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ON CORRECT SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR SOME CLASSES OF OPERATOR - DIFFERENTIAL EQUATIONS OF ODD ORDER WITH VARIABLE COEFFICIENTS

Abstract

In this paper the sufficient conditions of the correctness of the solvability of the boundary-value problems for the op erator-differential equations of order $4k-1$, *whose main parts are discontinuous, have been found, and the relations of this conditions with the estimations of the norms of operators of intermediate derivatives, have been shown.*

1. Let H be a separable Hilbert space, A-be a self-adjoint positive-defined operator in H. Let's consider the boundary-value problem for the operator differential equation of the higher odd order with discontinuous coefficients

$$A_0 u^{(4k-1)}(t)+\rho(t)A^{4k-1}u(t)+\sum_{j=1}^{4k-1}A_j(t)u^{(4k-1-j)}(t)=f(t), \quad t\in\mathcal{R}_+=[0;+\infty), \tag{1.1}$$

$$u^{(j)}(0)=0, \quad j=\overline{0,n}, \ n<4k-1, \tag{1.2}$$

where $f(t)\in L_2(\mathcal{R}_+;H)$, $u(t)\in W_2^{4k-1}(\mathcal{R}_+;H)$, $A_j(t)\left(j=\overline{1,4k-1}\right)$ are linear, generally speaking, unbounded operators determined almost for all $t\in R_+$, $A_0=\pm I$ (I-is a unit operator in H), $n=2k-2$ or $n=2k-1$ depending on the choice on choosing of the operator A_0, and $\rho(t)$ is a scalar bounded function, given by the following way:

$$\rho(t)=\begin{cases}\alpha_1, \ if \ 0\le t\le T_1,\\ \alpha_2, \ if \ T_1<t\le T_2,\\ \alpha_3, \ if \ T_2<t\le T_3,\\ \dots\dots\dots\dots\dots\dots\\ \alpha_s, \ if \ T_{s-1}<t<+\infty,\end{cases} \tag{1.3}$$

where $\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_s$ are positive, generally speaking, unequal numbers. Here and further the derivative $u^{(j)}\equiv\dfrac{d^j u}{dt^j}$ is regarded in the sense of the theory of generalized functions.

By $L_2(\mathcal{R}_+;H)$ it is denoted a Hilbert space of all vector functions determined in $\mathcal{R}_+$ with the values in H which have the finite norm:

$$\|f\|_{L_2(\mathcal{R}_+;H)}=\left(\int_0^{+\infty}\|f(t)\|_H^2dt\right)^{1/2},$$

and

$$W_2^{4k-1}(\mathcal{R}_+;H), \ \overset{0}{W}{}_2^{4k-1}(\mathcal{R}_+;H), \ \overset{00}{W}{}_2^{4k-1}(\mathcal{R}_+;H)$$

are the following Hilbert spaces:

$$W_2^{4k-1}(\mathcal{R}_+;H)=\left\{u(t)\Big|u^{(4k-1)}(t)\in L_2(\mathcal{R}_+;H), A^{4k-1}u(t)\in L_2(\mathcal{R}_+;H)\right\},$$

$$\overset{0}{W}{}_2^{4k-1}(\mathcal{R}_+;H)=\left\{u(t)\middle|u(t)\in W_2^{4k-1}(\mathcal{R}_+;H), u^{(j)}(0)=0, j=\overline{0,2k-2}\right\},$$

$$\overset{00}{W}{}_2^{4k-1}(\mathcal{R}_+;H)=\left\{u(t)\middle|u(t)\in W_2^{4k-1}(\mathcal{R}_+;H), u^{(j)}(0)=0, j=\overline{0,2k-1}\right\}$$

with the norm [1]

$$\|u\|_{W_2^{4k-1}(\mathcal{R}_+;H)}=\left(\left\|u^{(4k-1)}\right\|^2_{L_2(\mathcal{R}_+;H)}+\left\|A^{4k-1}u\right\|^2_{L_2(\mathcal{R}_+;H)}\right)^{1/2}.$$

Under $L(X;Y)$ we'll consider a set of linear bounded operators acting from Hilbert space X to another Y, and $L_\infty(\mathcal{R}_+;B)$ is the set of the B valued essentially bounded operator-functions in $\mathcal{R}_+$, where B is a Banach space.

Definition. *If the vector-function $u(t)$ from $W_2^{4k-1}(\mathcal{R}_+;H)$, satisfies the equation (1.1) almost everywhere in $\mathcal{R}_+$, and the boundary conditions (1.2) are fulfilled in the sense*

$$\lim_{t\to 0}\left\|A^{4k-j-\frac{1}{2}}u^{(j)}(t)\right\|_H=0,\quad j=\overline{0,n},$$

then $u(t)$ we'll be a regular solution of the boundary-value problem (1.1)-(1.2).

Further in order to avoid cumbersome records and for the sake of simplicity of the representations of results we'll assume the function $\rho(t)$ as explosive only at one pant, i.e.

$$\rho(t)=\begin{cases}\alpha, & if\ \ 0\le t\le T,\\ \beta, & if\ \ T<t<+\infty,\end{cases}\tag{1.4}$$

where α,β are positive, generally speaking unequal numbers. It is necessary to note that, the boundary value problems for the operator-differential equations of the form (1.1) with the continuous constant coefficients, i.e. the case $\rho(t)\equiv 1,\ t\in\mathcal{R}_+$, and $A_j(t)\equiv A_j\left(j=\overline{1,\ 4k-1}\right)$ are investigated in articles [2]-[4], with the variable coefficients in the case $\rho(t)\equiv 1, t\in\mathcal{R}_+$ have been studied in work [5]. In these papers the sufficient conditions the existence of the unique regular solution for the boundary-value problems, have been shown.

Note that, the correct and onevaluedly solvability of the boundary-value problems for the operator- differential equations of the form (1.1) on the constant operator coefficients and on the function $\rho(t)$ of the form (1.4), has been investigated in [6], [7] in case $k=1$.

The investigations on a semiaxis of the operator-differential equations of the higher even order with discontinuous coefficient $\rho(t)$ of the form (1.3) with different boundary-value conditions were carried out in [8], [9], [10]. But works [6] and [7] gave us the idea of construction of the general method of investigation for variable operator coefficients of boundary-value problem for some classes of differential equations of higher odd order, whose the main parts are discontinuous, that is one of the aims of the present paper.

In the present paper the sufficient conditions of the existence of the unique regular solutions of the boundary-value problems (1.1)-(1.2), which are closely related with found upper estimations of norms of the operators of intermediate derivatives by the main part of the equations (1.1) in the spaces $\overset{0}{W}{}_2^{4k-1}(R_+;H)$ and $\overset{00}{W}{}_2^{4k-1}(R_+;H)$, have been obtained.

The mentioned conditions of the solvability of the boundary-value problems (1.1)-(1.2) have been expressed only in terms of the operator coefficients of the equations (1.1).

2. Remind, that in led investigations we'll assume $\rho(t)$ as the function of the form (1.4). Let $A_0 = -I$ and $n = 2k-1$.

At first let's consider the main part of the equation (1.1)

$$-u^{(4k-1)}(t) + \rho(t)A^{4k-1}u(t) = f(t), \tag{2.1}$$

where $f(t) \in L_2(R_+;H)$, $u(t) \in \overset{00}{W}{}_2^{4k-1}(R_+;H)$.

By $\mathcal{L}_0$ we'll denote the operator acting from the space $\overset{00}{W}{}_2^{4k-1}(R_+;H)$ *to* $L_2(R_+;H)$ by the following way:

$$\mathcal{L}_0 u(t) \equiv -u^{(4k-1)}(t) + \rho(t)A^{4k-1}u(t),\ u(t) \in \overset{00}{W}{}_2^{4k-1}(R_+;H)$$

Then the following theorem is true.

Theorem 1. *The operator $\mathcal{L}_0$ realizes the isomorphism between the spaces $\overset{00}{W}{}_2^{4k-1}(R_+;H)$ and $L_2(R_+;H)$.*

Proof. It is not difficult, to see that the homogenous equation $\mathcal{L}_0 u(t) = 0$ has only zero solution from the space $\overset{00}{W}{}_2^{4k-1}(R_+;H)$. Let's build the proof of the given fact by more simple method unlike the traditional [see for example, [6]].

Assume the contrary, let the differ from zero solution $u(t) \in \overset{00}{W}{}_2^{4k-1}(R_+;H)$ exist. Then

$$-u^{(4k-1)} + \rho(t)A^{4k-1}u = 0$$

scalarly multiplying by $A^{4k-1}u$ we have:

$$\left(-u^{(4k-1)} + \rho(t)A^{4k-1}u, A^{4k-1}u\right)_{L_2(R_+;H)} = 0,$$

$$\left(-u^{(4k-1)}, A^{4k-1}u\right)_{L_2(R_+;H)} + \left(\rho(t)A^{4k-1}u, A^{4k-1}u\right)_{L_2(R_+;H)} = 0. \tag{2.2}$$

On the another hand subject to $u(t) \in \overset{00}{W}{}_2^{4k-1}(R_+;H)$ we find:

$$\operatorname{Re}\left(-u^{(4k-1)}, A^{4k-1}u\right)_{L_2(R_+;H)} = 0. \tag{2.3}$$

Therefore from the equality (2.2) we obtain:

$$\left(\rho(t)A^{4k-1}u, A^{4k-1}u\right)_{L_2(R_+;H)} = \left\|\rho^{1/2}(t)A^{4k-1}u\right\|^2_{L_2(R_+;H)} = 0.$$

Consequently, $u(t) = 0$.

Now, let's show that the equation $\mathcal{L}_0 u(t) = f(t)$ has the solution from $\overset{00}{W}{}_2^{4k-1}(R_+;H)$ for any $f(t) \in L_2(R_+;H)$

Really, consider the equation

$$\mathcal{L}_\alpha v(t) \equiv -v^{(4k-1)}(t) + \alpha A^{4k-1}v(t) = F(t), \tag{2.4}$$

in the space $W_2^{4k-1}(\mathcal{R};H)$ $\left(\mathcal{R}=(-\infty;+\infty)\right)$, where

$$F(t)=\begin{cases} f(t), if\ t\in[0,T), \\ 0,\ if\ t\in\mathcal{R}\setminus[0,T). \end{cases}$$

It is easy to see that the solution of the equation (2.4) from the space $W_2^{4k-1}(\mathcal{R};H)$ is represented in the form

$$v(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(i\lambda^{4k-1}I+\alpha A^{4k-1}\right)^{-1}\left(\int_0^T f(s)e^{-i\lambda s}ds\right)e^{it\lambda}d\lambda.$$

Really, by Plansherel theorem

$$\|v\|^2_{W_2^{4k-1}(\mathcal{R};H)}=\left\|v^{(4k-1)}\right\|^2_{L_2(\mathcal{R};H)}+\left\|\alpha A^{4k-1}v\right\|^2_{L_2(\mathcal{R};H)}=\left\|i\lambda^{4k-1}\hat{v}(\lambda)\right\|^2_{L_2(\mathcal{R};H)}+$$

$$+\left\|\alpha A^{4k-1}\hat{v}(\lambda)\right\|^2_{L_2(\mathcal{R};H)}\le\left\|i\lambda^{4k-1}\left(i\lambda^{4k-1}I+\alpha A^{4k-1}\right)^{-1}\right\|^2_{H\to H}\left\|\hat{f}(\lambda)\right\|^2_{L_2(\mathcal{R};H)}+$$

$$+\left\|\alpha A^{4k-1}\left(i\lambda^{4k-1}I+\alpha A^{4k-1}\right)^{-1}\right\|^2_{H\to H}\left\|\hat{f}(\lambda)\right\|^2_{L_2(\mathcal{R};H)}\le$$

$$\le const\left\|\hat{f}(\lambda)\right\|^2_{L_2(\mathcal{R};H)}=const\|f(t)\|^2_{L_2([0;T);H)},$$

where $\hat{v}(\lambda)$, $\hat{f}(\lambda)$ are Fourier transformations of the functions $v(t), f(t)$ respectively.
Further let's determine the contraction of the solution $v(t)$ on $[0;T)$ and denote it by $u_\alpha(t)$. Similarly we consider the equation

$$\mathcal{L}_\beta v(t)\equiv -v^{(4k-1)}(t)+\beta A^{4k-1}v(t)=F(t), \tag{2.5}$$

where

$$F(t)=\begin{cases} f(t), if\ t\in(T;+\infty), \\ 0, if\ t\in\mathcal{R}\setminus(T;+\infty) \end{cases}$$

and determine the solution $u_\beta(t)$ of the equation (2.5) from the space $W_2^{4k-1}((T;+\infty);H)$.
So the solution of the equation $\mathcal{L}_0 u(t)=f(t)$ from the space $\overset{00}{W}{}_2^{4k-1}(\mathcal{R}_+;H)$ *is represented* in the form

$$u(t)=\begin{cases} u_1(t)=u_\alpha(t)+\sum_{i=1}^{2k}\left(\exp\left(\alpha^{\frac{1}{4k-1}}\omega_i tA\right)\right)\varsigma_i+\sum_{i=2k+1}^{4k-1}\left(\exp\left(-\alpha^{\frac{1}{4k-1}}\omega_i(T-t)A\right)\right)\varsigma_i,\ if \\ 0\le t<T, \\ u_2(t)=u_\beta(t)+\sum_{i=1}^{2k}\left(\exp\left(\beta^{\frac{1}{4k-1}}\omega_i(t-T)A\right)\right)\varsigma_{4k-1+i},\ if\ \ T<t<+\infty, \end{cases}$$

where $\omega_1,\omega_2,\ldots,\omega_{4k-1}$ are the roots of the equation $-\lambda^{4k-1}+1=0$, however $\mathrm{Re}\,\omega_i<0,\ i=\overline{1,2k},\ \mathrm{Re}\,\omega_i>0,\ i=\overline{2k+1,4k-1},$ and the vectors $\varsigma_i\in D\left(A^{4k-\frac{3}{2}}\right)$,

$i=\overline{1,6k-1}$ (see, for example, [1]) are the elements from the space H one-valuedly determined from the condition $u(t)\in \overset{00}{W}{}_2^{4k-1}(\mathcal{R}_+;H)$ by the following relations:

$$\begin{cases} u^{(j)}(0)=u_1^{(j)}(0)=0, \ j=\overline{0,2k-1}, \\ u^{(j)}(T)=u_1^{(j)}(T)=u_2^{(j)}(T), \ \ j=\overline{0,4k-2}, \end{cases}$$

although it is necessary to surmount many technical difficulties.

And now let's show the boundedness of the operator $\mathcal{L}_0$.

In view of that

$$\left(u^{(4k-1)},\rho(t)A^{4k-1}u\right)_{L_2(\mathcal{R}_+;H)}+\left(\rho(t)A^{4k-1}u,u^{(4k-1)}\right)_{L_2(\mathcal{R}_+;H)}=$$
$$=2\operatorname{Re}\left(u^{(4k-1)},\rho(t)A^{4k-1}u\right)_{L_2(\mathcal{R}_+;H)}$$

we've

$$\left\|\mathcal{L}_0u\right\|^2_{L_2(\mathcal{R}_+;H)}=\left\|u^{(4k-1)}\right\|^2_{L_2(\mathcal{R}_+;H)}-2\operatorname{Re}\left(u^{(4k-1)},\rho(t)A^{4k-1}u\right)_{L_2(\mathcal{R}_+;H)}+$$
$$+\left\|\rho(t)A^{4k-1}u\right\|^2_{L_2(\mathcal{R}_+;H)}\le 2\left(\left\|u^{(4k-1)}\right\|^2_{L_2(\mathcal{R}_+;H)}+\left\|\rho(t)A^{4k-1}u\right\|^2_{L_2(\mathcal{R}_+;H)}\right)\le$$
$$\le 2\left(\left\|u^{(4k-1)}\right\|^2_{L_2(\mathcal{R}_+;H)}+\max\left(\alpha^2;\beta^2\right)\left\|A^{4k-1}u\right\|^2_{L_2(\mathcal{R}_+;H)}\right)\le$$
$$\le 2\max\left(1;\alpha^2;\beta^2\right)\left\|u\right\|^2_{W_2^{4k-1}(\mathcal{R}_+;H)},$$

from which the boundness of the operator $\mathcal{L}_0$ yields.

Thus the operator $\mathcal{L}_0$ is bounded and one-to-one acts from the space $\overset{00}{W}{}_2^{4k-1}(\mathcal{R}_+;H)$ to the space $L_2(\mathcal{R}_+;H)$. Further, applying Banach theorem on the inverse operator complete the proof of the theorem, i.e. the operator $\mathcal{L}_0$ isomorphically maps the space

$$\overset{00}{W}{}_2^{4k-1}(\mathcal{R}_+;H) \ \ to \ \ L_2(\mathcal{R}_+;H).$$

Theorem has been proved.

By $\mathcal{L}$ we denote the operator generated by the equation (1.1) and by the boundary value conditions (1.2), which acts from the space $\overset{00}{W}{}_2^{4k-1}(\mathcal{R}_+;H)$ in $L_2(\mathcal{R}_+;H)$.

The following theorem holds.

Theorem 2. *Let* $S_j(t)=A_j(t)A^{-j}\in L_\infty(\mathcal{R}_+;L(H;H)),\ j=\overline{1,4k-1}$. *Then operator* $\mathcal{L}$ *is a bounded operator from the space* $\overset{00}{W}{}_2^{4k-1}(\mathcal{R}_+;H)$ *in* $L_2(\mathcal{R}_+;H)$.

Proof. By virtue of that for any $u(t)\in \overset{00}{W}{}_2^{4k-1}(\mathcal{R}_+;H)$

$$\left\|\mathcal{L}u\right\|_{L_2(R_+;H)} \le \left\|\mathcal{L}_0 u\right\|_{L_2(R_+;H)} + \left\|\sum_{j=1}^{4k-1} A_j(t)u^{(4k-1-j)}\right\|_{L_2(R_+;H)} \le$$
$$\le \left\|\mathcal{L}_0 u\right\|_{L_2(R_+;H)} + \sum_{j=1}^{4k-1}\sup_t \left\|S_j(t)\right\|_{H\to H}\left\|A^j u^{(4k-1-j)}\right\|_{L_2(R_+;H)},$$

then applying theorem 1 and intermediate derivatives [1], from the last inequality we obtain

$$\left\|\mathcal{L}u\right\|_{L_2(R_+;H)} \le const\left\|u\right\|_{W_2^{4k-1}(R_+;H)}.$$

Theorem has been proved.

3. Now let's pass to the exact upper estimations of the norms of the operators of intermediate derivatives by the main part of the equation (1.1) in the space $\overset{00}{W}{}_2^{4k-1}(R_+;H)$, but before let's prove the following coercive inequality applied later

Lemma. *For any $u(t)\in \overset{00}{W}{}_2^{4k-1}(R_+;H)$ the following inequality holds:*

$$\left\|\rho^{-\frac{1}{2}}(t)u^{(4k-1)}\right\|^2_{L_2(R_+;H)} + \left\|\rho^{\frac{1}{2}}(t)A^{4k-1}u\right\|^2_{L_2(R_+;H)} \le$$
$$\le [\min(\alpha;\beta)]^{-1}\cdot\left\|\mathcal{L}_0 u\right\|^2_{L_2(R_+;H)}.$$

Proof. Since $u(t)\in \overset{00}{W}{}_2^{4k-1}(R_+;H)$, then with the help of the integration by parts it is easy to get the following equality:

$$\left(\mathcal{L}_0 u, A^{4k-1}u\right)_{L_2(R_+;H)} = \left(-u^{(4k-1)}, A^{4k-1}u\right)_{L_2(R_+;H)} +$$
$$+\left(\rho(t)A^{4k-1}u, A^{4k-1}u\right)_{L_2(R_+;H)} = \left\|\rho^{\frac{1}{2}}(t)A^{4k-1}u\right\|^2_{L_2(R_+;H)}, \tag{3.1}$$

$$\left(\mathcal{L}_0 u, -\rho^{-1}(t)u^{(4k-1)}\right)_{L_2(R_+;H)} = \left(-u^{(4k-1)}, -\rho^{-1}(t)u^{(4k-1)}\right)_{L_2(R_+;H)} -$$
$$-\left(A^{4k-1}u, u^{(4k-1)}\right)_{L_2(R_+;H)} = \left\|\rho^{-\frac{1}{2}}(t)u^{(4k-1)}\right\|^2_{L_2(R_+;H)}, \tag{3.2}$$

taking into consideration (2.3).

Further, summing the equalities (3.1) and (3.2) and taking into account the inequalities from analysis we obtain

$$\left\|\rho^{-\frac{1}{2}}(t)u^{(4k-1)}\right\|^2_{L_2(R_+;H)} + \left\|\rho^{\frac{1}{2}}(t)A^{4k-1}u\right\|^2_{L_2(R_+;H)} =$$
$$= \left(\mathcal{L}_0 u, A^{4k-1}u - \rho^{-1}(t)u^{(4k-1)}\right)_{L_2(R_+;H)} \le$$
$$\le \frac{1}{2\min(\alpha;\beta)}\left\|\mathcal{L}_0 u\right\|^2_{L_2(R_+;H)} + \frac{1}{2}\left\|\rho^{\frac{1}{2}}(t)A^{4k-1}u - \rho^{-\frac{1}{2}}(t)u^{(4k-1)}\right\|^2_{L_2(R_+;H)} =$$

$$=\frac{1}{2\min(\alpha;\beta)}\left\|\mathcal{L}_0 u\right\|^2_{L_2(R_+;H)}+\frac{1}{2}\left\|\rho^{\frac{1}{2}}(t)A^{4k-1}u\right\|^2_{L_2(R_+;H)}+\frac{1}{2}\left\|\rho^{-\frac{1}{2}}(t)u^{(4k-1)}\right\|^2_{L_2(R_+;H)}. \qquad (3.3)$$

By the same token, from the inequality (3.3) we obtain the correctness of the lemma. The lemma has been proved.

From theorem 1 it follows that the norm $\left\|\mathcal{L}_0 u\right\|_{L_2(R_+;H)}$ is equivalent to the norm $\|u\|_{W_2^{4k-1}(R_+;H)}$ in the space $\overset{00}{W}{}_2^{4k-1}(R_+;H)$ and as known the operators of intermediate derivatives

$$A^j\frac{d^{4k-1-j}}{dt^{4k-1-j}}:\overset{00}{W}{}_2^{4k-1}(R_+;H)\to L_2(R_+;H),\ j=\overline{1,4k-1}$$

are continuous therefore one can estimate the norms of these operators by $\left\|\mathcal{L}_0 u\right\|_{L_2(R_+;H)}$, and namely the following theorem is correct.

Theorem 3. *Let* $u(t)\in\overset{00}{W}{}_2^{4k-1}(R_+;H)$. *Then the following inequalities are correct*

$$\left\|A^j u^{(4k-1-j)}\right\|_{L_2(R_+;H)}\le\left(\frac{4k-1-j}{4k-1}\right)^{\frac{4k-1-j}{8k-2}}\cdot\left(\frac{j}{4k-1}\right)^{\frac{j}{8k-2}}\cdot C_j\left[\max(\alpha;\beta)\right]^{\frac{4k-1-j}{8k-2}}\times$$

$$\times\left[\min(\alpha;\beta)\right]^{-\frac{4k-1+j}{8k-2}}\cdot\left\|\mathcal{L}_0 u\right\|_{L_2(R_+;H)},\ j=\overline{1,4k-2}, \qquad (3.4)$$

$$\left\|A^{4k-1}u\right\|_{L_2(R_+;H)}\le\left[\min(\alpha;\beta)\right]^{-1}\left\|\mathcal{L}_0 u\right\|_{L_2(R_+;H)}, \qquad (3.5)$$

$$\left\|u^{(4k-1)}\right\|_{L_2(R_+;H)}\le\left[\frac{\max(\alpha;\beta)}{\min(\alpha;\beta)}\right]^{\frac{1}{2}}\left\|\mathcal{L}_0 u\right\|_{L_2(R_+;H)}, \qquad (3.6)$$

where $C_j=\begin{cases}2^{\frac{4k-1-j}{8k-2}(2k-1)(2k-2)}, & \text{if } j\le 2k-2,\\ 2^{\frac{j}{8k-2}[(4k-1)(3k-j)-(7k-1)]}, & \text{if } j>2k-2.\end{cases}$

Proof. The correctness of the inequalities (3.5), (3,6) is obvious. One can obtain them from the lemma. Now let's prove the inequality (3.4) since $u(t)\in\overset{00}{W}{}_2^{4k-1}(R_+;H)$, then integrating by parts and taking into consideration Buniakowski-Schwartz inequality we obtain:

$$\left\|A^{4k-1-j}u^{(j)}\right\|^2_{L_2(R_+;H)}\le\left\|A^{4k-j}u^{(j-1)}\right\|_{L_2(R_+;H)}\left\|A^{4k-2-j}u^{(j+1)}\right\|_{L_2(R_+;H)}, \qquad (3.7)$$
$$j=\overline{1,2k}.$$

On the another hand, for $j=\overline{2k+1,4k-2}$ the following relations hold:

$$\left\|A^{4k-1-j}u^{(j)}\right\|^2_{L_2(R_+;H)}\le 2\left\|A^{4k-j}u^{(j-1)}\right\|_{L_2(R_+;H)}\left\|A^{4k-2-j}u^{(j+1)}\right\|_{L_2(R_+;H)}. \qquad (3.8)$$

Assume that $r=\overline{1,4k-2}$, $p_1=\dfrac{4k-1-r}{4k-1}$, $p_j=jp_1\left(j=\overline{2,r}\right)$, $p_{r+1}=\dfrac{r}{4k-1}(4k-2-r)$,

$p_{r+2}=\frac{r}{4k-1}(4k-3-r),\ldots,\ \ p_{4k-2}=\frac{r}{4k-1}$, then from (3.7) and (3.8) we find

$$\prod_{j=1}^{4k-1}\left\|A^{4k-1-j}u^{(j)}\right\|_{L_2(R_+;H)}^{2p_j-p_{j-1}-p_{j+1}}\le 2^{p_{2k+1}+\ldots+p_{4k-2}}\left\|A^{4k-1}u\right\|_{L_2(R_+;H)}^{\frac{4k-1-r}{4k-1}}\times\left\|u^{(4k-1)}\right\|_{L_2(R_+;H)}^{\frac{r}{4k-1}}.$$

Hence

$$\left\|A^{4k-1-r}u^{(r)}\right\|_{L_2(R_+;H)}\le\left(\frac{4k-1-r}{4k-1}\right)^{\frac{4k-1-r}{8k-2}}\left(\frac{r}{4k-1}\right)^{\frac{r}{8k-2}}\cdot 2^{p_{2k+1}+\ldots+p_{4k-2}}\left\|\mathcal{L}_0u\right\|_{L_2(R_+;H)}\times$$

$$\times\left[\min(\alpha;\beta)\right]^{-\frac{4k-1-r}{4k-1}}\left[\frac{\max(\alpha;\beta)}{\min(\alpha;\beta)}\right]^{\frac{r}{8k-2}}=\left(\frac{4k-1-r}{4k-1}\right)^{\frac{4k-1-r}{8k-2}}\left(\frac{r}{4k-1}\right)^{\frac{r}{8k-2}}\times$$

$$\times 2^{p_{2k+1}+\ldots+p_{4k-2}}\left[\min(\alpha;\beta)\right]^{-\frac{8k-2-r}{8k-2}}\left[\max(\alpha;\beta)\right]^{\frac{r}{8k-2}}\cdot\left\|\mathcal{L}_0u\right\|_{L_2(R_+;H)}.$$

Taking $j=4k-1-r$, we find

$$\left\|A^{j}u^{(4k-1-j)}\right\|_{L_2(R_+;H)}\le\left(\frac{j}{4k-1}\right)^{\frac{j}{8k-2}}\left(\frac{4k-1-j}{4k-1}\right)^{\frac{4k-1-j}{8k-2}}C_j\left[\min(\alpha;\beta)\right]^{-\frac{4k+j-1}{8k-2}}\times$$

$$\times\left[\max(\alpha;\beta)\right]^{\frac{4k-1-j}{8k-2}}\left\|\mathcal{L}_0u\right\|_{L_2(R_+;H)},\ j=\overline{1,4k-2},$$

where

$$C_j=\begin{cases}2^{\frac{4k-1-j}{8k-2}(2k-1)(2k-2)}, & if\ j\le 2k-2,\\ 2^{\frac{j}{8k-2}[(4k-1)(3k-j)-(7k-1)]}, & if\ j>2k-2.\end{cases}$$

Theorem has been proved.

4. In the present item the main results of the work have been obtained, i.e. the conditions imposed only on the operator coefficients of the equations (1.1) which provide the existence of the unique regular solution of the boundary-value problems (1.1)-(1.2), are found. And namely, the following basic theorem holds:

Theorem 4. *Let* $A_0=-I,\ n=2k-1$, *the operators* $S_j(t)=A_j(t)A^{-j}$, $j=\overline{1,4k-1}$ *be bounded in H and the inequality*

$$\delta=\sum_{j=1}^{4k-2}\left[\left(\frac{4k-1-j}{4k-1}\right)^{\frac{4k-1-j}{8k-2}}\left(\frac{j}{4k-1}\right)^{\frac{j}{8k-2}}C_j\left[\max(\alpha,\beta)\right]^{\frac{4k-1-j}{8k-2}}\times\right.$$

$$\left.\times\left[\min(\alpha;\beta)\right]^{-\frac{4k-1+j}{8k-2}}\sup_t\left\|S_j(t)\right\|_{H\to H}\right]+\frac{1}{\min(\alpha;\beta)}\sup_t\left\|S_{4k-1}(t)\right\|_{H\to H}<1$$

be fulfilled, where the numbers C_j *are determined as in theorem 3. Then the boundary value problem (1.1)-(1.2) has the unique regular solution for any* $f(t)$ *from the space* $L_2(R_+;H)$.

Proof. Let's represent the boundary -value problem (1.1)-(1.2) in the form of the following operator equation:

$$\mathcal{L}_0 u(t) + (\mathcal{L} - \mathcal{L}_o)u(t) = f(t), \text{ where } f(t) \in L_2(\mathcal{R}_+;H),\ u(t) \in \overset{00}{W}{}_2^{4k-1}(\mathcal{R}_+;H).$$

From theorem 1 it follows that the operator $\mathcal{L}_0$ has the bounded inverse $\mathcal{L}_0^{-1}$ acting from the space $L_2(\mathcal{R}_+;H)$ to the space $\overset{00}{W}{}_2^{4k-1}(\mathcal{R}_+;H)$.

Then after substitution $u(t) = \mathcal{L}_0^{-1} v(t)$, where $v(t) \in L_2(\mathcal{R}_+;H)$ we get the following equation $\left(I + (\mathcal{L} - \mathcal{L}_o)\mathcal{L}_o^{\prime}\right)v(t) = f(t)$ in $L_2(\mathcal{R}_+;H)$.

Now, let's show that under the correctness of the conditions of the theorem the norm of the operator $(\mathcal{L} - \mathcal{L}_o)\mathcal{L}_0^{-1}$ is less than unt.

Really, using theorem 3 we obtain

$$\left\|(\mathcal{L} - \mathcal{L}_0)\mathcal{L}_0^{-1} v\right\|_{L_2(\mathcal{R}_+;H)\to L_2(\mathcal{R}_+;H)} = \left\|(\mathcal{L} - \mathcal{L}_0)u\right\|_{L_2(\mathcal{R}_+;H)} =$$

$$= \left\|\sum_{j=1}^{4k-1} A_j(t) u^{(4k-1-j)}\right\|_{L_2(\mathcal{R}_+;H)} \le \sum_{j=1}^{4k-1}\left\|A_j(t) u^{(4k-1-j)}\right\|_{L_2(\mathcal{R}_+;H)} \le$$

$$\le \sum_{j=1}^{4k-1}\left[\sup_t \left\|S_j(t)\right\|_{H\to H}\left\|A^j u^{(4k-1-j)}\right\|_{L_2(\mathcal{R}_+;H)}\right] \le$$

$$\le \sum_{j=1}^{4k-2}\left[\sup_t \left\|S_j(t)\right\|_{H\to H}\left(\frac{4k-1-j}{4k-1}\right)^{\frac{4k-1-j}{8k-2}}\left(\frac{j}{4k-1}\right)^{\frac{j}{8k-2}} C_j\left[\max(\alpha;\beta)\right]^{\frac{4k-1-j}{8k-2}} \times\right.$$

$$\left.\times\left[\min(\alpha;\beta)\right]^{-\frac{4k-1+j}{8k-2}}\left\|\mathcal{L}_0 u\right\|_{L_2(\mathcal{R}_+;H)}\right] + \sup_t \left\|S_{4k-1}(t)\right\|_{H\to H}\left[\min(\alpha;\beta)\right]^{-1} \times$$

$$\times\left\|\mathcal{L}_0 u\right\|_{L_2(\mathcal{R}_+;H)} = \delta\left\|v\right\|_{L_2(\mathcal{R}_+;H)}.$$

Thereby we find

$$\left\|(\mathcal{L} - \mathcal{L}_0)\mathcal{L}_0^{-1}\right\|_{L_2(\mathcal{R}_+;H)\to L_2(\mathcal{R}_+;H)} \le \delta < 1.$$

Therefore under correctness of this inequality the operator $I + (\mathcal{L} - \mathcal{L}_0)\mathcal{L}_0^{-1}$ has the inverse in the space $L_2(\mathcal{R}_+;H)$ and one can determine $u(t)$ by the following formula:

$$u(t) = \mathcal{L}_0^{-1}\left(I + (\mathcal{L} - \mathcal{L}_0)\mathcal{L}_0^{-1}\right)^{-1} f(t).$$

The theorem has been proved.

The corresponding results for the boundary-value problems (1.1)-(1.2) are found also in case $A_0 = I$ and $n = 2k-2$.

Theorem 5. *Let* $A_0 = I$, $n = 2k-2$, *the operators* $S_j(t) = A_j(t)A^{-j}$, $j = \overline{1,4k-1}$ *be bounded in H and the inequality*

$$\delta = \sum_{j=1}^{4k-2}\left[\left(\frac{4k-1-j}{4k-1}\right)^{\frac{4k-1-j}{8k-2}}\left(\frac{j}{4k-1}\right)^{\frac{j}{8k-2}} d_j\left[\max(\alpha;\beta)\right]^{\frac{4k-1-j}{8k-2}}\left[\min(\alpha;\beta)\right]^{-\frac{4k-1+j}{8k-2}} \times\right.$$

$$\left.\times \sup_t\left\|S_j(t)\right\|_{H\to H}\right] + \left[\min(\alpha;\beta)\right]^{-1}\sup_t\left\|S_{4k-1}(t)\right\|_{H\to H} < 1, \qquad (4.1)$$

where

$$d_j = \begin{cases} 2^{\frac{4k-1-j}{4k-1}(2k-1)k}, & if\ j \le 2k-1, \\ 2^{\frac{j}{8k-2}[(4k-1)(3k-j)-(3k-1)]}, & if\ j > 2k-1 \end{cases}$$

be fulfilled.

Then the boundary value problem (1.1)-(1.2) has the unique regular solution for any $f(t)$ *from* $L_2(\mathcal{R}_+;H)$.

In the inequality (4.1) the coefficients for $\sup_t \|S_j(t)\|_{H\to H}$, $j=\overline{1,4k-1}$ are the upper estimations of the norm of the operators $A^j \frac{d^{4k-1-j}}{dt^{4k-1-j}} : \overset{0}{W}{}_2^{4k-1}(\mathcal{R}_+;H) \to L_2(\mathcal{R}_+;H)$ by the main part of the operator -differential equation (1.1) in the space $\overset{0}{W}{}_2^{4k-1}(\mathcal{R}_+;H)$.

Remark 1. Similarly one can investigate the boundary-value problem (1.1)-(1.2) and obtain the corresponding results for them in case, when $\rho(t)$ represents any positive function having a finite number of break points.

Remark 2. The mentioned conditions of the solvability of the boundary value problems (1.1)-(1.2) are unimprovable in terms of operator coefficients.

Remark 3. In the present paper the applied method of the investigation of the boundary-value problems (1.1)-(1.2) are used also the investigations on a semiaxis of the operator differentiable equations of order 4k+1, whose main parts are discontinuous with 2k or 2k+1 boundary-value conditions at zero.

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