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ON BEHAVIOR OF SOLUTION OF THE INITIAL-BOUNDARY VALUE PROBLEM FOR THE ROSSBI WAVE EQUATION IN CYLINDRICAL DOMAIN AT $t \to +\infty$

Abstract

In this article the unique solvability of initial-boundary value problem for the Rossbi wave equation in cylindrical domain is investigated and the estimation of solution of the initial-boundary value problem at $t \to +\infty$ is received.

At studying dynamics of the acoustic, surface and intrinsic waves Rossbi equation was introduced. In [1]-[6] the Cauchy problem and some initial-boundary value problems for the Sobolev type equations was studied. But initial-boundary value problem for the Rossbi wave equation wasn't studied. We study unique solvability of the initial-boundary value problem for this equation in multidimensional cylindrical domain and receive the estimation of solution of this problem at $t \to +\infty$. For this Green function of corresponding stationary boundary value problem was constructed.

§1. Notations, definitions and uniqueness of solution of initial-boundary value problem for Rossbi equation.

Let $R_m(y)$ be m-dimensional Euclidean space with element $y=(y_1,y_2,\dots,y_m)$ and $R_n(x)$ is the same space with element $x=(x_1,x_2,\dots,x_m)$. Let $\amalg = R_n(x)\times\Omega$ be a cylindrical domain in $R_n(x)\times R_m(y)$, where Ω is a bounded domain in $R_m(y)$ with smooth boundary $\partial\Omega$. Let $Q=\amalg\times(0,\infty)$ We consider in Q the next problem

$$\frac{\partial}{\partial t}\Delta_{n+m}u(x,y,t)+\Delta_m u(x,y,t)=0 \qquad (1.1)$$

with the initial condition

$$u(x,y,0)=\varphi(x,y) \qquad (1.2)$$

and the boundary condition

$$u(x,y,t)\big|_{\partial\amalg\times(0,\infty)}=0, \qquad (1.3)$$

where Δ_{n+m} is the Laplacian on (x,y), Δ_n - on $x, \varphi(x,y)\in C_0^{0,\mu}(\amalg)$, μ - natural number. At $n=2$, $m=1$ the equation (1.1) describes Rossbi waves.

By $C^{(2,2,1)}$ we denote a class of functions $u(x,y,t)$, which is defined at $(x,y,t)\in\amalg\times[0,\infty)$, $D_x^\alpha D_y^\beta D_t^\gamma u(x,y,t)\in C^{0,0,0}(\amalg\times(0,\infty))$ and

$$\left|D_x^\alpha D_y^\beta D_t^\gamma u(x,y,t)\right|\le Ce^{(a-c(\varepsilon))|x|} \qquad (1.4)$$

uniformly with respect to y, where $|x|$ is Euclidean norm of x in $R_n(x)$, $c(\varepsilon)>0$ - some constant, $0\le\alpha\le 2$, $0\le\beta\le 2$, $0\le\gamma\le 1$.

Definition. *The function $u(x,y,t)$ we shall call a classical solution of problem (1.1)-(1.3), if $u(x,y,t)\in C^{2,2,1}(\amalg\times[0,\infty))\cap C^{1,1,1}(\bar{\amalg}\times[0,\infty))$, satisfies the equation (1.1) and conditions (1.2)-(1.3) in ordinary sense.*

Theorem 1. *The classical solution of the problem (1.1)-(1.3) is unique.*

Proof. We show that homogeneous problem, corresponding to the problem (1.1)-(1.3) has only trivial solution. Multiplying equation (1.1) to $u(x,y,t)$ and integrating on $Ш\times(0,t)$, we have

$$\int_0^t\int_Ш\left(\frac{\partial}{\partial t}\Lambda_{n+m}u(x,y,t)\right)u(x,y,t)d\,Ш\,dt+\int_0^t\int_Ш(\Delta_m u(x,y,t))u(x,y,t)d\,Ш\,dt=0. \qquad (1.5)$$

Let $\sigma_R(x)$ be the sphere with center at origin of coordinates and radius R in $R_n(x)$, $Ш_R=\Omega\times\sigma_R(x)$. The boundary of $Ш_R$ is

$$\partial Ш_R=\partial\Omega\times\sigma_R(x)\cup\Omega\times\partial\sigma_R(x).$$

Using the first Green's formula and boundary condition (1.3) we receive

$$\int_{Ш_R}\left(\frac{\partial}{\partial t}\Delta_{n+m}u\right)u\,d\,Ш=\int_{Ш_R}\left\{\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial t}\right)\frac{\partial u}{\partial x_i}+\sum\frac{\partial}{\partial y_j}\left(\frac{\partial u}{\partial t}\right)\frac{\partial u}{\partial y_j}\right\}d\,Ш-$$

$$-\int_{\Omega\times\partial\sigma_R(x)}u\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial t}\right)ds, \qquad (1.6)$$

where ds is the element of the surface $Ш_R$. In (1.6) tending $R\to\infty$ by virtue of condition (1.4) we have that integral on the surface $\Omega\times\partial\sigma_R(x)$ tends to zero. Then

$$\int_Ш\left(\Delta_{n+m}\frac{\partial u}{\partial t}\right)u\,d\,Ш=\frac{1}{2}\frac{\partial}{\partial t}\int_Ш\left[\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2+\sum_{j=1}^n\left(\frac{\partial u}{\partial y_j}\right)^2\right]d\,Ш. \qquad (1.7)$$

By analogy,

$$\int_Ш u(x,y,t)\Delta_m u(x,y,t)d\,Ш=\int_Ш\sum_{j=1}^m\left(\frac{\partial u}{\partial y_j}\right)^2d\,Ш. \qquad (1.8)$$

Denoting by

$$\int_Ш\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2d\,Ш=\|\nabla_x u\|^2_{L_2(Ш)},$$

$$\int_Ш\sum_{j=1}^m\left(\frac{\partial u}{\partial y_i}\right)^2d\,Ш=\|\nabla_y u\|^2_{L_2(Ш)}.$$

From (1.5)-(1.8) we receive

$$\frac{1}{2}\int_0^t\frac{d}{dt}\left(\|\nabla_x u\|^2_{L_2(Ш)}+\|\nabla_y u\|^2_{L_2(Ш)}\right)dt+\int_0^t\|\nabla_y u\|^2_{L_2(Ш)}dt=0. \qquad (1.9)$$

Denoted by $E(t)$ energy integral

$$E(t)=\|\nabla_x u\|^2_{L_2(Ш)}+\|\nabla_y u\|^2_{L_2(Ш)}$$

from (1.9) we have

$$\int_0^t\|\nabla_y u\|^2_{L_2(Ш)}dt+E(t)=E(0). \qquad (1.10)$$

Since for homogeneous problem $E(0)=0$, then from (1.10) we get

$$\int_0^t\|\nabla_y u\|^2_{L_2(Ш)}dt+E(t)=0. \qquad (1.11)$$

By virtue of non-negativity every term in (1.11) we have

$$\|\nabla_x u\|_{L_2(\Pi)} = 0, \quad \|\nabla_y u\|_{L_2(\Pi)} = 0.$$

From this and from $\varphi(x,y) \equiv 0$ for homogeneous problem, we receive that $u(x,y,t) \equiv 0$. Theorem is proved.

§2. Construction of Green's function of stationary problem.

By virtue of estimation (1.4) we accomplish Laplace transformation with respect to t in problem (1.1)-(1.3). Then obtain the next boundary value problem with complex parameter k

$$k\Delta_{n+m}\hat{u}(x,y,k) + \Delta_m \hat{u}(x,y,k) = \varphi(x,y), \tag{2.1}$$

$$\hat{u}(x,y,k)\big|_{\partial\Pi} = 0, \tag{2.2}$$

where $\operatorname{Re} k > 0$, $\hat{u}(x,y,k)$ is a Laplace transformation of $u(x,y,t)$. Now we construct Green's function for the problem (2.1)-(2.2). Accomplishing in (2.1)-(2.2) Fourier transformation with respect to x in this taking into account estimation (1.4) we have

$$(k+1)\Delta_m \tilde{\hat{u}}(s,y,k) - k|s|^2 \tilde{\hat{u}}(s,y,k) = \tilde{\varphi}(s,y), \tag{2.3}$$

$$\tilde{\hat{u}}(s,y,k)\big|_{\partial\Omega} = 0, \tag{2.4}$$

where $\tilde{\varphi}(s,y)$ denotes Fourier transformation of $\varphi(x,y)$.

We consider the differential operator L, generated by differential expression $\tilde{L} = \Delta_m$ with domain of definition

$$D(L) = \left\{V(y) : V(y) \in C^2(\Omega) \cap C(\overline{\Omega}),\ \Delta_m V(y) \in L_2(\Omega), V\big|_{\partial\Omega} = 0\right\}.$$

Operator L is a negative-definite self-adjoint operator. It is known [8] (p.177-178) that a spectrum of operator L is discrete and for its eigen-values λ_l are true the inequality

$$0 > \lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_l \ge \ldots, \quad \lim_{l\to\infty} \lambda_l = -\infty. \tag{2.5}$$

The eigen-functions $\psi_l(y)$ of the operator L, corresponding to eigen-values λ_l forms a basis in the space $L_2(\Omega)$. The next theorem takes place.

Theorem 2. *Green's function of problem (2.1)-(2.2) is an analytic function of the parameter k at $\operatorname{Re} k > 0$ and for its take place the next representation*

$$G(x,y,z,k) = \frac{(2\pi)^{-\left(\frac{n}{2}+1\right)}|x|^{1-\frac{n}{2}}}{4k^{\frac{1}{2}}} \sum_{l=1}^{\infty} \sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-1} \times$$
$$\times H^{(1)}_{\frac{n}{2}-1}\left(|x|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\psi_l(y)\psi_l(z)\right), \tag{2.6}$$

where $H^{(1)}_\nu(z)$ is a Hankel function of the first kind and order ν. The series in (2.6) at $|x| \ge \delta > 0$ converges uniformly with respect to k and (x,y,z) in every compactum of Π, $0 < \delta$ - some number.

Proof. For construction of Green's function of problem (2.1)-(2.2) we apply the method of paper [7]. Using theorem 3.6 from [8] (p.177) for the solution of problem (2.3)-(2.4) we have

$$\tilde{\tilde{u}}(s,y,k)=\sum_{l=1}^{\infty}\frac{c_l(s)\psi_l(y)}{(k+1)\lambda_l-k|s|^2},\tag{2.7}$$

where

$$C_l(s)=\int_{\Omega}\tilde{\varphi}(s,z)\psi_l(z)dz\,.$$

The solution of problem (2.1)-(2.2) is determined as the inverse Fourier transformation of $\tilde{\tilde{u}}(s,y,k)$ with respect to s :

$$\hat{u}(x,y,k)=\frac{1}{(2\pi)^n}\sum_{l=1}^{\infty}\psi_l(y)\int_{R_n}\frac{C_l(s)e^{-i(s,x)}}{(k+1)\lambda_l-k|s|^2}ds\,,\tag{2.8}$$

The integrating here is allowed by virtue of theorem 8 from [9] (p.253). Taking into account

$$\tilde{\varphi}(s,y)=\mathcal{F}(\varphi(x,y)),$$

where $\mathcal{F}$ is a Fourier transformation, from (2.8) w obtain

$$\hat{u}(x,y,k)=\frac{1}{(2\pi)^n}\sum_{l=1}^{\infty}\psi_l(y)\int_{R_n}\varphi_l(\xi)\left[\int_{R_n}\frac{e^{i(s,\xi-x)}}{(k+1)\lambda_l-k|s|^2}\right]d\xi\,,\tag{2.9}$$

where

$$\varphi_l(\xi)=\int_{\Omega}\varphi(\xi,z)\psi_l(z)dz\,.\tag{2.10}$$

Denote by $\tau=\xi-\tau$ we calculate interior integral in (2.9)

$$J_l(k,\tau)=\frac{1}{(2\pi)^n}\lim_{N\to\infty}\int_{|s|\le N}\frac{e^{i(s,\tau)}ds}{(k+1)\lambda_l-k|s|^2}\equiv\frac{1}{(2\pi)^n}\lim_{N\to\infty}J_{l,N}(k,\tau).\tag{2.11}$$

Passing on to spherical coordinates, in this taking into account spherical symmetry of integrand in (2.11) we obtain

$$J_{l,N}(k,\tau)=\frac{\tau^{1-\frac{n}{2}}}{(2\pi)^{\left(\frac{n}{2}+1\right)}}\int_0^N\frac{|s|^{\frac{n}{2}}\dot{J}_{\frac{n}{2}-1}(|\tau|\,|s|)}{(k+1)\lambda_l-k|s|^2}d|s|,\tag{2.12}$$

where $\dot{J}_v(z)$ is the Bessel function of v order. We now calculate integral in (2.12). Let n be an add number. Then $z^{\frac{n}{2}}\dot{J}_{\frac{n}{2}-1}(z)$ is an even integral function. Continuing this function on interval $(-N,0)$ in even form and using formula

$$\dot{J}_{\frac{n}{2}-1}(z)=\frac{1}{2}\left(H^{(1)}_{\frac{n}{2}-1}(z)+H^{(2)}_{\frac{n}{2}-1}(z)\right)\tag{2.13}$$

we obtain

$$J_{l,N}(k,\tau)=\frac{|\tau|^{1-\frac{n}{2}}}{4(2\pi)^{\left(1+\frac{n}{2}\right)}}\left\{\int_N^N\frac{|s|^{\frac{n}{2}}H^{(1)}_{\frac{n}{2}-1}(|\tau|\,|s|)}{(k+1)\lambda_l-k|s|^2}d|s|+\right.$$

$$\left.+\int_{N}^{N}\frac{|s|^{\frac{n}{2}}H^{(2)}_{\frac{n}{2}-1}(|\tau||s|)}{(k+1)\lambda_l-k|s|^2}d|s|+\right\}\equiv J^{(1)}_{l,N}(k,\tau)+J^{(2)}_{l,N}(k,\tau). \qquad (2.14)$$

Poles of the integrand in (2.14) are

$$|s|_{1,2}=+\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}.$$

Taking into account analyticity of integrand in (2.14) and the asymptotic behavior of the Hankel function as $|s|\to\infty$ [10] (p.219), applying theorem of residues and turn N to infinity, we obtain

$$J_l(k,\tau)=\frac{|\tau|^{1-\frac{n}{2}}}{4(2\pi)^{\left(\frac{n}{2}+1\right)}}\left\{\frac{1}{2\sqrt{k}}\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-1}\left[H^{(1)}_{\frac{n}{2}-1}\left(|\tau|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right)+\right.\right.$$

$$\left.\left.+(-1)^{\frac{n}{2}-1}H^{(2)}_{\frac{n}{2}-1}\left(-|\tau|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right)\right]\right\}. \qquad (2.15)$$

Taking into account [10] (p.218)

$$H^{(2)}_{\frac{n}{2}-1}(-z)=(-1)^{\frac{n}{2}-1}H^{(2)}_{\frac{n}{2}-1}(z), \qquad (2.16)$$

from (2.15) for $J_l(k,\tau)$ we receive

$$J_l(k,\tau)=\frac{|\tau|^{1-\frac{n}{2}}}{4(2\pi)^{\left(\frac{n}{2}+1\right)}}\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}\left(|\tau|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right). \qquad (2.17)$$

Now let n be an even number. Then $z^{\frac{n}{2}}J_{\frac{n}{2}-1}(z)$ is an even integral function. As above expressing in (2.12) the Bessel function by the Hankel functions $H^{(1,2)}_{\frac{n}{2}-1}(z)$ on formula (2.13) making section $(-\infty,0)$ and using formula (2.16), we obtain

$$J_{l,N}(k,\tau)=\frac{|\tau|^{1-\frac{n}{2}}}{2(2\pi)^{\left(\frac{n}{2}+1\right)}}\int_{-N}^{N}\frac{|s|^{\frac{n}{2}}H^{(1)}_{\frac{n}{2}-1}(|\tau||s|)}{(k+1)\lambda_l-k|s|^2}d|s|. \qquad (2.18)$$

Further, applying theorem of residues to integral (2.18) and turn N to infinity at even n for $J_l(k,\tau)$ again we obtain formula (2.17). Putting expression of $J_l(k,\tau)$ from (2.17) in (2.9), changing order of integration and summation, we have

$$\hat{u}(x,y,k)=\frac{1}{4(2\pi)^{\left(\frac{n}{2}+1\right)}\sqrt{k}}\int_{\Pi}|x-\xi|^{1-\frac{n}{2}}\sum_{l=1}^{\infty}\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-1},$$

$$H^{(1)}_{\frac{n}{2}-1}\left(|x-\xi|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right)\psi_l(y)\psi_l(z)\varphi(\xi,z)d\Pi,$$

or

$$\hat{u}(x,y,k)=G(x,y,z,k)*\varphi(\xi,z), \tag{2.19}$$

where convolution is accomplished on the cylinder Π. From this for the Green's function $G(x,y,z,k)$ of problem (2.1)-(2.2) we receive expression (2.6).

Now we study the convergence of series in (2.6). For this aim we establish some estimations of the eigen-functions $\psi_l(y)$, which are necessary for further.

In [7] it is shown that

$$\|\psi_l(y)\|_{W_2^{\left(\left[\frac{m}{2}\right]+1\right)}(\Omega)} \le C|\lambda_l|^{\frac{1}{2}\left(\left[\frac{m}{2}\right]+1\right)},$$

where $[\sigma]$ denotes integral part of σ. From this by Sobolev's imbedding theorem we obtain

$$\|\psi_l(y)\|_{C(\bar{\Omega})} \le C|\lambda_l|^{\frac{1}{2}\left(\left[\frac{m}{2}\right]+1\right)}. \tag{2.20}$$

It is known, that [9] (p.200)

$$c_0 l^{\frac{2}{m}} < |\lambda_l| < c_1 l^{\frac{2}{m}}, \tag{2.21}$$

where c_0, c_1 are some constants, which do not depend on l. Then from (2.20)-(2.21), it follows that

$$\|\psi_l(y)\|_{C(\bar{\Omega})} \le C l^{\frac{\left[\frac{m}{2}\right]+1}{m}}. \tag{2.22}$$

Since $\Delta^{\nu}\psi_l(y)$ (ν-natural number) is on eigen-function of operator L, corresponding eigen-values λ_l^{ν}, then, as above, we can show that

$$\|\psi_l(y)\|_{C^{(\nu)}(\bar{\Omega})} \le C l^{\frac{\left[\frac{m}{2}\right]+1+\nu}{m}}. \tag{2.23}$$

It can be shown, that at $\operatorname{Re}k \ge \varepsilon > 0$

$$\operatorname{Re}\left(1+\frac{1}{k}\right)^{\frac{1}{2}} \ge \frac{\sqrt{2}}{2}\left(1+\frac{1}{|k|^2}\right)^{\frac{1}{4}} \ge \frac{\sqrt{2}}{2}. \tag{2.24}$$

Since λ_l satisfies the inequality (2.5), then considering the asymptotic behavior of the Hankel function $H^{(1)}_{\frac{n}{2}-1}(z)$ at $z \to \infty$ [10] (p.219), from (2.22)-(2.24) we receive that the series in (2.6) converges uniformly with respect to k and (x,y,z) in every compactum of Π. Its at $x \ne 0$ can be differentiated arbitrary time with respect to (x,y,z,k). Theorem is proved.

Lemma 1. *Green's function $G(x,y,z,k)$ of problem (2.1)-(2.2) admits analytic continuation by k to left half plane on exterior of interval $[-1,0]$, which composes continuous spectrum of problem (2.1)-(2.2).*

Proof. Since every term of series (2.6) has singularity at points of interval $[-1,0]$, then making section $[-1,0]$ for square root we choose a branch which is positive for positive value radicand. Thus we receive one-valued functions which admits analytic continuation by k to left half plane. Denote by D_δ an exterior of angle 2δ with vertex on origin, for with negative semi-axis is bisectrix. Now we show, that at $|x| \geq \sigma_0 > 0$ on exterior of interval $[-1,0]$ on left half plane the series (2.6) convergences uniformly with respect to $k \in D_\delta$ and (x, y, z) in every compactum of Π. For this it is necessary to estimate $\operatorname{Re}\sqrt{1+\frac{1}{k}}$ from below.

Let $r = |k|$ and $\varphi = \arg k$. Then

$$\left|1+\frac{1}{k}\right| = \sqrt{1+\frac{1}{r^2}+\frac{2\cos\varphi}{r}} \geq \sqrt{1+\frac{1}{r^2}-\frac{1}{r}}. \qquad (2.25)$$

The radicand function in (2.25) its minimum value, which is equal to $\frac{1}{4}$, receives at $r = 2$. Since

$$\frac{-\pi+\delta}{2} \leq \arg\left(1+\frac{1}{k}\right)^{\frac{1}{2}} \leq \frac{\pi-\delta}{2}$$

then

$$\operatorname{Re}\left(1+\frac{1}{k}\right)^{\frac{1}{2}} \geq \frac{1}{2}\sin\frac{\delta}{2}. \qquad (2.26)$$

Taking into account (2.5) and the asymptotics of the Hankel function $H^{(1)}_{\frac{n}{2}-1}(z)$ for $z \to \infty$ from (2.22), (2.26) we receive uniform convergence of series (2.6) in indicated domain.

The interval $[-1,0]$ composes continuous spectrum of problem (2.1)-(2.2), since in points of interval $(-1,0)$ and $k \to -1+0$ series (2.6) diverges. Lemma is proved.

§3. The behavior of the solution of non-stationary problem (1.1)-(1.3) at $t \to +\infty$.

The solution $u(x, y, t)$ of non-stationary problem (1.1)-(1.3) is determined as the inverse Laplace transformation of $\hat{u}(x, y, k)$, that is

$$u(x, y, t) = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{kt}\hat{u}(x, y, k)dk, \qquad (3.1)$$

where $\varepsilon > 0$ is an arbitrary small number and integral in (3.1) is understood in main sense. Now we will study behavior of $u(x, y, t)$ at $t \to +\infty$. Then following theorem is true.

Theorem 3. *Let $\partial\Omega \in C^\mu$, $\varphi(x, y)$ be a finite function, continuous with respect to x and differentiable to order $\mu = \left[\frac{n-1}{2}\right] + \left[\frac{m}{2}\right] + m$ with respect to y. Then for solution of initial-boundary value problem (1.1)-(1.3) at $t \to +\infty$ it holds the estimation*

$$u(x, y, t) = O\left(t^{-\nu}\right)$$

uniformly on (x, y) in arbitrary-compactum of Π.

Proof. From (2.19) we receive that

$$\hat{u}(x,y,k)=\frac{(2\pi)^{-\left(\frac{n}{2}+1\right)}}{4k}\int\limits_{R_n}|x-\xi|^{1-\frac{n}{2}}\sum_{l=1}^{\infty}\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-1}\times$$
$$\times H^{(1)}_{\frac{n}{2}-1}\left(|x-\xi|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right)\psi_l(y)\varphi_l(\xi)d\xi, \tag{3.2}$$

where $\varphi_l(\xi)$ is defined by formular (2.10). By virtue of uniform convergence of series in (3.2) it can be integrated term by term on k. Changing the order of integration, we have

$$u(x,y,t)=-\frac{(2\pi)^{-\left(\frac{n}{2}+2\right)}}{4}\sum_{l=1}^{\infty}\psi_l(y)\int\limits_{R_n}|x-\xi|^{1-\frac{n}{2}}\varphi_l(\xi)\times$$
$$\times\left[\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{e^{kt}}{\sqrt{k}}\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}\left(|x-\xi|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right)dk\right]d\xi, \tag{3.3}$$

where the interior integral is understood in the main sense. Denote

$$T_l(x,t)=\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{e^{kt}}{\sqrt{k}}\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}\left(|x-\xi|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right)dk, \tag{3.4}$$

we shall regularize the integral in (3.4) in the next form. Denote in the complex plane k by L_0^+ ray outgoing from origin and making with the positive imaginary axis an angle of $\frac{\pi}{6}$ and by L_0^- same ray making with the negative imaginary axis an angle of $-\frac{\pi}{6}$. We make section $(-\infty,0)$ on the plane k and for square root choose such a branch, which is real for the positive arguments. Denote by C_ε a circle of radius ε with center at origin in the complex plane k and by

$$\widetilde{C}_\varepsilon=\left\{k:k\in C_\varepsilon,-\frac{2\pi}{3}\le\arg k\le\frac{2\pi}{3}\right\},$$
$$L_\varepsilon=L_\varepsilon^-\cup\widetilde{C}_\varepsilon\cup L_\varepsilon^+,\quad L_0=L_0^-\cup L_0^+$$

where L_ε^-, L_ε^+ are the parts of rays L_0, L_0^+ at exterior of C_ε respectively.

Since integrand in (3.4) decreases at $k\to\infty$, then by Cauchy theorem contour of integration can be substructed on contour L_ε. From the asymptotics of the Hankel function $H^{(1)}_{\frac{n}{2}-1}(z)$ at $z\to\infty$ it follows that, integrand in (3.4) tends to zero faster than arbitrary degree of k at $k\to 0$, $-\frac{2\pi}{3}\le\arg k\le\frac{2\pi}{3}$. Therefore in integral (3.4) taken on contour L_ε we can pass to limit at $\varepsilon\to 0$. Thus

$$T_l(x,t)=\int\limits_{L_0}\frac{e^{kt}}{\sqrt{k}}\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}\left(|x|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right)dk. \tag{3.5}$$

We study now $T_l(x,t)$ at $t\to+\infty$, $0<|x|\le A$, where A is a constant. Integrating in (3.5) ν times by parts in this integrating e^{kt} and taking into consideration a differentiation formula of Hankel functions [10] (p.183) we have

$$T_l(x,t)=\frac{(-1)^\nu}{2^\nu t^\nu}\left\{\nu!!\int_{L_0}\frac{e^{kt}}{k^{\nu+\frac{1}{2}}}\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}\left(|x|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right)dk+\right.$$

$$\left.\ldots+\left(|x|\lambda_l\right)^\nu\int_{L_0}\frac{e^{kt}}{k^{2\nu+\frac{1}{2}}}\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-\nu-1}H^{(1)}_{\frac{n}{2}-\nu-1}\left(|x|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right)dk\right\}. \qquad (3.6)$$

Now we estimate integrals in (3.6). Since they are estimated by the same form, then we estimate the last integral.

Lemma 2. *At* $k\in L_0^\pm,\ |k|=r\geq\frac{1}{2}$

$$\operatorname{Re}\sqrt{1+\frac{1}{k}}\geq\frac{1}{2}\left(\frac{3}{4}\right)^{\frac{1}{4}}, \qquad (3.7)$$

and $r<\frac{1}{2}$

$$\operatorname{Re}\sqrt{1+\frac{1}{k}}\geq 2^{-\frac{5}{4}}r^{-\frac{1}{2}}.$$

Proof. Consider the case $k\in L_0^+$, the case $k\in L_0^-$ is done by analogy

$$1+\frac{1}{k}=1+r^{-1}\left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right).$$

Then

$$f(r)\equiv\left|1+\frac{1}{k}\right|=\sqrt{1+\frac{1}{r^2}-\frac{1}{r}}.$$

At $r<\frac{1}{2}$

$$\sqrt{1+\frac{1}{r^2}-\frac{1}{r}}>\frac{\sqrt{2}}{2}r.$$

From this

$$\operatorname{Re}\sqrt{1+\frac{1}{k}}=\left|1+\frac{1}{k}\right|^{\frac{1}{4}}\cos\frac{\pi}{3}>2^{-\frac{5}{4}}r^{-\frac{1}{2}}.$$

Let now $r\geq\frac{1}{2}$. Then the function $f(r)$ takes its minimal value at $r=2$ and

$$\min_{r\geq\frac{1}{2}}f(r)=\frac{3}{4}.$$

Then

$$\operatorname{Re}\sqrt{1+\frac{1}{k}}\geq\frac{1}{2}\left(\frac{3}{4}\right)^{\frac{1}{4}}.$$

Lemma is proved.

Consider

$$T_l^{(\nu)}(x,t)=\int_{L_0}\frac{e^{kt}}{k^{2\nu+\frac{1}{2}}}\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-\nu-1}H^{(1)}_{\frac{n}{2}-\nu-1}\left(|x|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right)dk. \qquad (3.8)$$

We estimate $T_l^{(\nu)}(x,t)$ at $|x|\geq\delta>0,\lambda_l\to-\infty,t\to\infty$. By virtue of asymptotic form of the Henkel function $H^{(1)}_{\frac{n}{2}-1}(z)$ at $z\to\infty$ we have

$$T_l^{(\nu)}(x,t)=\int_{L_0}\frac{e^{kt}}{k^{2\nu+\frac{1}{2}}}\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}^{\frac{n}{2}-\nu-1}e^{-|x||\lambda_l|^{\frac{1}{2}}\sqrt{1+\frac{1}{k}}}dk. \qquad (3.9)$$

Integrand in (3.9) at point $k=0$ has a zero of infinity order and at $k\to\infty$ on contour L_0 tends to zero exponentially. Estimating modulo by virtue of (2.5), (3.7) we obtain

$$\left|T_l^{(\nu)}(x,t)\right|\leq C|\lambda_l|^{\frac{1}{2}\left(\frac{n}{2}-\nu-\frac{3}{2}\right)}|x|^{-\frac{1}{2}},$$

where C is a constants. From this and (3.6) it follows that

$$|u(x,y,t)|\leq\frac{C(\nu)}{t^{\nu}}\sum_{l=1}^{\infty}|\lambda_l|^{\frac{1}{2}\left(\frac{n}{2}-\nu-\frac{3}{2}\right)}\|\psi_l(Y)\|_{C(\bar{\Omega})}\int_{R_n}|x-\xi|^{\frac{1-n}{2}+\nu}\varphi_l(\xi)d\xi, \qquad (3.10)$$

where $C(\nu)$ is a constant, depending of ν.

Taking into account estimation (2.20) and that $\varphi(\xi,z)$ is finite function with respect to ξ applying Cauchy's-Bunyakovcki's inequality to (3.10) we obtain

$$|u(x,y,t)|\leq\frac{C(\nu)}{t^{\nu}}\left\{\sum_{l=1}^{\infty}|\lambda_l|^{-m}+\sum_{l=1}^{\infty}|\lambda_l|^{\frac{n-1}{2}+\nu+\left[\frac{m}{2}\right]+m}\int_D|\varphi_l(\xi)|^2d\xi\right\} \qquad (3.11)$$

uniformly with respect to x in every compactum of R_n. Further by B.Levi's theorem

$$\sum_{l=1}^{\infty}|\lambda_l|^{\mu}\int_D|\varphi_l(\xi)|^2d\xi=\int_D\left(\sum_{l=1}^{\infty}|\lambda_l|^{\mu}|\varphi_l(\xi)|^2\right)d\xi. \qquad (3.12)$$

Since function $\varphi(\xi,z)$ no z satisfies to conditions of theorem 8 of [9] (p.253) then

$$\sum_{l=1}^{\infty}|\lambda_l|^{\mu}|\varphi_l(\xi)|^2\leq\|\Phi(\xi,z)\|^2_{H^{\mu}(\Omega)}. \qquad (3.13)$$

From (3.11)-(3.13) we get

$$|u(x,y,t)|\leq\frac{C(\nu)}{t^{\nu}}\left\{\sum_{l=1}^{\infty}|\lambda_l|^{-m}+\int_D\|\varphi(\xi,z)\|^2_{H^{\mu}(\Omega)}d\xi\right\} \qquad (3.14)$$

uniformly with respect to (x,y) in every compactum of Π. By virtue of estimation (2.21) series in (3.14) converges. From (3.14) we get

$$U(x,y,t)=O(t^{-\nu})$$

uniformly with respect to (x,y) in every compactum of Π. Theorem 3 if proved.

§4. The estimation of the solution of non-stationary problem (1.1)-(1.3).

Now we receive the estimation (1.4) for the solution of non-stationary problem (1.1)-(1.3). Estimating modulo in (3.3) where the contour of integration $(\varepsilon-i\infty,\varepsilon+i\infty)$ is substituted by L_ε we get

$$|u(x,y,t)|\le C\sum_{l=1}^{\infty}\|\psi_l(y)\|_{C(\bar\Omega)}\int_{R_n}|x-\xi|^{1-\frac{n}{2}}|\varphi_l(\xi)|\left[\int_{l_\varepsilon}H^{(1)}_{\frac{n}{2}-1}\left(|x-\xi|\sqrt{\lambda_l\left(1+\frac{1}{k}\right)}\right)|d_k|\right]d\xi\,. \quad (4.1)$$

It can be show that at $k\in\tilde C_\varepsilon$

$$\mathrm{Re}\left(1+\frac{1}{k}\right)\ge\frac{1}{2}\left(1+\frac{1}{2\varepsilon^2}\right)^{\frac{1}{4}}=c_0(\varepsilon)\,. \quad (4.2)$$

By virtue of estimations (2.20), (4.2) and the asymptotic of the Hankel function at infinity from (4.1) we have

$$|u(x,y,t)|\le Ce^{\varepsilon t}\sum_{l=1}^{\infty}|\lambda_l|^{\frac{1}{2}\left(1+\left[\frac{m}{2}\right]\right)}\int_{R_n}|x-\xi|^{1-\frac{n}{2}}e^{-c_0(\varepsilon)|\lambda_l||x-\xi|}|\varphi_l(\xi)|d\xi\le$$

$$\le Ce^{\varepsilon t-c(\varepsilon)|x|}\sum_{l=1}^{\infty}|\lambda_l|^{\frac{1}{2}\left(1+\left[\frac{m}{2}\right]\right)}\int_{R_n}|x-\xi|^{1-\frac{n}{2}}e^{c(\varepsilon)|\xi|}|\varphi_l(\xi)|d\xi\,,$$

where $c(\varepsilon)=c_0(\varepsilon)|\lambda_1|$, D is the compact support of $\varphi(\xi,z)$ on ξ. Applying Cauchy's-Bunyakovcki's inequality we obtain

$$|u(x,y,t)|\le Ce^{\varepsilon t-c(\varepsilon)|x|}\left(\sum_{l=1}^{\infty}|\lambda_l|^{-m}+\sum_{l=1}^{\infty}|\lambda_l|^{1+m+\left[\frac{m}{2}\right]}\right)\left(\int_D|\varphi_l(\xi)|^2d\xi\right). \quad (4.3)$$

Further by B.Levi's theorem [11] (p.134) from (4.3) we get

$$|u(x,y,t,)|\le Ce^{\varepsilon t-c(\varepsilon)|x|}\left(\sum_{l=1}^{\infty}|\lambda_l|^{-m}+\int_D\left(\sum_{l=1}^{\infty}|\lambda_l|^{1+m+\left[\frac{m}{2}\right]}|\varphi_l(\xi)|^2\right)d\xi\right). \quad (4.4)$$

From (3.15) and (4.4) we have

$$|u(x,y,t,)|\le Ce^{\varepsilon t-c(\varepsilon)|x|}\left(\sum_{l=1}^{\infty}|\lambda_l|^{-m}+\int_D\|\varphi(\xi,z)\|^2_{H^{1+m+\left[\frac{m}{2}\right]}(\Omega)}d\xi\right).$$

By analogy we estimate derivatives of $u(x,y,t)$, contained in (1.4). In this in formula (3.3) the $\psi_l(y)$ substituted by $D_y^\beta\psi_l(y)$ and $\varphi_l(\xi)$ by $D_\xi^\alpha\varphi_l(\xi)$. Then we get

$$\left|D_x^\alpha D_y^\beta D_t^\gamma u(x,y,t)\right|\le Ce^{\varepsilon t-c(\varepsilon)|x|}\left(\sum_{l=1}^{\infty}|\lambda_l|^{-m}+\int_D\left\|D_\xi^\alpha\varphi(\xi,z)\right\|^2_{H^{1+m+\left[\frac{m}{2}\right]+\beta}}d\xi\right).$$

The estimation (1.4) is proved.

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Received September 4, 2000; Revised February 29, 2001.
Translated by authors.