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ON THE EXISTENCE OF A GENERALIZED SOLUTION OF A BOUNDARY VALUE PROBLEM OF ONE CLASS OPERATOR-DIFFERENTIAL EQUATIONS

Abstract

In the paper the sufficient conditions ensuring the existence and uniqueness of generalized solutions for operator-differential equations, whose main parts have multiple characteristic are obtained. Simultaneously the exact values of the norms of intermediate derivatives of operators in some spaces are obtained.

Let H be a separable Hilbert space, A be a positive definite self-adjoint operator in H with the domain of definition $D(A)$. Denote by H_γ a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$ $(\gamma \geq 0)$, $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in D(A^\gamma)$. Denote by $L_2(a,b:H)$ $(-\infty \leq a < b \leq \infty)$ a Hilbert space of vector-functions determined in (a,b) almost everywhere with values in H which have the norm

$$\|f\|_{L_2(a,b:H)} = \left(\int_a^b \|f(t)\|_H^2 \, dt \right)^{1/2}$$

and assume $L_2(R:H) \equiv L_2(-\infty,\infty:H)$, $L_2(R_+:H) \equiv L_2(0,\infty:H)$.

Further we determine the Hilbert space

$$W_2^m(a,b:H) = \left\{ u \middle| u^{(m)} \in L_2(a,b:H),\ u \in L_2(a,b:H_m) \right\}$$

with the norm

$$\|u\|_{W_2^m(a,b:H)} = \left(\left\|u^{(m)}\right\|^2_{L_2(a,b:H)} + \|u\|^2_{L_2(a,b:H_m)} \right)^{1/2}.$$

Here and later on the derivatives $u^{(j)}(t) \equiv \dfrac{d^j u}{dt^j}$ $\left(j = \overline{1,m}\right)$ are understood in the sense of the distribution theory. Here we assume $W_2^m(R:H) \equiv W_2^m(-\infty,\infty:H)$, $W_2^m(R_+:H) = W_2^m(0,\infty:H)$. Further we determine the space

$$W_2^m\left(R_+:H;\{k\}_{k=0}^{m-1}\right) = \left\{\psi : \psi \in W_2^m(R_+:H),\ u^{(k)}(0) = 0,\ k = \overline{0,m-1}\right\}.$$

It is obvious that in the theorem on traces [1] the space $W_2^m\left(R_+:H;\{k\}_{k=0}^{m-1}\right)$ is a closed subspace of the Hilbert spaces $W_2^m(R_+:H)$.

Let's determine the space $D(a,b:H_\gamma)$ a set of infinity-differentiable functions with values in H_γ having a compact support in $[a,b]$. As it is known the linear set $D(a,b:H_m)$ is everywhere dense in the space $W_2^m(a,b:H)$ [1]. From the theorem of traces it follows that the space

$$D\left(R_+:H_{2m};\{k\}_{k=0}^{m-1}\right) = \left\{u \middle| u \in D(R_+:H_{2m}),\ u^{(k)}(0) = 0,\ k = \overline{0,m-1}\right\}$$

is also everywhere dense in the space $W_2^m\left(R_+:H;\{k\}_{k=0}^{m-1}\right)$.

Consider the following boundary value problem in the space H

$$P\left(\frac{d}{dt}\right)u(t)=\left(-\frac{d^2}{dt^2}+A^2\right)^m u(t)+\sum_{j=0}^{n}A_j u^{(n-j)}(t)=0,\ t\in R_+ , \qquad (1)$$

$$u^{(k)}(0)=\varphi_k,\quad k=\overline{0,m-1}, \qquad (2)$$

where A is a positive-definite self-adjoint operator, $A_0,A_1,A_2,...,A_n$ are linear operators in H, φ_k $\left(k=\overline{0,m-1}\right)$ are some vectors from H.

We'll investigate the existence of generalized solutions of the problem (1)-(2). Note that the generalized solutions of the problem for operator-differentiable equations are investigated by many authors. For example, in S.S.Mirzoyev's [2] and M.B.Obrazov's [3] paper the analogous problem is investigated, when the main part of the equation(1) has the form $(-1)^m \frac{d^{2m}}{dt^{2m}}+A^{2m}$. For $m=2$ the condition of the existence of a generalized solution is investigated in paper [4].

Later on we need the following.

Lemma 1 [2]. *Let A be a positive-definite self-adjoint operator, the operators $B_j=A_jA^{-j}$, $\left(j=\overline{0,m}\right)$ and $D_j=A^{-m}A_jA^{m-j}$, $\left(j=\overline{m+1,2m}\right)$ are bounded in H. Then the bilinear functional*

$$\mathcal{P}_1(u:\psi)\equiv\left(P_1(d/dt)u,\psi\right)_{L_2(R_+:H)},$$

defined for all vector-functions $u\in D(R_+:H)$ and $\psi\in D\left(R_+:H;\{k\}_{k=0}^{m-1}\right)$ is continued on the space $W_2^m(R_+:H)\oplus W_2^m\left(R_+:H;\{k\}_{k=0}^{m-1}\right)$ to the bilinear functional $\mathcal{P}_1(u:\psi)$ operating by the following form

$$\mathcal{P}_1(u:\psi)=\sum_{j=0}^{m}(-1)^m\left(A_ju^{(m-j)},\psi^{(m)}\right)_{L_2(R_+:H)}+$$

$$+\sum_{j=m+1}^{2m}(-1)^m\left(A_ju^{(2m-j)},\psi\right)_{L_2(R_+:H)}.$$

Definition. *The vector-function $u(t)\in W_2^m(R_+:H)$ is called a generalized solution of the problem (1)-(2), if*

$$\lim_{t\to 0}\left\|u^{(k)}(t)\right\|_{H_{m-k-1/2}}=0,\quad k=\overline{0,m-1}$$

and for any $\psi(t)\in W_2^m\left(R_+:H;\{k\}_{k=0}^{m-1}\right)$ the following identity is satisfied

$$(u,\psi)_{W_2^m}+\sum_{p=1}^{m-1}\binom{m}{p}\left(A^pu^{(m-p)},A^p\psi^{(m-p)}\right)_{L_2(R_+:H)}+\mathcal{P}_1(u:\psi)=0,$$

here $\binom{m}{p}=C_m^p=\frac{m(m-1)\dots(m-p+1)}{p!}$.

At first consider the problem

$$P_0\left(\frac{d}{dt}\right)u(t)=\left(-\frac{d^2}{dt^2}+A^2\right)^m u(t)=0,\ t\in R_+, \qquad (3)$$

$$u^{(k)}(0)=\varphi_k,\ k=\overline{0,m-1}. \qquad (4)$$

It holds

Theorem 1. *For any set $\varphi_k\in H_{m-k-1/2}$ $\left(k=\overline{0,m-1}\right)$ the problem (3)-(4) has a unique generalized solution.*

Proof. Let the vectors $c_0, c_1, \ldots c_{m-1} \in H_{m-1/2}$, e^{-At} be a holomorphic semi-group of bounded operators generalized by the operator $(-A)$. Then the vector function

$$u_0(t) = e^{-tA}\left(c_0 + \frac{t}{1!}Ac_1 + \ldots \frac{t^{m-1}}{(m-1)!}A^{m-1}c_{m-1}\right)$$

belongs to the space $W_2^m(R_+ : H)$, since every term $\frac{t^{m-k}}{(m-k)!}A^{m-\nu}e^{-tA}c_k \in W_2^m(R_+ : H)$ for $c_k \in H_{m-1/2}$, $\left(k = \overline{0, m-1}\right)$. On the other hand $u_0(t)$ is a general solution of the equation (3), therefore from the condition (4) we have to determine the vectors c_k, $\left(k = \overline{0, m-1}\right)$. It is easy to see that for determination of the vectors c_k, $\left(k = \overline{0, m-1}\right)$ the following system of the equations is obtained

$$\begin{bmatrix} E & 0 & 0 & \ldots & 0 \\ -E & E & 0 & \ldots & 0 \\ E & -E & E & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{m-1}E & \binom{m-1}{1}(-1)^{m-2}E & \binom{m-2}{2}(-1)^{m-3}E & \ldots & E \end{bmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{m-1} \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ A^{-1}\varphi_1 \\ A^{-2}\varphi_2 \\ \vdots \\ A^{-(m-1)}\varphi_m \end{pmatrix}, \qquad (5)$$

where E is a unit operator in H. Since the base operator determinant is reversible, then we can identically determine c_k, $\left(k = \overline{0, m-1}\right)$. It is obvious that $c_k \in H_{m-1/2}$. Really for any k, $\left(k = \overline{0, m-1}\right)$, $A^{-(m-k)}\varphi_k \in H_{m-1/2}$ since $\varphi_k \in H_{m-k-1/2}$.

Since the vector in the right hand side of the equation (5) belongs to the space $\underbrace{H_{m-1/2} \oplus H_{m-1/2} \oplus \ldots \oplus H_{m-1/2}}_{m \text{ times}} = \left(H_{m-1/2}\right)^m$, then taking into account that the base operator matrix as a product reversible by a scalar matrix $\tilde{E}$, where $\tilde{E}$ is a unit matrix in $\left(H_{m-1/2}\right)^m$, is reversible. Since any vector is a linear combination of the elements $A^{-(m-k)}\varphi_k \in H_{m-1/2}$, then $c_k \in H_{m-1/2}$, $\left(k = \overline{0, m-1}\right)$. Further, it is easily verified that $u_0(t)$ is a generalized solution of the equation (3), i.e.

$$(u_0, \psi)_{W_2^m} + \sum_{p=1}^{m-1}\binom{m}{p}\left(A^{m-p}u_0^{(p)}, A^{m-p}\psi^{(p)}\right) = 0$$

for any $\psi \in W_2^m\left(R_+ : H; \{k\}_{k=0}^{m-1}\right)$. The theorem is proved.

In the space $W_2^m\left(R_+ : H; \{k\}_{k=0}^{m-1}\right)$ we'll determine the new norm

$$|||u|||_{W_2^m(R_+:H)} = \left(\|u\|_{W_2^m(R_+:H)} + \sum_{p=1}^{m-1}\binom{m}{p}\left\|A^{m-p}u^{(p)}\right\|^2_{L_2(R_+:H)}\right)^{1/2}.$$

By the intermediate derivatives theorem [1, p.29] the norms $|||u|||_{W_2^m}$ and $\|u\|_{W_2^m}$ are equivalent in the space $W_2^m\left(R_+ : H; \{k\}_{k=0}^{m-1}\right)$. Therefore the numbers

$$N_j\left(R_+ : \{k\}_{k=0}^{m-1}\right) = \sup_{0 \neq u \in W_2^m\left(R_+ : H; \{k\}_{k=0}^{m-1}\right)} \left\|A^{m-j}u^{(j)}\right\|_{L_2(R_+:H)} |||u|||^{-1}_{W_2^m(R_+:H)}, \ j = \overline{0, m}$$

are finite.

Using the methods of work [2] we find exact values of these numbers.

Lemma 2. *The number* $N_j\left(R_+ : \{k\}_{k=0}^{m-1}\right)$ *is determined by the following form*

$$N_j\left(R_+ : \{k\}_{k=0}^{m-1}\right) = d_{m,j}^{m/2}$$

where

$$d_{m,j} = \begin{cases} \left(\frac{j}{m}\right)^{j/m}\left(\frac{m-j}{m}\right)^{\frac{m-j}{m}}, & if \ \ j=\overline{1,m-1}, \\ 1, & if \ \ j=\overline{0,m} \ . \end{cases}$$

Proof. Since $u \in W_2^m\left(R_+ : H; \{k\}_{k=0}^{m-1}\right)$, then we can continue it to negative semi-axis as a zero vector function from the class $W_2^m(R_+ : H)$. At first we prove the correctness of the exact inequality

$$\left\|A^{m-j}u^{(j)}\right\|_{L_2(R:H)}^2 \le d_{m,j}^m \|u\|_{W_2^m(R:H)}^2 \tag{6}$$

for the vector-functions $u \in W_2^m(R:H)$. To this end we assume that $u \in D(R:H_{2m})$. Then for $\beta \in \left(0, d_{m,j}^{-m}\right)$ and $u \in D(R:H_{2m})$ by the Plansherel theorem it holds the equality

$$\begin{aligned} \left|\|u\|_{W_2^m(R:H)}^2 - \beta\left\|A^{m-j}u^{(j)}\right\|_{L_2(R:H)}^2\right. = \int_{-\infty}^{+\infty}\Big[\left((-i\xi)^m \hat{u}(\xi), (-i\xi)^m \hat{u}(\xi)\right) + \\ \left(A^m u(\xi), A^m u(\xi)\right) + \sum_{p=1}^{m}\binom{m}{p}\left((-i\xi)^p A^{m-p}\hat{u}(\xi), (-i\xi)^p A^{m-p}\hat{u}(\xi)\right)_H - \\ - \beta\left((-i\xi)^j A^{m-j}\hat{u}(\xi), (-i\xi)^j A^{m-j}\hat{u}(\xi)\right)\Big]d\xi = \int_{-\infty}^{+\infty}\left(P_j(\xi;\beta;A)\hat{u}(\xi), u(\xi)\right)d\xi \ , \end{aligned} \tag{7}$$

where $\hat{u}(\xi)$ is a Fourier representation of the vector-function $u(t)$,

$$P_j(\xi;\beta;A) = \left(\xi^2 E + A^2\right)^m - \beta\xi^{2j}A^{2m-2j}, \tag{8}$$

Since for $\sigma \in \sigma(A)$ $(\sigma \ge \mu_0 > 0)$ and $\beta \in \left(0, d_{m,j}^{-m}\right)$

$$P_j(\xi;\beta;\sigma) = \left(\xi^2 + \sigma^2\right)^m - \beta\xi^{2j}\sigma^{2m-2j} = \left(\xi^2 + \sigma^2\right)^m\left[1 - \beta\frac{\xi^{2j}\sigma^{2m-2j}}{\left(\sigma^2 + \xi^2\right)^m}\right] \ge$$

$$\ge \left(\xi^2 + \sigma^2\right)^m\left[1 - \beta\sup_{\substack{\xi \in R \\ \sigma \ge \mu_0}}\frac{\xi^{2j}\sigma^{2m-2j}}{\left(\sigma^2 + \xi^2\right)^m}\right] = \left(\xi^2 + \sigma^2\right)^m\left(1 - \beta d_{m,j}^{-m}\right) > 0,$$

then from spherical expansion of the operator A it follows that the operator bundle $P_j(\xi;\beta;A)$ for $\beta \in \left(0, d_{m,j}^{-m}\right)$ satisfies the inequality

$$P_j(\xi;\beta;A) > \left(1 - \beta d_{m,j}^{-m}\right)\left(\xi^2 E + A^2\right)^m > 0 .$$

Thus from the equality (7) it follows that for all $\beta \in \left(0, d_{m,j}^{-m}\right)$ and $u \in W_2^m(R:H)$ the inequality

$$\left|\|u\|_{W_2^m(R:H)}^2 - \beta\left\|A^{m-j}u^{(j)}\right\|_{L_2(R:H)}^2\right. > 0$$

is valid.

Passing to the limit $\beta \to d_{m,j}^{-m}$ we obtain the correctness of the inequality (6).

Now we'll prove that this inequality is exact. Let's fix $\varepsilon > 0$ and show that there exists the vector function $\vartheta_\varepsilon(t) = g_\varepsilon(t)\varphi_\varepsilon$, where $g_\varepsilon(t)$ is a scalar function from the space $W_2^m(R:C)$, $\varphi_\varepsilon \in H_{2m}$, $\|\varphi_\varepsilon\| = 1$, for which

$$\boldsymbol{\mathcal{E}}(\vartheta_\varepsilon) \|u\|^2_{W_2^m(R:H)} - \left(d_{m,j}^{\,m} + \varepsilon\right)\left\|A^{m-j}u^{(j)}\right\|^2_{L_2(R:H)} < \varepsilon \tag{9}$$

or in the equivalent form

$$\boldsymbol{\mathcal{E}}(\vartheta_\varepsilon) = \int_{-\infty}^{+\infty}\left(P_j\left(\xi; d_{m,j}^{-m} + \varepsilon; A\right)\varphi_\varepsilon, \varphi_\varepsilon\right)\left|\hat{g}_\varepsilon(t)\right|^2 d\xi < 0, \tag{10}$$

where $P_j\left(\xi; d_{m,j}^{\,m} + \varepsilon; A\right)$ is an operator polynomial from the equality (8) for $\beta = d_{m,j}^{-m} + \varepsilon$. If the operator A has if only one eigenvalue $\mu > \mu_0$, then for φ_ε we choose the corresponding eigenvector φ, i.e.. $A\varphi_\varepsilon = \mu\varphi_\varepsilon$ $\left(\|\varphi_\varepsilon\| = 1\right)$. Then it is easy to see that

$$\begin{aligned}\left(P_j\left(\xi; d_{m,j}^{-m} + \varepsilon; A\right)\varphi_\varepsilon, \varphi_\varepsilon\right) = P_j\left(\xi; d_{m,j}^{-m} + \varepsilon; \mu\right) = \left(\xi^2 + \mu^2\right)^m - \\ -\xi^{2j}\left(d_{m,j}^{-m} + \varepsilon\right)\mu^{2m-2j} = \mu^{2m}\left(\xi^2/\mu^2 + 1\right)^m\left[1 - \left(d_{m,j}^{-m} + \varepsilon\right)\frac{\left(\xi^2/\mu^2\right)^{2j}}{\left(\xi^2/\mu^2 + 1\right)^m}\right].\end{aligned} \tag{11}$$

For $j = \overline{0, m-1}$ we find the point $\tau_0 = \xi_0/\mu$ such that

$$d_{m,j}^{-m} = \sup_{\tau \in R}\left|\tau^{2j}\left(\tau^2 + 1\right)^{-m}\right| = \left|\tau_0^{2j}\left(\tau_0^2 + 1\right)^{-m}\right|.$$

Then at the point $\xi = \xi_0 = \tau_0/\mu$ from (11) it follows that

$$P_j\left(\xi; d_{m,j}^{-m} + \varepsilon; \mu\right) = \mu^{2m}\left(\tau_0^2 + 1\right)\left(1 - \left(d_{m,j}^{-m} + \varepsilon\right)d_{m,j}^{\,m}\right) > 0. \tag{12}$$

If the operator A has no eigenvalue, then for $\mu \in \sigma(A)$ and for the sufficiently small $\delta > 0$ we can to construct the vector $\varphi_\delta \in H_{2m}$ $\left(\|\varphi_\delta\| = 1\right)$, such that

$$A^l\varphi_\delta = \mu^l\varphi_\delta + o(1, \delta), \quad \text{for } \delta \to 0,\ l = 1, 2, \ldots.$$

In this case the inequality

$$\left(P_j\left(\xi; d_{m,j}^{-m} + \varepsilon; A\right)\varphi_\delta, \varphi_\delta\right) = P_j\left(\xi; d_{m,j}^{-m} + \varepsilon; \mu\right) + 0(1, \delta) < 0$$

for $\xi = \tau_0/\mu$ is valid too. For $j = m$ it is also easy to prove that at some point the inequality (12) is valid. Thus for any $\xi = \tau_0/\mu$ we can construct the vector $\varphi_\varepsilon \in H_{2m}$ such that

$$\left(P_j\left(\xi; d_{m,j}^{-m} + \varepsilon; A\right)\varphi_\varepsilon, \varphi_\varepsilon\right) < 0 \qquad \left(j = \overline{0, m}\right) \tag{13}$$

at the some point $\xi = \tau_0/\mu$. Since the function $\left(P_j\left(\xi; d_{m,j}^{-m} + \varepsilon; A\right)\varphi_\varepsilon, \varphi_\varepsilon\right)$ is a continuous function of the argument ξ, then the inequality is valid and for some interval (η_0, η_1). Now we can construct $g_\varepsilon(t)$.

Let $\hat{g}(\xi)$ be an infinitely-differentiable finite function with a support in the interval (η_0, η_1). Denote its inverse Fourier transformation by $g_\varepsilon(t)$, i.e.

$$g_\varepsilon(t) = \frac{1}{\sqrt{2\pi}}\int_{\eta_1}^{\eta_0}\hat{g}(\xi)e^{i\xi t}d\xi.$$

It's obvious that $g_\varepsilon(t) \in W_2^m(R:C)$ and $\vartheta_\varepsilon(t) = g_\varepsilon(t)\varphi_\varepsilon$ from inequality (10) the next inequality follows

$$\mathcal{E}(\vartheta_\varepsilon)=\mathcal{E}(g_\varepsilon(t)\varphi_\varepsilon)=\int_{\eta_0}^{\eta_1}\left(P_j\left(\xi;d_{m,j}^{m}+\varepsilon;A\right)\varphi_\varepsilon,\varphi_\varepsilon\right)\left|\hat{g}_\varepsilon(t)\right|^2 d\xi<0 .$$

Thus we showed that the inequality is precise for the vector-function $u(t)$ from the class $W_2^m(R:H)$. We prove that it is exact and for the vector-functions $u(t)$ from the class $W_2^m\left(R_+:H;\{k\}_{k=0}^{m-1}\right)$.

Since $D(R:H_m)$ is dense in $W_2^m(R:H)$ and the functional $\mathcal{E}(\cdot)$ is continuous in the space $W_2^m(R:H)$, then we can find the vector-function $\omega_\varepsilon(t)\in D(R:H_m)$ for which the inequality (9) is satisfied. Then there exists the interval $(-N;N)\subset R$ outside of which $\omega_\varepsilon(t)=0$. Assuming $\omega_\varepsilon(t)=\omega_\varepsilon(t-2N)$, we obtain that $u_\varepsilon(t)\in W_2^m\left(R_+:H;\{k\}_{k=0}^{m-1}\right)$ and $\mathcal{E}(u_\varepsilon(t))<0$. Consequently, $N_j\left(R_+:\{k\}_{k=0}^{m-1}\right)=d_{m,j}^{m/2}$. The lemma is proved.

Now we prove a theorem on the solvability of the problem (1)-(2).

Theorem 2. *Let A be a positive-definite self-adjoint operator, the operators $B_j=A_jA^{-j}\left(j=\overline{0,m}\right)$ and $D_j=A^{-m}A_jA^{m-j}\left(j=\overline{m+1,2m}\right)$ be bounded in H and the inequality*

$$L=\sum_{j=0}^{k-1}d_{m,m-j}^{m/2}\left\|B_j\right\|+\frac{1}{2}\left\|B_k\right\|+\sum_{j=m+1}^{2m}d_{m,2m-j}^{m/2}\left\|D_j\right\|<1 .$$

holds.

Then the problem (1)-(2) has unique generalized solution for any set $\varphi_k\in H_{m-k-1/2}\left(k=\overline{0,m-1}\right)$, where the inequality

$$\|u\|_{W_2^m(R_+:H)}\le const\sum_{k=0}^{m-1}\|\varphi\|_{m-k-1/2}$$

is valid.

Proof. Let's show that for $L<1$ for all the vector-functions $\psi\in W_2^m\left(R_+:H;\{k\}_{k=0}^{m-1}\right)$ the inequality

$$\left|\left(\mathcal{P}(d/dt)\psi,\psi\right)\right|=\left|\|\psi\|_{W_2^m(R_+:H)}+\mathcal{P}_1(\psi,\psi)\right|\ge const\|\psi\|^2_{W_2^m(R_+:H)} \tag{14}$$

holds.

It's obvious that

$$\left|\left(\mathcal{P}(d/dt)\psi,\psi\right)\right|\ge\|\psi\|^2_{W_2^m(R_+:H)}-\mathcal{P}(\psi,\psi) . \tag{15}$$

On the other hand

$$\left|\mathcal{P}(\psi,\psi)\right|\le\sum_{j=0}^{k-1}\left|\left(A_j\psi^{(m-j)},\psi^{(m)}\right)_{L_2}\right|+\left|A_m\psi,\psi^{(m)}\right|_{L_2}+\sum_{j=m+1}^{2m}\left|\left(A_j\psi^{(2m-j)},\psi\right)_{L_2}\right| . \tag{16}$$

Since for $j=m$ the next inequality is valid

$$\begin{aligned}\left|\left(A_m\psi,\psi^{(m)}\right)_{L_2}\right|&=\left(A_mA^{-m}\psi,\psi^{(m)}\right)_{L_2}=\left(B_mA^m\psi,\psi^{(m)}\right)_{L_2}\le\\ &\le\|B_m\|\left\|A^m\psi\right\|_{L_2}\left\|\psi^{(m)}\right\|_{L_2}\le\frac{1}{2}\|B_m\|\left(\left\|A^m\psi\right\|^2_{L_2}\left\|\psi^{(m)}\right\|^2_{L_2}\right)\le\frac{1}{2}\|B_m\|\|\psi\|^2_{W_2^m} .\end{aligned} \tag{17}$$

For $j=\overline{0,m-1}$ by lemma 2 we find

$$\left|A_j\psi^{(m-j)},\psi^{(m)}\right|_{L_2}\le\left\|B_j\right\|\left\|A^j\psi^{(m-j)}\right\|_{L_2}\left\|\psi^{(m)}\right\|_{L_2}\le d_{m,2m-j}^{m/2}\left\|B_j\right\|\|\psi\|^2_{W_2^m} \tag{18}$$

For $j = \overline{m+1,2m}$ we analogously find

$$\left|\left(A_j\psi^{(2m-j)},\psi\right)_{L_2}\right| \le d_{m,2m-j}^{m/2}\|D_j\|\|\psi\|_{W_2^m}^2. \qquad (19)$$

Taking into account the inequalities (17), (18), and (19) in the inequality (15) we obtain

$$\left(P(d/dt)\psi,\psi\right)_{L_2(R_+:H)} \ge (1-\alpha)\|\psi\|_{L_2(R_+:H)}^2. \qquad (20)$$

Now we search a generalized solution of the generalized solution of (1)-(2) in the form of $u(t) = \vartheta_0(t) + \vartheta_0(t)$, where $\vartheta_0(t)$ is a generalized solution of the problem (3)-(4), and $\vartheta_0(t) \in W_2^m\left(R_+ : H; \{k\}_{k=0}^{m-1}\right)$, for determination of $\vartheta(t)$ we obtain

$$<\vartheta,\psi> = (\vartheta,\psi)_{W_2^m(R_+:H)} + \sum_{p=1}^{m-1}\binom{p}{m}\left(A^{m-p}\vartheta, A^{m-p}\psi\right) + P_1(\vartheta,\psi) = -P_1(\vartheta_0,\psi). \qquad (21)$$

As the right hand side of the equality (21) is a continuous functional in $W_2^m\left(R_+ : H; \{k\}_{k=0}^{m-1}\right)$ and the left hand side $<\vartheta,\psi>$ is a bilinear functional in the space $W_2^m\left(R_+ : H; \{k\}_{k=0}^{m-1}\right) \oplus W_2^m\left(R_+ : H; \{k\}_{k=0}^{m-1}\right)$, then it by the inequality (20) satisfies Lax-Milgram theorem. Consequently, there exists a unique vector-function $\vartheta(t) \in W_2^m\left(R_+ : H; \{k\}_{k=0}^{m-1}\right)$, which satisfies the equality (21) and $u(t) = \vartheta_0(t) + \vartheta_0(t)$ is a generalized solution of the problem (1)-(2).

Further, denote by $\Im(R_+ : H)$ a set of generalized solutions of the problem (1)-(2) and determine the operator $\Gamma : \Im(R_+ : H) \to \widetilde{H} = \bigoplus_{k=0}^{m-1} H_{m-k-1/2}$ operating by the following form $\Gamma u = \left(u^{(k)}(0)\right)_{k=0}^{m-1}$. It's obvious that $\Im(R_+ : H)$ is a closed set and by the theorem on traces $\|\Gamma u\|_{\widetilde{H}} \le c\|u\|_{W_2^2(R_+:H)}$. Then by the Banach theorem on an inverse operator the inverse bounded operator $\Gamma^{-1} : \widetilde{H} \to \Im(R_+ : H)$ exists, consequently

$$\|u\|_{W_2^2(R_+:H)} \le const\sum_{k=0}^{m-1}\|\varphi\|_{m-k-1/2}.$$

The theorem is proved.

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