}\overset{\circ}{T}_{+}(k_{j,s})}+\frac{1}{2\pi}\int_0^{+\infty}|\Phi_{+}(-i\sigma)|^2d\sigma+\frac{1}{2\pi}\int_0^{+\infty}|\Phi_{+}(i\sigma)|^2d\sigma=0$$

and since $-\overline{C}_{j,s}^{*}\overset{\circ}{T}_{+}(k_{j,s})>0$, then from here we obtain $\Phi_{+}(k)=0$, i.e. $P(k)=M_{+}(k)=M_{-}(k)=0$. Consequently, $f_{+}(t)=f_{-}(t)=0$.

The theorem is proved.

§3. Example.

Consider an important particular case when we can obtain explicit formulas for the coefficients $p(x)$ and $q(x)$ of the equation (1). Assume that $R_+(\alpha\sigma)=R+(\alpha^2\sigma)=0$, $\mu_{1,1}=\mu$ and $m_{1,1}=m$ are arbitrary positive numbers. Later on we'll need the following formulas connected the Mittag-Leffler function $E_\rho(z;\mu)=\sum_{s=0}^{\infty}\frac{z^s}{\Gamma\left(\mu+\frac{s}{\rho}\right)}$ ([17], p.127 and [18])

$$E_{3/2}\left(-z;\frac{2}{3}\right)=\frac{3}{4\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{e^{k^{3/2}}k^{1/2}}{k+z}dk-\frac{3}{4\pi i}\int_{\Gamma_{-\frac{2\pi}{3}}}\frac{e^{k^{3/2}}k^{1/2}}{k+z}dk\,, \qquad (23)$$

$$\int_0^\infty e^{-\xi^{3/2}t}E_{3/2}\left(-zt^{3/2};\frac{2}{3}\right)t^{-\frac{1}{3}}dt=\frac{1}{\xi+z}\,, \qquad \xi,\, z\in\Omega^- . \qquad (24)$$

By the formulas (15) we'll calculate the free members of a system of the equations (14). Since (12)

$$T_+(k)=\frac{k-i\alpha\mu}{k-i\alpha^2\mu}\,, \qquad T_-(k)=\frac{k-i\alpha^2\mu}{k-i\alpha\mu}\,,$$

then using the formula (23) we have

$$b^-(x,t)+c^-(x,t)=\frac{3}{4\pi i}\int_{\Gamma_{-\frac{2\pi}{3}}}\frac{i(\alpha-\alpha^2)}{k+i\alpha^2\mu}k^{1/2}e^{tk^{3/2}}dk-\frac{3}{4\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{i(\alpha-\alpha^2)}{k+i\alpha^2\mu}k^{1/2}e^{tk^{3/2}}dk=$$

$$=i(\alpha^2-\alpha)t^{-\frac{1}{3}}E_{3/2}\left(-i\alpha^2\mu t^{2/3};\frac{2}{3}\right)=\sqrt{3}e_{3/2}(t;i\alpha^2\mu),$$

$$b^+(x,t)+c^+(x,t)=\sqrt{3}\mu e_{3/2}(t,-i\alpha\mu),$$

$$a^+(x,t)=a^-(x,t)=0\,.$$

Here $e_\rho(t,\lambda)=t^{\frac{1}{\rho}-1}E_\rho\left(-\lambda t^{1/\rho};\frac{1}{\rho}\right)$.

Allowing for these relations we'll search a solution of a system of the equations (14), (16) in the form of

$$\mathsf{F}_+(x,t)=A(x)e_{3/2}(t,i\alpha^2\mu), \qquad \mathsf{F}_-(x,t)=B(x)e_{3/2}(t,-i\alpha\mu).$$

Using the formulas (23) and (24) we have

$$\int_0^{+\infty}\mathsf{F}_+(x,\xi)\left[b^+(x,t+i\xi)+c^+(x,t-i\xi)\right]d\xi=A(x)\int_0^\infty e_{3/2}(t,i\alpha^2\mu)\times$$

$$\times\left[\frac{3}{4\pi i}\int_{\Gamma_{-\frac{2\pi}{3}}}\frac{k-i\alpha^2\mu}{k-i\alpha\mu}k^{1/2}e^{(t+i\xi)k^{3/2}}dk-\frac{3}{4\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{k-i\alpha^2\mu}{k-i\alpha\mu}k^{1/2}e^{(t+i\xi)k^{3/2}}dk\right]d\xi=$$

$$=A(x)\left[\frac{3}{4\pi i}\int_{\Gamma_{-\frac{2\pi}{3}}}\frac{k-i\alpha^2\mu}{k-i\alpha\mu}k^{1/2}e^{tk^{3/2}}\int_0^\infty\xi^{-\frac{1}{3}}E_{3/2}\left(-i\alpha^2\mu\xi^{2/3};\frac{2}{3}\right)e^{i\xi k^{3/2}}d\xi dk-\right.$$

$$-\frac{3}{4\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{k-i\alpha^2\mu}{k-i\alpha\mu}k^{1/2}e^{tk^{3/2}}\int_0^\infty \xi^{-\frac{1}{3}}E_{3/2}\left(-i\alpha^2\mu\xi^{2/3};\frac{2}{3}\right)e^{i\xi k^{3/2}}d\xi dk\Bigg]=$$

$$=A(x)\left[-\frac{3}{4\pi i}\int_{\Gamma_{-\frac{2\pi}{3}}}\frac{k^{1/2}e^{tk^{3/2}}}{k-i\alpha\mu}+\frac{3}{4\pi i}\int_{\Gamma_{\frac{2\pi}{3}}}\frac{k^{1/2}e^{tk^{3/2}}}{k-i\alpha\mu}dk\right]=A(x)e_{3/2}(t,-i\alpha\mu),$$

$$\int_0^{+\infty}F_-(x,\xi)\left[b^-(x,t+i\xi)+c^-(x,t-i\xi)\right]d\xi=B(x)e_{3/2}\left(t,i\alpha^2\mu\right).$$

Therefore the system of the equation (14) is converted to one equation

$$A(x)+B(x)+\sqrt{3}\mu=0\,. \qquad (25)$$

On the other hand, since

$$\overset{\circ}{T}_+(i\alpha\mu)=\frac{1}{i\left(\alpha-\alpha^2\right)\mu}=-\frac{1}{\sqrt{3}\mu}\,,$$

$$C^+=-\frac{1}{m^2\overset{\circ}{T}_+(i\alpha\mu)}=\frac{\sqrt{3}\mu}{m^2}\,,$$

then the equation (16) gets the form

$$1+\frac{A(x)}{\sqrt{3}\mu}=\frac{\sqrt{3}\mu}{m^2}e^{\sqrt{3}\mu x}\left(1+\frac{B(x)}{\sqrt{3}\mu}\right). \qquad (26)$$

Solving the system of the equations (25)-(26) we have

$$A(x)=-\frac{m^2e^{-\sqrt{3}\mu x}}{\sqrt{3}\mu+m^2e^{-\sqrt{3}\mu x}}\,,\quad B(x)=-\frac{3\mu^2}{\sqrt{3}\mu+m^2e^{-\sqrt{3}\mu x}},$$

i.e. in the considered case the system of the basic equations (14), (16) has the solution

$$F_+(x,t)=-\frac{m^2e^{-\sqrt{3}\mu x}}{\sqrt{3}\mu+m^2e^{-\sqrt{3}\mu x}}e_{3/2}\left(t,i\alpha^2\mu\right),$$

$$F_-(x,t)=-\frac{3\mu^2}{\sqrt{3}\mu+m^2e^{-\sqrt{3}\mu x}}e_{3/2}(t,-i\alpha\mu)\,.$$

Consequently, by the formula (4) we can find the functions $p(x)$ and $q(x)$

$$q(x)=\frac{3}{2}\frac{d}{dx}\left(I_t^{\frac{1}{3}}F_-(x,t)\right)_{t=0}=-\frac{9\mu^2}{2}\frac{d}{dx}\left(\frac{1}{\sqrt{3}\mu+m^2e^{-\sqrt{3}\mu x}}\right)=-\frac{9\mu^2}{2}ch^{-2}\frac{\sqrt{3}\mu}{2}(x+a)\,,$$

$$a=\frac{1}{\sqrt{3}\mu}\ln\left(\frac{\sqrt{3}\mu}{m^2}\right),$$

$$p(x)=-iq'(x)+\frac{4i}{3}q(x)\int_{-\infty}^{x}q(\xi)d\xi-3i\frac{d}{dx}\left(I_t^{\frac{1}{3}}D_t^{\frac{2}{3}}F_-\right)_{t=0}=$$

$$=\frac{9\mu^3}{2}\frac{d}{dx}\left(\frac{1}{\sqrt{3}\mu+m^2e^{-\sqrt{3}\mu x}}\right)=\frac{9\mu^3}{2}ch^{-2}\frac{\sqrt{3}\mu}{2}(x+a)\,.$$

The case $\mu < 0$ and also a general non-reflective case are analogously considered.

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