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FINITE DIMENSIONAL INVERSE PROBLEMS

Abstract

In the present paper the inverse problems of spectral analysis for finite dimensional thridiagonal matrix are solved. We use the approach suggested in papers [1], [2]. Unlike differential operators the solutions of inverse problems in given case are strongly simplified. It is possible to consider the results of the present paper as introduction to the theory of inverse problems and to use by integrating the initial-boundary value problem for finite Tod's chain [3].

§1. Direct problem of spectral analysis.

Consider the equations system

$$\begin{cases} b_0 y_0 + a_0 y_0 = \lambda y_0, \\ a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n = 1,2,\dots,N-2, \\ a_{N-2} y_{N-2} + b_{N-1} y_{N-1} = \lambda y_{N-1}, \end{cases} \tag{1}$$

where $\{y_n\}_1^N$ is a desired solution, λ is a complex parameter and

$$a_n > 0, Jmb_n = 0,\ n = 0,1,\dots,N-1. \tag{2}$$

The problem (1) is equivalent to the determination of the vector $\{y_n\}_{n=1}^N$ satisfying the equation

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n = 0,1\dots,N-1 \tag{3}$$

and the boundary condition

$$y_{-1} = y_N = 0. \tag{4}$$

We can write the equations system (1) or the problem (3)-(4) in the form of

$$Ly = \lambda y,$$

where $y = (y_0,\dots,y_{N-1})^T$, L is the Jacobian thridiagonal matrix

$$L = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \dots & 0 & 0 \\ a_0 & b_1 & a_1 & 0 & \dots & 0 & 0 \\ 0 & a_1 & b_2 & a_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & 0 & \dots & a_{N-2} & b_{N-1} \end{pmatrix}. \tag{5}$$

Denote by $l_2(0, N-1)$ a complex Hilbert space of the vectors $y = \{y_n\}_{n=0}^{N-1}$ with the scalar product

$$(y,z) = \sum_{j=0}^{N-1} y_j \bar{z}_j .$$

Since the matrix L generates a self-adjoint operator in the space $l_2(0, N-1)$, it has N number (subject to the multiplicity) real eigenvalues $\lambda_1,\dots,\lambda_N$ and N number eigenvectors $e_1,\dots,e_N$, which form an orthonormalized basis. We remind the algorithm of structure for the matrix L of eigenvalues and eigenvectors. Let $P_n(\lambda)$ be a solution of the Cauchy difference problem

$$a_{n-1}P_{n-1}(\lambda)+b_nP_n(\lambda)+a_nP_{n+1}(\lambda)=\lambda P_n(\lambda),\ \ n=0,1,\dots,N-1, \tag{6}$$

$$P_{-1}(\lambda)=0,\ \ P_0(\lambda)=1. \tag{7}$$

Note that $P_n(\lambda)$ is a polynomial of degree n. By the condition (4) the roots of the equation

$$P_N(\lambda)=0$$

determine the eigenvalues λ_k, and the eigen vectors corresponding them will be

$$\mathbf{P}(\lambda_k)=(P_0(\lambda_k),P_1(\lambda_k),\dots,P_{N-1}(\lambda_k))^T.$$

Assuming $e_k=\frac{1}{\sqrt{\alpha_k}}\mathbf{P}(\lambda_k)$, where $\alpha_k=\|\mathbf{P}(\lambda_k)\|^2=\sum_{j=0}^{N-1}P_j^2(\lambda_k)$ we obtain the complete orthonormalized system of eigen vectors of the matrix L. The numbers α_k are called normed numbers of the boundary problem (3)-(4) (or of the matrix L).

The structure of the matrix L allows to make the properties of eigen numbers more precise

Lemma 1. *The eigenvalues λ_k are different.*

Proof. Since the numbers λ_k are the roots of the polynomial $P_N(\lambda)$, it is sufficient to prove that $\dot{P}_N(\lambda_k)\neq 0$ (we denote by points the differentiation by λ). Differentiating the equation (6) by λ, we see that $\dot{P}_n(\lambda)$ satisfies such equation

$$a_{n-1}\dot{P}_{n-1}(\lambda)+b_n\dot{P}_n(\lambda)+a_n\dot{P}_{n+1}(\lambda)=\lambda\dot{P}_n(\lambda)+P_n(\lambda),\ n=0,1,\dots,N-1.$$

We multiply this identity by $P_n(\lambda)$, and the relation (6) by the polynomial $\dot{P}_n(\lambda)$ and substract the second from the first. As a result we obtain

$$a_{n-1}\left(\dot{P}_{n-1}(\lambda)P_n(\lambda)-P_{n-1}(\lambda)\dot{P}_n(\lambda)\right)-a_n\left(\dot{P}_{n+1}(\lambda)\dot{P}_n(\lambda)-\dot{P}_{n+1}(\lambda)P_n(\lambda)\right)=P_n^2(\lambda)$$

Assuming here $\lambda=\lambda_k$ and summing by n from 0 to $N-1$, taking into account the conditions (7) and $P_N(\lambda_k)=0$ we have

$$a_{N-1}\dot{P}_N(\lambda_k)P_{N-1}(\lambda_k)=\sum_{n=0}^{N-1}P_n^2(\lambda_k). \tag{8}$$

Consequently $\dot{P}_N(\lambda_k)\neq 0$. The lemma is proved.

Not restricting the generality we assume that $\lambda_1<\lambda_2<\dots<\lambda_N$. The following Lemma is also easily proved.

Lemma 2. *The Parceval equality equivalent to the expansion formula*

$$\sum_{j=1}^{N}\frac{1}{\alpha_j}P_n(\lambda_j)P_k(\lambda_j)=\delta_{nk},\ \ n,k=0,1,\dots,N-1, \tag{9}$$

where δ_{nk} is Cronecker's symbol, is valid.

Assuming $n=k=0$ in the equality (9) and using (7) we have

Corollary. *The normed numbers α_k satisfy the equality*

$$\sum_{j=1}^{N}\alpha_j^{-1}=1. \tag{10}$$

Thus for the given operator L we construct its spectrum $\{\lambda_k\}$ and the basis from the eigenvectors e_k, i.e. we solve the direct problem of spectral analysis.

On the contrary knowing the spectrum $\{\lambda_k\}_{k=1}^N$ and the eigenvectors $\{e_k\}_{k=1}^N$, it is possible to restore the operator L by the formula

$$Lu = \sum_{k=1}^{N} \lambda_k (u, e_k) e_k, \quad u \in l_2(0, N-1).$$

However the matrix L has a special thridiagonal structure, it depends on $2N-1$ parameters, and N^2 parameters are contained in the spectral information $\{\lambda_k, e_k\}$. Therefore it is natural to expect that we can restore the matrix L using only a part of the spectral information $\{\lambda_k, e_k\}$. It is obvious that a spectrum $\{\lambda_k\}$ is not sufficient for identical restoration of the operator L. Therefore it is necessary to have some more information about eigenvectors. It appears that as an additional information we can take a set of functionals from e_k - normalized numbers α_k.

We will call the quantities $\{\lambda_k, \alpha_k\}$ the spectral data of the matrix L.

Further we'll need the denotation of elements of the matrix L by the polynomial $P_n(\lambda)$. Multiplying the equality (6) for $\lambda = \lambda_j$ by $\frac{1}{\alpha_j} P_k(\lambda_j)$, then summing by j and using the relations (9) we obtain

$$b_n = \sum_{j=1}^{N} \frac{\lambda_j}{\alpha_j} P_n^2(\lambda_j), \quad n = 0,1,..,N-1. \tag{11}$$

Analogously the following formulas are found

$$a_n = \sum_{j=1}^{N} \frac{\lambda_j}{\alpha_j} P_n(\lambda_j) P_{n+1}(\lambda_j), \quad n = 0,1,..,N-2. \tag{12}$$

§2. Inverse problem of spectral analysis.

The inverse problem of spectral analysis is in restoration of the matrix L by the spectral data $\{\lambda_k, \alpha_k\}$.

Theorem 1. *For the sets of numbers $\{\lambda_k, \alpha_k\}_{k=1}^N$ to be spectral data of the matrix as (5) with elements satisfing the conditions (2), it is necessary and sufficient that the conditions*

$$1)\ \lambda_k \neq \lambda_j \quad 2)\ \sum_{j=1}^{N} \alpha_j^{-1} = 1 \quad \textit{and} \quad a_k > 0, \quad k = 1,2,...,N$$

are fulfilled.

The necessity is proved in §1.

It is possible to prove the sufficiency using the following well known lemma ([4], p.316) and the formulas (11), (12).

Lemma 2. *Let such different real numbers $\lambda_1, \lambda_2, ..., \lambda_N$ and the positive numbers $\alpha_1, \alpha_2, ..., \alpha_N$ be given that $\sum_{j=1}^{N} \alpha_j^{-1} = 1$. Then there exist unique polynomials $P_0(\lambda), P_1(\lambda), ..., P_{N-1}(\lambda)$ with $\deg P_k(\lambda) = k$ and positive leading coefficients satisfying the conditions (9).*

The theorem is proved.

Such an approach to the solution of the inverse problem (i.e. analogue lemma 2) is not suitable for differential operators. So we give another method – the Gelphand-Levitan-Marchenko method which is widely used in the theory of inverse problems and for another operators.

Denote by $\overset{\circ}{P}_n(\lambda)$ a solution of the equation (3) satisfying the conditions $\overset{\circ}{P}_{-1}(\lambda)=0,\ \overset{\circ}{P}_0(\lambda)=1$, in the case $a_n\equiv 1,\ b_n\equiv 0$. Since $P_n(\lambda)$ is a polynomial of degree n and $\overset{\circ}{P}_j(\lambda),\ j=0,1,...,n$ from a basis in the set of polynomials of degree not more than n, then the following one-valued representation (transformation operator) holds.

$$P_n(\lambda)=\gamma_n\left[\overset{\circ}{P}_n(\lambda)+\sum_{m=0}^{n-1}K_{nm}\overset{\circ}{P}_m(\lambda)\right],\ n=0,1,...,N, \tag{13}$$

where the coefficients of the equation (3) and the quantities α_n, K_{nm} are connected among themselves by the equalities

$$\begin{aligned} a_n&=\gamma_n/\gamma_{n+1},\ \ \gamma_0=1,\\ b_n&=K_{n,n-1}-K_{n+1,n},\\ a_n^2&=1+K_{n,n-2}-b_nK_{n,n-1}-K_{n+1,n-1},\\ a_{n-1}^2K_{n-1,m}+b_nK_{nm}+K_{n+1,m}&=K_{n,m+1}+K_{n,m-1}.\end{aligned} \tag{14}$$

Using (13) we have

$$\sum_{j=1}^{N}\frac{1}{\alpha_j}P_n(\lambda_j)\overset{\circ}{P}_s(\lambda_j)=\gamma_n\left[F_{ns}+\sum_{m=0}^{n-1}K_{nm}F_{ms}\right], \tag{15}$$

where

$$F_{ns}=\sum_{j=1}^{N}\frac{1}{\alpha_j}\overset{\circ}{P}_n(\lambda_j)\overset{\circ}{P}_s(\lambda_j). \tag{16}$$

Since the expansion

$$\overset{\circ}{P}_s(\lambda)=\sum_{k=0}^{s}c_k^{(s)}P_k(\lambda)$$

holds, then by virtue of (9) we obtain

$$\sum_{j=1}^{N}\frac{1}{\alpha_j}P_n(\lambda_j)\overset{\circ}{P}_s(\lambda_j)=\frac{1}{\gamma_n}\delta_{ns},\ \ n\ge 0,\ \ s=0,1,...,n.$$

Taking into account these relations in (15) we have

$$F_{ns}+\sum_{m=0}^{n-1}K_{nm}F_{ms}=0,\ \ 0\le s<n,\ n\ge 1, \tag{17}$$

$$\frac{1}{\gamma_n^2}=F_{nn}+\sum_{m=0}^{n-1}K_{nm}F_{mn},\ \ n=0,1,...,N-1. \tag{18}$$

The equation (17) is the basic equation of the inverse problem. This equation allows to solve the inverse problem. Really, let the quantities $\{\lambda_j,\alpha_j\}$ satisfying the conditions of theorem 1 be given. Let's determine the quantities F_{ns} of the formula (16) and consider a non-homogeneous system of the equations (17) with the unknowns $K_{n0},K_{n1},...,K_{n,n-1}$.

Lemma 3. *For any fixed n the system of equations (17) is identically solvable.*

Proof. It is obvious that the vectors

$$\hat{e}_n = \left(\frac{1}{\sqrt{\alpha_1}} \overset{\circ}{P}_n(\lambda_1), \ldots, \frac{1}{\sqrt{\alpha_N}} \overset{\circ}{P}_n(\lambda_N) \right), \quad n = 0,1,\ldots,N-1$$

are linear independent and from (16) we have

$$F_{ns} = (\hat{e}_n, \hat{e}_s).$$

The basic determinant of the system of the equations (17) is the Gram determinant of the vectors $\hat{e}_n$

$$\det \{F_{ij}\}_{i,j=0}^{n-1} = \det \{(\hat{e}_i, \hat{e}_j)\}_{i,j=0}^{n-1},$$

consequently, it is positive.

Lemma 3 is proved.

Lemma 4. *Let* K_{nm}, $m = 0,1,\ldots,n-1$ *be a solution of the equations system (17). Then*

$$F_{nn} + \sum_{m=0}^{n-1} K_{nm} F_{mn} > 0, \quad n = 0,1,\ldots,N-1.$$

Proof. It follows from the formula

$$F_{nn} + \sum_{m=0}^{n-1} K_{nm} F_{mn} = \frac{\Delta_n}{\Delta_{n-1}},$$

where

$$\Delta_k = \det \{F_{ij}\}_{i,j=0}^{k} = \det \{(\hat{e}_i, \hat{e}_j)\}_{i,j=0}^{k} > 0.$$

Thus we can find the coefficients a_n, b_n of the desired equation from the formula (14), where γ_n is determined by the formula (18).

§3. Inverse problem on two spectrums.

Consider the boundary value problem

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \; n = 1,2,\ldots,N-1, \tag{19}$$

$$y_0 = y_N = 0. \tag{20}$$

We can write this problem in the form of

$$L_1 y = \lambda y,$$

where L_1 is a matrix which is obtained by truncation of the first column and first row of the matrix L (see [5]). The matrix L_1 has the same structure as L. Therefore it has $N-1$ number different real eigenvalues $\mu_1 < \mu_2 < \ldots < \mu_{N-1}$. If we denote by $\{Q_n(\lambda)\}$ a solution of the equation (19) satisfying the conditions $Q_0(\lambda) = 0$, $Q_1(\lambda) = 1$, then it is obvious that the eigenvalues μ_j will be zeros of the polynomial $Q_N(\lambda)$, i.e. $Q_N(\mu_j) = 0$, $j = 1,2,\ldots,N-1$.

In the present item the problem of determination of the matrix L on two spectrums i.e. on eigenvalues $\{\lambda_j\}$ and $\{\mu_j\}$ is considered.

At first we'll prove the following lemma on mutual disposition of the spectrums of the matrixes L and L_1.

Lemma 5. *The eigenvalues of the boundary value problems (3)-(4) and (19)-(20) alternate, i.e.*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \ldots < \lambda_{N-1} < \mu_{N-1} < \lambda_N .$$

At first show that $\lambda_i \neq \mu_j$ $(i = 1,2,\ldots,N;\ j = 1,\ldots,N-1)$. Assume the contrary. Let $\lambda_{i_0} = \mu_{j_0} (= \lambda^*)$, i.e. $P_N(\lambda^*) = Q_N(\lambda^*) = 0$. Then for the Wronskian solution $\{Q_n\}$ and $\{P_n\}$ we have

$$W\{Q_n, P_n\} \overset{def}{=} a_{n-1}\left(Q_{n-1}(\lambda^*)P_n(\lambda^*) - Q_n(\lambda^*)P_{n-1}(\lambda^*)\right) =$$
$$= a_{N-1}\left(Q_{N-1}(\lambda^*)P_N(\lambda^*) - Q_N(\lambda^*)P_{N-1}(\lambda^*)\right) = 0,$$

i.e. the solutions $Q_n(\lambda^*)$ and $P_n(\lambda^*)$ are linear dependent, that is not possible, since by virtue of the mutual conditions $Q_0(\lambda^*) = 0,\ P_0(\lambda^*) = 1$. Thus, $\lambda_i \neq \mu_j$.

We assume $f_n(\lambda) = Q_n(\lambda) + m(\lambda)P_n(\lambda)$ and require that $f_N(\lambda) = 0$. Hence it follows that

$$m(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{21}$$

From this formula it is obvious that $m(\lambda)$ is a meromorphic function, where its poles and zeros coincide with the eigenvalues of the problem (3)-(4) and (19)-(20) respectively.

Differentiating the equation

$$a_{n-1}f_{n-1}(\lambda) + b_n f_n(\lambda) + a_n f_{n+1}(\lambda) = \lambda f_n(\lambda) \tag{22}$$

by λ, we see that the function $\dot{f}_n(\lambda)$ satisfies the equation

$$a_{n-1}\dot{f}_{n-1}(\lambda) + b_n \dot{f}_n(\lambda) + a_n \dot{f}_{n-1}(\lambda) = f_n(\lambda) + \lambda \dot{f}_n(\lambda).$$

Hence it follows that

$$\sum_{n=1}^{N-1} f_n^2(\lambda) = a_0\left(f_1(\lambda)\dot{f}_0(\lambda) - \dot{f}_1(\lambda)f_0(\lambda)\right) =$$
$$= a_0\left[(1 + m(\lambda)P_1(\lambda))\dot{m}(\lambda) - \left(\dot{m}(\lambda)P_1(\lambda) + m(\lambda)\dot{P}_1(\lambda)\right)m(\lambda)\right] =$$
$$= a_0\left[\dot{m}(\lambda) - m^2(\lambda)\frac{1}{a_0}\right] = a_0\dot{m}(\lambda) - f_0^2(\lambda)$$

or

$$\sum_{n=0}^{N-1} f_n^2(\lambda) = a_0\dot{m}(\lambda).$$

Taking into account that for real λ the left hand side of obtained equality is positive, we get that the function $m(\lambda)$ monotonically increases in the intervals $(-\infty, \lambda_1), (\lambda_1, \lambda_2)$, $\ldots, (\lambda_{N,+\infty})$. On the other hand it is obvious that

$$\lim_{\lambda\to\infty} m(\lambda) = 0, \quad \lim_{\lambda\to\lambda_j+0} m(\lambda) = +\infty, \quad \lim_{\lambda\to\lambda_j-0} m(\lambda) = -\infty .$$

Therefore in the interval $(-\infty, \lambda_1)$ the function $m(\lambda)$ has not solutions and in each interval it has one root $(\lambda_k, \lambda_{k+1})$.

The lemma is proved.

Now we deduce a formula with the help of which it is possible to determine the number α_n - normed numbers of the matrix L, on known two spectrums $\lambda_1,\ldots,\lambda_N$ and $\mu_1,\ldots,\mu_{N-1}$ of the matrices L and L_1 respectively.

We multiply the relation (22) by $P_n(\lambda_k)$ and the relation (6) by the polynomial $f_n(\lambda)$ for $\lambda=\lambda_k$ and substract the second from the first and sum by n. As a result we get

$$(\lambda-\lambda_k)\sum_{n=1}^{N-1} f_n(\lambda)P_n(\lambda_k)=\sum_{n=1}^{N-1}\{a_{n-1}(f_{n-1}(\lambda)P_n(\lambda_k)-P_{n-1}(\lambda_k)f_n(\lambda))-$$
$$-a_n(f_n(\lambda)P_{n+1}(\lambda_k)-f_{n+1}(\lambda)P_n(\lambda_k))\}$$

or

$$(\lambda-\lambda_k)\sum_{n=0}^{N-1} f_n(\lambda)P_n(\lambda_k)=-a_0 .$$

Assuming here $\lambda\to\lambda_k$ we have

$$\alpha_k=\frac{a_0\dot{P}_N(\lambda_k)}{Q_N(\lambda_k)} .$$

On the other hand since

$$P_N(\lambda)=\frac{1}{a_0a_1\ldots a_{N-1}}(\lambda-\lambda_1)\ldots(\lambda-\lambda_N),\ \ Q_N(\lambda)=\frac{1}{a_1\ldots a_{N-1}}(\lambda-\mu_1)\ldots(\lambda-\mu_{N-1})$$

we finally obtain

$$\alpha_k=\frac{\prod\limits_{j=1}^{N}{}'(\lambda_k-\lambda_j)}{\prod\limits_{j=1}^{N-1}(\lambda_k-\mu_j)}, \tag{23}$$

where the symbol Π' means that in product the member with the numbers k is omitted.

Theorem 2. *For two sets of the real numbers* $\{\lambda_k\}_{k=1}^{N}$, $\{\mu_k\}_{k=1}^{N-1}$ *to be the spectrums of the matrixes* L *and* L_1, *where* L *is a matrix as (5) with elements substituting the conditions (2), the matrix* L_1 *is obtained from the matrix* L *by truncating the first column and the first row, it is necessary and sufficient that they are alternated, i.e.*

$$\lambda_1<\mu_1<\lambda_2<\ldots<\mu_{N-1}<\lambda_N .$$

Proof. The necessity follows from Lemma 5.

Sufficiency. Let two sets of the numbers $\{\lambda_k\}$ and $\{\mu_k\}$ with indicated above properties be given. We determine the number α_k by the formula (23). From alternation condition it follows that all α_k are positive. On the other hand for sufficiently large R we have

$$\sum_{k=1}^{N}\frac{1}{\alpha_k}=\sum_{k=1}^{N}\operatorname*{Res}_{\lambda=\lambda_k}\frac{(\lambda-\mu_1)\ldots(\lambda-\mu_{N-1})}{(\lambda-\lambda_1)\ldots(\lambda-\lambda_N)}=\frac{1}{2\pi i}\oint_{|\lambda|=R}\frac{(\lambda-\mu_1)\ldots(\lambda-\mu_{N-1})}{(\lambda-\lambda_1)\ldots(\lambda-\lambda_N)}d\lambda=$$
$$=\frac{1}{2\pi i}\oint_{|\lambda|=R}\frac{\lambda^{N-1}+\ldots}{\lambda^N+\ldots}d\lambda=\frac{1}{2\pi i}\oint_{|\lambda|=R}\left[\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right]d\lambda=1+\frac{1}{2\pi i}\oint_{|\lambda|=R}O\left(\frac{1}{\lambda^2}\right)d\lambda .$$

Assuming here $R\to\infty$ we obtain that $\sum\limits_{k=1}^{N}\frac{1}{\alpha_k}=1$.

Thus the sets of the numbers $\{\lambda_k,\alpha_k\}$ satisfy the conditions of theorem 1 consequently there exists the matrix L as (5) with elements satisfying the conditions (2) for which $\{\lambda_k,\alpha_k\}$ are its spectrum data. Denote by L_1 matrix which is obtained from the matrix L by truncating the first column and the first row. Let $\tilde{\mu} < ... < \tilde{\mu}_{N-1}$ be eigenvalues of the matrix L_1. To complete the proof of the theorem it is sufficient to show that $\mu_k = \tilde{\mu}_k$, $k = 1,...,N-1$.

On the one side we have the formulas (23). Further, analogously we obtain that

$$\alpha_k = \frac{\prod\limits_{j=1}^{N}{}'(\lambda_k - \lambda_j)}{\prod\limits_{j=1}^{N-1}(\lambda_k - \tilde{\mu}_j)}.$$

Comparing these two formulas we arrive at equalities

$$\prod_{j=1}^{N-1}(\lambda_k - \mu_j) = \prod_{j=1}^{N-1}(\lambda_k - \tilde{\mu}_j), \quad k = 1,2,...,N$$

Thus, the polynomials $Q(\lambda) = \prod\limits_{j=1}^{N-1}(\lambda_k - \mu_j)$, $\tilde{Q}(\lambda) = \prod\limits_{j=1}^{N-1}(\lambda - \tilde{\mu}_j)$ of degree $N-1$ coincide in N different points. Therefore $Q(\lambda) \equiv \tilde{Q}(\lambda)$, i.e. $\mu_j = \tilde{\mu}_j$, $j = 1,2,...,N-1$

The theorem is proved.

Remark 1. Analogously we can investigate the inverse problems for the difference equation(3) with indivisible boundary conditions, in particular with periodic and antiperiodic boundary conditions.

Remark 2. Another variants of the inverse problems for the matrix L are considered in article [5].

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Received December 1, 2000; Revised February 14, 2001.
Translated by Mamedova V.A.