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A HARNACK INEQUALITY FOR SOLUTIONS OF SECOND ORDER NON-UNIFORMLY DEGENERATE PARABOLIC EQUATIONS

Abstract

In the paper a class of second order parabolic equations of divergent structure with non-uniformly power degeneration is considered. For non-negative weak solutions of the mentioned equations the Harnack inequality is proved.

Introduction. Let R_{n+1} be an $(n+1)$-dimensional Euclidean space of the points $(x,t)=(x_1,...,x_n,t)$, $Q_T \subset R^{n+1}$ be a bounded cylindrical domain located in R_{n+1}, where the point $(0,0)$ lies on the upper foundation of Q_T.

Consider the following equation in Ω

$$Lu = \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) = 0 \qquad (1)$$

in assumption that $\|a_{ij}(x,t)\|$ is real, symmetric matrix with measurable in Ω elements, where for any n-dimensional vector $\xi \in E_n$ and all $(x,t)\in\Omega$ the condition

$$\mu\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \le \mu^{-1}\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2 \qquad (2)$$

is satisfied, here $\mu\in(0,1]$ is a constant, $\lambda_i(x,t)=\left(|x|_\alpha + \sqrt{|t|}\right)^{\alpha_i}$

$$|x|_\alpha = \sum_{i=1}^{n}|x_i|^{\overline{\alpha}_i}, \quad \overline{\alpha}_i = \frac{2}{2+\alpha_i}, \quad \alpha=(\alpha_1,...,\alpha_n), \ 0\le\alpha_i<\frac{2}{n-1}, \ i=1,...,n .$$

The aim of the present paper is the proof of the Harnack inequality for non-negative solutions of the equations (1). Note that for uniformly parabolic equations of divergent structure the analogous result was obtained in [1-2]. Relative to parabolic equations in non-divergent form we note papers [3-6]. For second order divergent parabolic equations with uniformly degeneration the Harnack inequality was proved in [7], and for the equations with weakly ("logarithmic") degeneration- in [8]. Note that the existence and uniqueness of a weak solution of the first boundary value problem for the equation (1) on fulfillment of the condition (2) was established in [9].

1^0. The imbedding theorem with weight.

We'll keep the following denotations: $E_R^{x^0}(k)$ is an ellipsoid $\left\{x : \sum_{i=1}^{n}\frac{(x_i-x_i^0)^2}{R^{\alpha_i}} < (kR)^2\right\}$, Q_T is a cylinder $\Omega\times(-T_0,T)$. Denote by $A(Q_T)$ a set of all functions $u(x,t)\in C^\infty(\overline{Q}_T)$ such that for each of them there will be found the domain $\Omega(u)$, $\overline{\Omega}(u)\subset\Omega$ and $\operatorname{supp} u \subset \Omega(u)\times[-T_0,T]$. Let further $\overset{\circ}{W}{}_{2,\alpha}^{1,0}(Q_T)$ and $\overset{\circ}{W}{}_{2,\alpha}^{1,1}(Q_T)$ be completion of $A(Q_T)$ by the norms

$$\|u\|_{\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)}=\left(vrai\max_{t\in[-T_0,T]}\int_\Omega u^2dx+\int_{Q_T}\sum_{i=1}^n\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2dxdt\right)^{1/2},$$

and

$$\|u\|_{\overset{\circ}{W}{}^{1,1}_{2,\alpha}(Q_T)}=\left(\int_{Q_T}\left(u^2+\sum_{i=1}^n\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right)dxdt\right)^{1/2}$$

respectively.

The function $u(x,t)\in\overset{\circ}{W}{}^{1,0}_{2,\alpha}(\Omega)$ is called a weak solution of the equation (1) in Ω, if for any function $\eta(x,t)\in C_0^\infty(\Omega)$ the integral identity

$$\int_\Omega\left(\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial\eta}{\partial x_j}+\frac{\partial u}{\partial t}\eta\right)dxdt=0\,, \qquad (3)$$

$$Q(\rho)=\left(-\rho^2R^2,0\right)\times E^0_{\rho R}(1),\ S(\rho)=\left(-\left(\frac{1}{3}+\rho\right)R^2,-\left(\frac{3}{4}-\rho\right)R^2\right)\times E^0_{\rho R}(1)$$

is satisfied.

Theorem 1. *Let* $C=E^0_R(1)\times(t_1,t_2),\ u\in A(C),\ \sigma=\frac{2k-1}{k}$. *Then there exists* R_0 *such that for any* $R\le R_0$

$$\left(\oint\!\!\oint_C u^{2\sigma}dxdt\right)^{1/\sigma}\le c_1\left(\max_{t_1\le t\le t_1}\oint_{E^0_R(1)}u^2dx+R^2\oint\!\!\oint_C\sum_{i=1}^n\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2dxdt\right), \qquad (4)$$

where

$$\oint\!\!\oint_C u\,dxdt=\frac{1}{mes\,C}\iint_C udxdt\,.$$

Proof. By lemma 2 [10] there exists the constants $k(\alpha,n)>1$ and $c_2(\alpha,n)$ such that $u\in C_0^\infty\left(E^0_R(1)\right)$ the estimation

$$\left(\oint_{E^0_R(1)}u^{2k}dx\right)^{1/k}\le c_2R^2\oint_{E^0_R(1)}\sum_{i=1}^n\lambda_i\left(\frac{\partial u}{\partial x_i}\right)^2dx\,,$$

is valid. Then for an arbitrary function $u\in A(C)$

$$\oint_{E^0_R(1)}u^{2\sigma}dx\le\left(\oint_{E^0_R(1)}u^{2k}dx\right)^{1/k}\left(\oint_{E^0_R(1)}u^2dx\right)^{\sigma-1}\le$$

$$\le c_2R^2\left(\max_{t_1\le t\le t_2}\oint_{E^0_R(1)}u^2dx\right)^{\sigma-1}\oint_{E^0_R(1)}\sum_{i=1}^n\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2dx\,.$$

Integrating the last inequality with respect to t from t_1 to t_2 and raising to the power $\frac{1}{\sigma}$ the both side we obtain

[Guseynov S.T.]

$$\left(\iint_C u^{2\sigma}dx\right)^{1/\sigma}\le c_2\left(\max_{t_1\le t\le t_2}\oint_{E_R^0(1)}u^2dx\right)^{\frac{\sigma-1}{\sigma}}\left(R^2\iint_C\sum_{i=1}^n\lambda_i\left(\frac{\partial u}{\partial x_i}\right)^2dxdt\right)^{1/\sigma}.$$

Now applying the Young inequality we obtain

$$\left(\iint_C u^{2\sigma}dxdt\right)^{1/\sigma}\le c_3\left(\max_{t_1\le t\le t_2}\oint_{E_R^0(1)}u^2dx+\iint_C\sum_{i=1}^n\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2dxdt\right).$$

The theorem is proved.

2^0. Some auxiliary estimations.

Lemma 1. *If $u(x,t)$ is a positive solution of the equation (1) then for τ_1,τ_2, $-R^2\le\tau_1<\tau_2\le 0$ the estimations*

1) $\beta\ne 0,-1$; $\vartheta(x,t)=u^{\frac{\beta+1}{2}}(x,t)$

$$\frac{sign\beta}{\beta+1}\int_{E_{R(1)}^0}\vartheta^2\eta^2dx\Big|_{\tau_1}^{\tau_2}+\frac{2|\beta|}{\mu(\beta+1)^2}\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i=1}^n\lambda_i\left(\frac{\partial\vartheta}{\partial x_i}\right)^2\eta^2dx\le$$

$$\le\frac{2}{|\beta+1|}\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\vartheta^2\eta\left|\frac{\partial\eta}{\partial t}\right|dx+\frac{2\mu^3}{|\beta|}\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i=1}^n\left(\frac{\partial\eta}{\partial x_i}\right)^2\vartheta^2\,dx\,; \tag{5}$$

2) $\beta=-1$; $\vartheta(x,t)=-\ln u(x,t)$

$$\int_{E_{R(1)}^0}\vartheta\eta^2dx\Big|_{\tau_1}^{\tau_2}+\frac{1}{2\mu}\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i=1}^n\lambda_i\left(\frac{\partial\vartheta}{\partial x_i}\right)^2\eta^2dx\le$$

$$\le 2\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\vartheta\eta\left|\frac{\partial\eta}{\partial t}\right|dx+2\mu^3\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i=1}^n\lambda_i\left(\frac{\partial\eta}{\partial x_i}\right)^2dx \tag{6}$$

are valid, where $\eta(x,t)$ is any non-negative function from $A(Q(1))$.

Proof. Let $\beta\ne 0,-1$. Since $u(x,t)$ is a solution of the equation (1), then for any function $\varphi\in A(Q(1))$, $-R^2\le\tau_1<\tau_2\le 0$

$$\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\varphi\frac{\partial u}{\partial t}dx=\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)\varphi dx\,. \tag{7}$$

Assume here $\varphi=\eta^2u^\beta sign\beta$, where $\eta\in A(Q(1))$, $\eta(x,t)\ge 0$. We have

$$\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i,j=1}^n\frac{\partial x}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)\eta^2u^\beta sign\beta\,dx=-2\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\eta\frac{\partial\eta}{\partial x_i}u^\beta sign\beta dx-$$

$$-\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\beta\cdot u^{\beta-1}\eta^2sign\beta\,dx\le 2\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sqrt{\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}}\sqrt{\sum_{i,j=1}^n a_{ij}\frac{\partial\eta}{\partial x_i}\frac{\partial\eta}{\partial x_j}}\times$$

$$\times\eta\cdot u^\beta dx-\frac{|\beta|}{\mu}\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i=1}^n\lambda_i\left(\frac{\partial u}{\partial x_i}\right)^2u^{\beta-1}\eta^2dx\le 2\mu\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sqrt{\sum_{i=1}^n\lambda_i\left(\frac{\partial u}{\partial x_i}\right)^2}\sqrt{\sum_{i=1}^n\lambda_i\left(\frac{\partial\eta}{\partial x_i}\right)^2}\times$$

$$\times\eta\cdot u^{\beta}dx-\frac{|\beta|}{\mu}\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i=1}^{n}\lambda_i\left(\frac{\partial u}{\partial x_i}\right)^2 u^{\beta-1}\eta^2dx\le\frac{2\mu^3}{|\beta|}\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}u^{\beta+1}\sum_{i=1}^{n}\lambda_i\left(\frac{\partial\eta}{\partial x_i}\right)^2dx-$$

$$-\frac{|\beta|}{2\mu}\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i=1}^{n}\lambda_i\left(\frac{\partial u}{\partial x_i}\right)^2 u^{\beta-1}\eta^2dx=\frac{2\mu^3}{|\beta|}\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i=1}^{n}\lambda_i\left(\frac{\partial\eta}{\partial x_i}\right)^2\vartheta^2dx-$$

$$-\frac{2|\beta|}{\mu(\beta+1)^2}\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\sum_{i=1}^{n}\lambda_i\left(\frac{\partial\vartheta}{\partial x_i}\right)^2\eta^2dx\,,$$

where $\vartheta=u^{\frac{\beta+1}{2}}$. Further

$$\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\frac{\partial u}{\partial t}\eta^2u^{\beta}sign\beta dx=\frac{sign\beta}{\beta+1}\int_{E_R^0(1)}\vartheta^2\eta^2dx\Bigg|_{\tau_1}^{\tau_2}-\frac{2sign\beta}{\beta+1}\int_{\tau_1}^{\tau_2}dt\int_{E_R^0(1)}\vartheta^2\eta\frac{\partial\eta}{\partial t}dx\,.$$

Allowing for the obtained estimation in (7) we obtain the statement of the lemma when $\beta\ne0,-1$. If $\beta=-1$ then we must lead the same reasoning assuming $\varphi=\eta^2u^{-1}$, $\eta\in C_0^{\infty}(Q(1))$, $\eta(x,t)\ge0$ and $\vartheta=-\ln u$.

Lemma 2. *Let* $r_0=\sigma^{-\nu}(1+\sigma)^{-1}$, $\nu=0,1,2,\ldots$, $\sigma=\dfrac{2k-1}{k}$, *u be a positive solution of the equation (1). Then the inequalities*

$$\max_{S(\rho')}u^{r_0}\le c_4\frac{1}{(\rho-\rho')^{n+1}}\left(\iint_{S(\rho)}u^{2r_0}dxdt\right)^{1/2}=c_4\frac{1}{(\rho-\rho')^{n+1}}\left\|u^{r_0};S(\rho)\right\|_{2,2};$$

$$\max_{Q(\rho')}u^{-r_0}\le c_5\frac{1}{(\rho-\rho')^{n+1}}\left\|u^{-r_0};Q(\rho)\right\|_{2,2}$$

are valid, where $\frac{1}{3}\le\rho'<\rho\le\frac{1}{2}$ *and constants* c_4,c_5 *depend only on* μ, λ *and* n.

Proof. Let $\eta(x,t)$ is a cut off function: $\eta(x,t)=1$ in $S(\rho')$, $\eta(x,t)=0$ outside of $S(\rho)$, $0\le\eta(x,t)\le1$, $\operatorname{supp}\eta\subset\overline{S(\rho)}\subset\overline{S\left(\frac{1}{2}\right)}$,

$$\left|\frac{\partial\eta}{\partial t}\right|\le\frac{2}{(\rho-\rho')R^2}\,,\quad\left|\frac{\partial\eta}{\partial x_i}\right|\le\frac{2}{(\rho-\rho')R^{1+\alpha_i/2}}\,.$$

Now if we apply lemma 1 assuming $\beta>0$, $\tau_1=-\frac{5}{6}R^2$, $\tau_2=\tau$, then we obtain

$$\frac{2\beta}{\mu(\beta+1)^2}\iint_{S(\rho)}\sum_{i=1}^{n}\lambda_i\left(\frac{\partial\vartheta}{\partial x_i}\right)^2\eta^2dxdt\le\frac{2}{\beta+1}\frac{2}{(\rho-\rho')^2R^2}\times$$

$$\times\iint_{S(\rho)}\vartheta^2\eta dxdt+\frac{2\mu^3}{\beta}\frac{4n}{(\rho-\rho')^2R^2}\iint_{S(\rho)}\vartheta^2dxdt\le\frac{C_6}{\beta(\rho-\rho')^2R^2}\iint_{S(\rho)}\vartheta^2dxdt\,.$$

On the other hand

$$\frac{1}{\beta+1}\int_{E_{\rho R}^0(1)}\vartheta^2\eta^2dx\Bigg|_{t=\tau}\le\frac{c_7}{\beta(\rho-\rho')^2R^2}\iint_{S(\rho)}\vartheta^2dxdt\,.$$

Thus we have

$$\left.\begin{aligned}\|\eta\vartheta;S(\rho)\|_{2,\infty}^2 &\le c_8\left(\frac{\beta+1}{\beta}\right)^2\frac{1}{(\rho-\rho')^2}\|\vartheta;S(\rho)\|_{2,2}^2\\ \|\eta\vartheta_x;S(\rho)\|_{2,2}^2 &\le c_9\left(\frac{\beta+1}{\beta}\right)^2\frac{1}{(\rho-\rho')^2R^2}\|\vartheta;S(\rho)\|_{2,2}^2\end{aligned}\right\} \tag{8}$$

where

$$\|\eta\vartheta;S(\rho)\|_{2,\infty}=\left(\max_t\oint_{E^0_{\rho R}(1)}\eta^2\vartheta^2dx\right)^{1/2},$$

$$\|\eta\vartheta_x;S(\rho)\|_{2,2}=\left(\oiint_{S(\rho)}\sum_{i=1}^n\lambda_i\left(\frac{\partial\vartheta}{\partial x_i}\right)^2\eta^2dxdt\right)^{1/2},$$

let now $\beta\in(-1,0)$. Assume in lemma 1 $\tau_1=\tau$, $\tau_2=-\frac{1}{4}R^2$. Then we obtain

$$\left.\begin{aligned}\|\eta\vartheta;S(\rho)\|_{2,\infty}^2 &\le \frac{c_{10}}{|\beta|^2(\rho-\rho')^2}\|\vartheta;S(\rho)\|_{2,2}^2,\\ \|\eta\vartheta_x;S(\rho)\|_{2,2}^2 &\le \frac{c_{11}}{|\beta|^2(\rho-\rho')^2R^2}\|\vartheta;S(\rho)\|_{2,2}^2.\end{aligned}\right\} \tag{9}$$

Assume now $\beta=\frac{2\sigma^m}{\sigma^\nu(1+\sigma)}-1$, where m,ν are any non-negative integers. Therefore for all considered β

$$\beta\ge\frac{1}{2n+1}.$$

From this estimation, from the inequalities (8), (9) and from theorem 1 it follows

$$\left\|\vartheta^\sigma;S(\rho')\right\|_{2,2}^{2/\sigma}\le c_{12}\left(\oiint_{S(\rho)}\vartheta^{2\sigma}\eta^{2\sigma}dxdt\right)^{1/\sigma}\le\frac{c_{13}}{(\rho-\rho')^2}\iint_{S(\rho)}\vartheta^2dxdt. \tag{10}$$

We determine the sequences

$$\rho'=\rho'_m=\rho'\left(1+\frac{\rho-\rho'}{2^{m+1}\rho'}\right),\quad \rho=\rho_m=\rho\left(1+\frac{\rho-\rho'}{2^m\rho'}\right),\ \beta=\beta_m=\frac{2\sigma^m}{\sigma^\nu(1+\sigma)}-1,$$

$$\vartheta=\vartheta_m=u^{\frac{\beta_m+1}{2}}.$$

Then from (10) it follows

$$\Phi_{m+1}=\left(\oiint_{S(\rho_{m+1})}\vartheta_{m+1}^2dxdt\right)^{1/\sigma^{m+1}}=\left(\iint_{S(\rho'_m)}\vartheta_m^{2\sigma}dxdt\right)^{1/\sigma^{m+1}}\le\left[\frac{c_{14}}{(\rho_m-\rho'_m)^2}\|\vartheta_m;S(\rho_m)\|_{2,2}^2\right]^{\sigma^{-m}}\le$$

$$\le c_{14}^{\sigma^{-m}}\cdot 2^{2m\cdot\sigma^{-m}}\frac{1}{(\rho-\rho')^{2\sigma-m}}\Phi_m\le\ldots\le c_{14}^{\sum_{j=0}^{m}\sigma^{-j}}\cdot 2^{\sum_{j=0}^{m}2j\sigma^{-j}}\frac{1}{(\rho-\rho')^{2\sum_{j=0}^{m}\sigma^{j}}}\Phi_0\le$$

$$\le\frac{c_{15}}{(\rho-\rho')^{2(n+1)}}\Phi_0\ .$$

It means that there exists a subsequence $\{\Phi_{m_k}\}$ such that $\lim\limits_{k\to\infty}\Phi_{m_k}=B\le\frac{c_{15}}{(\rho-\rho')^{2(n+1)}}\Phi_0$.

It's established that $B\ge\max\limits_{S(\rho')}u^{2r_0}=A$.

Really, let it not be like this. We take $C\in(B,A)$. Denote by E a set $\{x,t\in S(\rho'):u^{2r_0}(x,t)>C\}$. By proposition $mesE>0$ and

$$\Phi_{m_k}=\left(\iint\limits_{S(\rho_{m_k})}\left(u^{2r_0}\right)^{\sigma^{m_k}}dxdt\right)^{\sigma^{-m_k}}\ge C\left(\frac{mesE}{mesS(\rho_{m_k})}\right)^{\sigma^{-m_k}}\ge C\left(\frac{mesE}{mesS(\rho)}\right)^{\sigma^{-m_k}}.$$

And it means that $B=\lim\limits_{k\to\infty}\Phi_{m_k}\ge C$.

The obtained contradiction proves the lemma.

The second inequality in statements of the lemma is proved in exactly the same way in addition instead of $S(\rho)$ we consider $Q(\rho)\subset Q\left(\frac{1}{2}\right)$, the estimation in lemma 1 is used for $\beta<-1$

$$\tau_1=-\frac{1}{4}R^2\ ,\quad \tau_2=\tau\ ,\quad \beta=\beta_m=-1-\frac{2\sigma^m}{\sigma^{\nu}(1+\sigma)}.$$

3^0. Estimations of maximum of solutions.

Lemma 3. *At the same assumptions of lemma 2 the estimation*

$$\max_{Q\left(\frac{1}{3}\right)}u\le c_{16}\left\|u;Q\left(\frac{1}{2}\right)\right\|_{2,2}$$

is valid.

The lemma is proved by the same scheme as previous one only instead of $S(\rho)$ we take $Q(\rho)\subset Q\left(\frac{1}{2}\right)$ and lemma 1 is used when

$$\beta=\beta_m=2^{\sigma^m}-1\ ,\quad m=0,1,\ldots\ ,\quad \tau_1=-\frac{1}{4}R^2\ ,\quad \tau_2=t\ .$$

Lemma 4. *Let $u(x,t)$ be a positive solution of the equation (1). Then there exist the constants a_1 and a_2 such that for any $S>0$*

$$mes\{(x,t)\in D_1:\ln u>S+a_1\}\le c_{17}\frac{R^2 mes\mathrm{E}_R^0(1)}{S},$$

$$mes\{(x,t)\in D_2:\ln u<-S+a_2\}\le c_{18}\frac{R^2 mes\mathrm{E}_R^0(1)}{S},$$

where

$$D_1 : \left(-R^2;\frac{1}{2}R^2\right)\times E^0_{\frac{R}{2}}(1),$$

$$D_2 : \left(-\frac{1}{2}R^2,0\right)\times E^0_{\frac{R}{2}}(1).$$

Proof. We use lemma 1, let $\eta(x,t)=\omega(t)\xi(x)$, where $\omega(t)=1$ when $t\le -\tau_1 R^2$, $\omega(t)=0$ when $t\ge -\frac{\tau_1}{2}R^2$, $0\le\omega(t)\le 1$, $\left|\frac{\partial\omega}{\partial t}\right|\le\frac{2}{\tau_1 R^2}$, $0<\tau_1<\tau_2\le 1$; $\xi(x)=1$ in $E^0_{\frac{R}{2}}(1)$, $\xi(x)=0$ outside of $E^0_{\frac{5}{6}R}(1)$, $\xi(x)\in[0,1]$, $\left|\frac{\partial\xi}{\partial x_i}\right|\le\frac{3}{R^{1+\alpha_i/2}}$, $i=1,2,...,n$.

We choose the function $\xi(x)$ such that for C the set $\{x:\xi(x)\ge C\}$ were convex. Let further $\vartheta=-\ln u$, from lemma 1 we obtain

$$\int_{E^0_{\frac{5}{6}R}(1)}\vartheta\xi^2dx\Bigg|_{-\tau_2R^2}^{-\tau_1R^2}+\frac{1}{2\mu}\int_{-\tau_2R^2}^{-\tau_1R^2}dt\int_{E^0_{\frac{5}{6}R}(1)}\sum_{i=1}^n\lambda_i\left(\frac{\partial\vartheta}{\partial x_i}\right)^2\xi^2dx\le c_{19}(\tau_2-\tau_1)mesE^0_R(1). \qquad (11)$$

Consider the averaging

$$V(t)=\int_{E^0_R(1)}\vartheta(x,t)\xi^2(x)dx\Big/\int_{E^0_R(1)}\xi^2(x)dx$$

and dispersion corresponding to this average

$$D(t)=\int_{E^0_R(1)}(\vartheta(x,t)-V(x,t))^2\xi^2(x)dx\Big/\int_{E^0_R(1)}\xi^2(x)dx.$$

By lemma 2

$$\left(\int_{E^0_R(1)}\xi^2(x)dx\right)^2D(t)\le c_{11}R^2mesE^0_R(1)\int_{E^0_R(1)}\sum_{i=1}^n\lambda_i\left(\frac{\partial\vartheta}{\partial x_i}\right)^2\xi^2(x)dx.$$

From (11) we obtain

$$V(-\tau_1R^2)-V(-\tau_2R^2)+c_{20}\frac{1}{R^2mesE^0_R(1)}\int_{-\tau_2R^2}^{-\tau_1R^2}dt\int_{E^0_{\frac{R}{2}}(1)}(\vartheta-V)^2dx\le c_{21}(\tau_2-\tau_1).$$

Let $t=-\tau_1R^2$ and $\tau_2=\tau_1$ then we have

$$R^2\frac{dV}{dt}+c_{20}\frac{1}{mesE^0_R(1)}\int_{E^0_{\frac{R}{2}}(1)}(\vartheta-V)^2dx\le c_{21}. \qquad (12)$$

Now we introduce the functions

$$w(x,t)=\vartheta(x,t)+\frac{c_{21}}{R^2}\left(-\frac{R^2}{2}-t\right),$$

$$W(t)=V(t)+\frac{c_{21}}{R^2}\left(-\frac{R^2}{2}-t\right).$$

Then we write the inequality (12) in the following form

$$R^2\frac{dW}{dt}+c_{20}\frac{1}{mes\mathrm{E}_R^0(1)}\int_{\mathrm{E}_{\frac{R}{2}}^0(1)}(w-W)^2dx\le 0\,. \qquad (13)$$

From (12) it follows that the function $W(t)$ is monotone non-increasing, therefore for any $t\in\left(-R^2,-\frac{R^2}{2}\right)$, $W(t)\ge W\left(-\frac{R^2}{2}\right)=V\left(-\frac{R^2}{2}\right)$, and for any $t\in\left(-\frac{R^2}{2},0\right)$, $W(t)\le$ $\le W\left(-\frac{R^2}{2}\right)$. Let

$$E_1(t)=\left\{x\in\mathrm{E}_{\frac{R}{2}}^0(1):w(x,t)<S_1\right\}.$$

Then for $t\in\left(-R^2,-\frac{R^2}{2}\right)$

$$0\le R^2\frac{dW}{dt}+c_{20}\frac{1}{mes\mathrm{E}_{\frac{R}{2}}^0(1)}\int_{E_1(t)}(w-W)^2dx\ge R^2\frac{dW}{dt}+c_{20}\frac{mesE_1(t)}{\mathrm{E}_{\frac{R}{2}}^0(1)}(W(t)-S_1)^2\,.$$

Hence we obtain

$$R^2\int_{-R^2}^{\frac{R^2}{2}}\frac{dW}{(W-S_1)^2}\le -c_{20}\frac{1}{mes\mathrm{E}_{\frac{R}{2}}^0(1)}\int_{-R^2}^{-\frac{R^2}{2}}mesE_1(t)dt=-\frac{c_{20}}{mes\mathrm{E}_R^0(1)}m_1(S_1)$$

and further

$$-R^2\frac{1}{W(t)-S_1}\Bigg|_{-R^2}^{-\frac{R^2}{2}}\le -\frac{c_{20}}{mes\mathrm{E}_R^0(1)}m_1(S_1)\,.$$

Thus

$$m_1(S_1)\le\frac{R^2mes\mathrm{E}_R^0(1)}{c_{20}\left(V\left(-\frac{R^2}{2}\right)-S_1\right)}\,.$$

The last inequality means that for any $S>0$

$$mes\left\{(x,t)\in D_1:\ln u>S-V\left(-\frac{R^2}{2}\right)+c_{22}\frac{\left(-\frac{R^2}{2}-t\right)}{R^2}\right\}\le\frac{R^2mes\mathrm{E}_R^0(1)}{S\cdot c_{20}}\,.$$

Since $t\in\left(-R^2,-\frac{R^2}{2}\right)$ then

$$mes\{(x,t)\in D_1:\ln u>S+a_1\}\le\frac{R^2mes\mathrm{E}_R^0(1)}{S\cdot c_{20}}\,,$$

where $a_1=-V\left(-\frac{R^2}{2}\right)+\frac{c_{22}}{2}$.

It's in exactly same way it is proved that if $S_2 > V\left(-\frac{R^2}{2}\right)$ and $m_2(S_2) = mes\{(x,t) \in D_2 : w > S_2\}$, then

$$m_2(S_2) \le \frac{R^2 mes E_R^0(1)}{c_{20}\left(S_2 - V\left(-\frac{R^2}{2}\right)\right)},$$

$$mes\{(x,t) \in D_2 : \ln u < -S + a_2\} \le \frac{R^2 mes E_R^0(1)}{S \cdot c_{20}}$$

where $a_2 = -V\left(-\frac{R^2}{2}\right) - \frac{c_{22}}{2}$. In addition $a_1 - a_2 = c_{22}$.

The lemma is proved.

Now we introduce the function $w_1(x,t) = u(x,t)\, e^{-a_1}$, $w_2(x,t) = (u(x,t))^{-1} \cdot e^{a_2}$, where $u(x,t)$ is a positive solution of the equation (1). From lemmas 2 and 4 it follows that for $\frac{1}{3} \le \rho' < \rho \le \frac{1}{2}$, $r_0 = \frac{1}{\sigma^\nu(1+\sigma)}$, $\nu = 0,1,...,$ $j = 1,2$

$$\max w_j^{r_0} \le c_{23} \frac{1}{(\rho - \rho')^{n+1}} \left\| w_j^{r_0}; S_j(\rho) \right\|_{2,2}, \tag{14}$$

$$mes\left\{(x,t) \in S_j\left(\frac{1}{2}\right) : \ln w_j > S\right\} \le c_{24} \frac{R^2 mes E_R^0(1)}{S}, \tag{15}$$

where $S_1(\rho) = S(\rho)$, $S_2(\rho) = Q(\rho)$.

From (14), (15) in turn it's derived.

Lemma 5. *The inequalities*

$$\max_{S_j\left(\frac{1}{3}\right)} w_j(x,t) \le c_{25} \quad (j = 1,2). \tag{16}$$

Proof. The inequalities (14), (15) are samely written for the functions w_1 and w_2. Therefore it's sufficient to prove (16) for $j = 1$. Let

$$\varphi(\rho) = \max_{S(\rho)} \ln w_1(x,t),$$

$$K = \max(2c_{23}, c_{24}, 1).$$

The function $\varphi(\rho)$ is monotone non-decreasing. If $\varphi\left(\frac{1}{3}\right) \le 3k$, then (16) holds with $c_{25} = e^{3k}$. Thus it's possible to consider the case $\varphi(\rho) > 3k$, $\rho \in \left[\frac{1}{3}, \frac{1}{2}\right]$. We show that in addition for any ρ', ρ, $\rho' < \rho$ the inequality

$$\varphi(\rho') < \frac{3}{4}\varphi(\rho) + c_{26} \frac{1}{(\rho - \rho')^{s(n+1)}} \tag{17}$$

is valid.

We'll divide $S(\rho)$ into two parts referring it at the first point, where

$$\frac{1}{2}\varphi(\rho') < \ln w_1(x,t) \le \varphi(\rho)$$

at the second point, where

$$\ln w_1(x,t) \le \frac{1}{2}\varphi(\rho).$$

We have

$$\left\| w_1^{r_0}; S(\rho) \right\|_{2,2}^2 = \oint\!\!\!\oint_{S(\rho)} w_1^{2r_0}\, dxdt \le \frac{k}{\varphi(\rho)} e^{2r_0\varphi(\rho)} + e^{r_0\varphi(\rho)}.$$

We choose the non-negative number ν so large that

$$r_0\sigma = \frac{\sigma}{\sigma^{\nu}(1+\sigma)} > \frac{1}{\varphi(\rho)} \ln\frac{\varphi(\rho)}{k}, \tag{18}$$

since $\frac{\varphi(\rho)}{k} > 3$ and the function $\frac{\ln x}{x}$ decreases, i.e.

$$\frac{1}{\varphi(\rho)} \ln\frac{\varphi(\rho)}{k} \le \frac{\ln 3}{3} < \frac{1}{2}.$$

At such $r_0 = r_0(\rho)$ from (14) it follows

$$\varphi(\rho') = \max_{S(\rho')} \ln w_1(x,t) \le \frac{1}{2r_0} \ln\left(c_{23} \frac{1}{(\rho-\rho')^{2(n+1)}} \right) + \frac{\varphi(\rho)}{2}.$$

Using now (18) we obtain

$$\varphi(\rho') \le \frac{1}{2}\varphi(\rho) \left\{ \frac{\sigma}{\ln\frac{\varphi(\rho)}{k}} \ln\left(c_{23} \frac{1}{(\rho-\rho')^{2(n+1)}} \right) + 1 \right\}. \tag{19}$$

From (19), (17) follows.

Really, if the first addend being in (19)in parenthesis, doesn't exceed $\frac{1}{2}$, then from (19) it follows

$$\varphi(\rho') \le \frac{3}{4}\varphi(\rho).$$

If

$$\frac{\sigma}{\ln\frac{\varphi(\rho)}{k}} \ln\left(c_{23} \frac{1}{(\rho-\rho')^{2(n+1)}} \right) > \frac{1}{2},$$

then

$$\ln\frac{\varphi(\rho)}{k} < 2\sigma \ln\left(c_{23} \frac{1}{(\rho-\rho')^{2(n+1)}} \right) \le 4 \ln\left(c_{23} \frac{1}{(\rho-\rho')^{2(n+1)}} \right).$$

Hence we conclude that

$$\varphi(\rho') \le \varphi(\rho) < c_{26}^4 \frac{1}{(\rho-\rho')^{8(n+1)}}.$$

The lemma is proved.

4^0. A Harnack inequality.

Theorem 2. *Let* $u(x,t)$ *be any non-negative solution of the equation (1) from* $\dot{W}_{2,2}^{1,0}(Q(1))$. *Then the inequality*

$$\sup_{S\left(\frac{1}{3}\right)} u(x,t) \le c_{27} \inf_{Q\left(\frac{1}{3}\right)} u(x,t)$$

is valid.

Proof. From lemma 5 it follows

$$\max_{S\left(\frac{1}{3}\right)} w_1(x,t) \max_{Q\left(\frac{1}{3}\right)} w_2(x,t) = e^{-a_1+a_2} \max_{S\left(\frac{1}{2}\right)} u(x,t) \max_{Q\left(\frac{1}{3}\right)} (u(x,t))^{-1} \le c_{25}^2 .$$

Thus

$$\max_{S\left(\frac{1}{3}\right)} u(x,t) \le c_{28} \min_{Q\left(\frac{1}{3}\right)} u(x,t) .$$

The theorem is proved.

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