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REDUCTION OF ONE HYDROELASTICITY PROBLEM TO THE SOLUTION OF STURM-LIOUVILLE BOUNDARY VALUE PROBLEM

Abstract

In frames of hydraulic approximation equations the problem of pulsating flow of viscous non compressible liquid contained in semi-infinite linear-visco-elastic tube with permeable walls is investigated. In addition the permeability coefficient is a function of a longitudinal coordinate. The proposed hydroelastic model is described by integro-differential equation.

The solution of the given problem is reduced to the solution of Sturm-Liouville singular boundary value problem which for one's turn is led to the solution of Wolterra type integral equations.

Most amount investigations have been devoted to the studying of the wave motion of liquid in deformable tubes. However permeability influence of the walls of a tube is sufficiently little studied and is always valid at most or least.

In this connection the represented paper in frames of hydraulic approximation equations is devoted to the statement and mathematical study of the problem on the propagation of small amplitude in viscous non compressible liquid contained in semi-infinite linear-visco-elastic tube with permeable walls. It is assumed that the tube is rigidly attached to environment and thus the displacement in axial direction is absent. In addition it is adopted that the permeability coefficient is a function of a longitudinal coordinates. The solution of the formulated problem is reduced to the solution of the Sturm-Liouville singular boundary value problem.

The vital function of any organism is characterized by totality of set of processes, and mechanical processes are one of them. It is known that the ideas and representations of continuum mechanics are systematically used as a bases for uncovering regularities of functioning of a system of circulation of blood. By investigating blood's flow in large blood-vessel (in artery and vein) the various hydrodynamic problems, whose solution has both theoretical and practical value arise.

Such problems as for example: the pulsating flow of liquid in deformable tubes (theory of pulse-waves) subject to the permeability of walls of vessel are concerned here. The latter in specific sense takes into account 0 branching of vessels and their contraction that can model unique arterial way from heart to capillary channel [1].

1. Let a straight, thin-shelled and cylindrical tube with the constant radius R be given. Assume that the material of the wall possesses visco-elastic properties described by the operator of complete type $E^{\vee}$, where following [2] $E^{\vee} = \left(1 - \Gamma^*\right)$. Here E is a Young module, Γ^* is a relaxation operator

$$\Gamma^* g = \int_{-\infty}^{t} \Gamma(t-\tau) g(\tau) \alpha\tau \,, \qquad (1.1)$$

and $\Gamma(t-\tau)$ is a difference kernel of relaxation. Further, we shall assume that the walls of the tube are permeable, and the permeability coefficient α is a function of longitudinal coordinate x. The liquid (blood) is taken as viscous, homogeneous, and non-compressible (with the density ρ and kinematic viscosity coefficient χ). In one-dimensional approximation it is supposed that the hydrodynamic pressure (surplus) is $p = p(x,t)$, and the radial displacement of the walls is $w = w(x,t)$. *Then the flow of* liquid may be represented by the longitudinal component $u = u(x,t)$ and the equation of momentum is [3]

$$\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial u}{\partial t} + \frac{8\chi}{R^2}u = 0. \tag{1.2}$$

For real wave processes the general $\mathcal{P}$ is made up of prescribed constant p_0 and hydrodinamic one. According to this statement $\mathcal{P}(x,t) = p(x,t) + p_0$.

Taking into outflow (filtration) of liquid by wall is realized at the expense of radical components, we write the equation of conservation of mass in the form of [4]

$$\frac{\partial u}{\partial x} + \frac{2}{R}\left\{\alpha(x)[\mathcal{P} - p_c] + \frac{\partial w}{\partial t}\right\} = 0, \tag{1.3}$$

where $p_c < p_0$ is a prescribed constant pressure in environment. Combining the equations (1.2) and (1.3) we can write

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} + \frac{16\chi}{R^3}\left\{\alpha(x)[\mathcal{P} - p_c] + \frac{\partial w}{\partial t}\right\} - \frac{2}{R}\alpha(x)\frac{\partial p}{\partial t} - \frac{2}{R}\frac{\partial^2 w}{\partial t^2} = 0. \tag{1.4}$$

Ignoring the dynamic effects in a domain of cross-section of tubes, we complete the equation (1.4) by means of the dependence [5]

$$\mathcal{P} - p_c = \frac{h}{R^2}E^{\vee} w. \tag{1.5}$$

By virtue of the obvious equalities

$$\frac{\partial^2 p}{\partial x^2} = \frac{h}{R^2}E^{\vee}\frac{\partial^2 w}{\partial x^2} = \frac{h}{R^2}E\frac{\partial^2 w}{\partial x^2} - \frac{h}{R^2}E\int_{-\infty}^{t}\Gamma(t-\tau)\frac{\partial^2 w}{\partial x^2}d\tau,$$

$$\frac{\partial p}{\partial t} = \frac{h}{R^2}\frac{\partial}{\partial t}E^{\vee} w = \frac{h}{R^2}E\frac{\partial w}{\partial t} - \frac{h}{R^2}E\frac{\partial}{\partial t}\int_{-\infty}^{t}\Gamma(t-\tau)w\,d\tau$$

and the formula (1.1), the equation (1.4) is reduced to an integro-differential equation with variable coefficients with respect to the deflection function w

$$\frac{\partial^2 w}{\partial x^2} \int_{-\infty}^{t}\Gamma(t-\tau)\frac{\partial^2 w}{\partial x^2}d\tau - \frac{16\chi\rho}{R^3}\alpha(x)w +$$

$$+\frac{16\chi\rho}{R^3}\alpha(x)\int_{-\infty}^{t}\Gamma(t-\tau)w\,d\tau - \frac{8\chi}{R^2C_0^2}\frac{\partial w}{dt} - \frac{2\rho}{R}\alpha(x)\frac{\partial w}{\partial t} +$$

$$+\frac{2\rho}{R}\alpha(x)\frac{\partial}{\partial t}\int_{-\infty}^{t}\Gamma(t-\tau)w\omega d\tau = \frac{1}{C_0^2}\frac{\partial^2 w}{\partial t^2} = 0 \quad ,$$

in which $C_0^2 = Eh/2\rho R$ is a velocity of propagation of Kortveg-Rezal's waves. Now the considered system is completely determined excluding the boundary conditions which are necessary for the solution of concrete problems.

The proposed here hydroelastic model may be reduced in some particular cases to the known in references one-dimensional models. For the case of ideal liquid and elastic tube we obtain the result of work [4]. If we reject the terms considering the permeability, then we arrive at the formulation of a problem on determination of pulse-waves in a visco-elastic tube. Disregarding further the viscosity of liquid and material of a wall we obtain a model for which the velocity of propagation of waves is determined by Kortveg-Rezal classical formula.

2. To describe the complicated impulses that are characteristic for a system of circulation of blood, the harmonic analysis is used, i.e. the impulses of a complicated form are distributed to the sinusoidal components generating a Fourier series. By virtue of linearity and homogeneity of defining equations, the advancing of every harmonics with the frequency $n\omega$ is traced, where n is a natural number, and to determine the form of impulse at any point of a system, the components are summed according to the given point. From the noted we can conclude the following: the principal value has a consideration of purely sinusoidal oscillation with one given frequency ω. In this case using the method of separation of variables, we represent the solution of the equation (1.6) in the following form

$$w(x,t) = y(x)\exp(i\omega t) \quad \left(i = \sqrt{-1}\right). \tag{2.1}$$

Here $y(x)$ is a desired function of a coordinate. As a result of substitution (2.1) in (1.6) and introducing the following designation

$$R^*(t) = \int_{-\infty}^{t} \Gamma(t-\tau)e^{i\omega\tau}d\tau$$

we obtain

$$\left\{e^{i\omega t} - R^*(t)\right\}y'' + \left\{\frac{16\chi\rho}{R^3}\alpha(x)e^{i\omega t} + \frac{16\chi\rho}{R^3}\alpha(x)R^*(t) - \right.$$
$$\left. - \frac{8\chi}{R^2C_0^2}i\omega e^{i\omega t} - \frac{2}{R}i\omega\alpha(x)e^{i\omega t} + \frac{2\rho}{R}\alpha(x)\frac{\partial R^*(t)}{\partial t} + \frac{\omega^2}{C_0^2}e^{i\omega t}\right\}y = 0. \tag{2.2}$$

The prime means a differentiation with respect to the coordinate x. Further using the replacement $t - \tau = \theta$, it is easy to show that $R^*(t)$ is determined by the formula

$$R^*(t) = \zeta e^{i\omega t}, \tag{2.3}$$

in which

$$\zeta = \int_0^{\infty} \Gamma(\theta)e^{-i\omega\theta}d\theta. \tag{2.4}$$

From the expression (2.4) it follows that it is valid for any known kinds of difference reelection kernels [2]. Taking into account now the equality (2.3) in the equation (2.2) for the function $y(x)$ we can write

$$y'' + \varphi(x)y = 0. \tag{2.5}$$

It addition the function $\varphi(x)$ has the form

$$\varphi(x) = \eta_0(x) - i\eta_1(x),$$

where

$$\eta_0(x) = \frac{\dfrac{\omega^2}{C_0^2} - \dfrac{16\chi\rho}{R^3}(1-\zeta)\alpha(x)}{1-\zeta},$$

$$\eta_1(x) = \frac{\dfrac{8\chi}{R^2C_0^2}\omega + \dfrac{2\rho}{R}\omega\alpha(x) - \dfrac{2\rho}{R}\omega\zeta\alpha(x)}{1-\zeta}.$$

It is easy to conclude that the velocity of wave propagation $C = \omega/\delta_0$ and the damping δ_1 for

$$\delta_0 = \sqrt{\frac{m+\eta_0}{2}}, \quad \delta_1 = \sqrt{\frac{m-\eta_0}{2}} \qquad \left(m = \sqrt{\eta_0^2 + \eta_1^2}\right)$$

depend on the coordinate x and are local wave characteristics.

For the consequent reasoning we'll accept that on infinity the permeability coefficient is constant. This statement allows the existence of the next limit

$$\lim_{x\to\infty}\alpha(x)=\alpha_0 .$$

Without loss of generality for the simplicity of the notation later on we'll assume $\alpha_0 = 0$. Then

$$\lim_{x\to\infty}\varphi(x)=\frac{\omega^2}{C_0^2(1-\zeta)}-i\frac{8\chi\omega}{R^3C_0^2(1-\zeta)}=\lambda^2 .$$

Denoting

$$q(x)=1-\frac{\varphi(x)}{\lambda^2}$$

we reduce the equation (2.5) to the form

$$y''+\lambda^2 y=\lambda^2 q(x)y .$$

Later on we assume that $Jm\lambda<0$, and the potential $q(x)$ satisfies the condition

$$\int_0^\infty |q(x)|dx<+\infty . \tag{2.6}$$

Thus the hydroelasticity problem is reduced to the solution of the Sturm-Liouville singular boundary value problem

$$y''+\lambda^2 y=\lambda^2 q(x)y , \tag{2.7}$$

$$y(0)=y_0 , \tag{2.8}$$

$$\lim_{x\to\infty}y(x)=0 . \tag{2.9}$$

The question of physical interpretation of the quantity y_0 will discussed below.

3. It is appropriate to reduce the solution of the differential equation (2.7) to the solution of an integral equation. The homogeneous equation (2.7)

$$y''+\lambda^2 y=0 . \tag{3.1}$$

has a fundamental system of solutions

$$y_1=e^{-i\lambda x},\quad y_2=e^{i\lambda x} .$$

Considering (3.1) as a non-homogeneous equation with the known right hand part $\lambda^2 q(x)y$ and applying the method of variation of arbitrary constants, the solution of the problem (2.7)-(2.8) is reduced to the solution of the equivalent integral equation

$$y(x,-\lambda)=Ce^{-i\lambda x}+\lambda\int_x^\infty \sin\lambda(\Lambda-x)q(\Lambda)y(\Lambda,-\lambda)d\Lambda . \tag{3.2}$$

Here the constant C is defined as

$$C=\frac{y_0}{f(0,-\lambda)}\quad\text{and}\quad y=y_0\frac{f(x,-\lambda)}{f(0,-\lambda)},$$

where the function $f(x,-\lambda)$ is found from the solution of the following integral equation

$$f(x,-\lambda)=e^{-i\lambda x}+\lambda\int_x^\infty \sin\lambda(\Lambda-x)q(\Lambda)\cdot f(\Lambda,-\lambda)d\Lambda , \tag{3.3}$$

which is a Volterra type equation and so it may be solved by the successive approximations method

$$f(x,-\lambda)=\sum_{k=0}^{\infty}\lambda^k f_k(x,-\lambda) \tag{3.4}$$

in addition

$$f_0(x,-\lambda)=e^{-i\lambda x},...f_k(x,-\lambda)=\int_x^\infty \sin\lambda(\Lambda-x)q(\Lambda)f_{k-1}(\Lambda,-\lambda)d\Lambda .$$

By virtue of the accepted inequality $Jm\lambda<0$, we write

$$|f_0(x,-\lambda)|=\exp(Jm\lambda x)\le 1 .$$

By the induction method we prove the estimation

$$|f_n(x,-\lambda)-f_{n-1}(x,-\lambda)|\le\frac{B_\lambda^n(x)}{n!}, \tag{3.5}$$

where

$$B_\lambda(x)=|\lambda|\int_x^\infty|q(\Lambda)|d\Lambda .$$

Note that

$$B_\lambda(x)=|\lambda|\int_x^\infty|q(\Lambda)|d\Lambda\le|\lambda|\int_0^\infty|q(\Lambda)|d\Lambda=B_\lambda(0)<+\infty$$

and consequently

$$|f_n(x,-\lambda)-f_{n-1}(x,-\lambda)|\le\frac{B_\lambda^n(0)}{n!}.$$

For $n=1$, we have

$$|f_1(x,-\lambda)-f_0(x,-\lambda)|=\left|\lambda\int_x^\infty \sin\lambda(\Lambda-x)q(\Lambda)e^{-i\lambda\Lambda}d\Lambda\right|\le$$

$$\le|\lambda|e^{Jm\lambda x}\int_x^\infty|q(\Lambda)|d\Lambda\le B_\lambda(x).$$

Let the estimation (3.5) be valid for $n=m$. Let's prove its correctness for $n=m+1$

$$|f_{m+1}(x,-\lambda)-f_m(x,-\lambda)|=\left|\lambda\int_x^\infty \sin\lambda(\Lambda-x)q(\Lambda)\{f_m(\Lambda,-\lambda)-f_{m-1}(\Lambda,-\lambda)\}d\Lambda\right|\le$$

$$\le|\lambda|\int_x^\infty|\sin\lambda(\Lambda-x)|\,|f_m(\Lambda,-\lambda)-f_{m-1}(\Lambda,-\lambda)||q(\Lambda)|\le$$

$$\le\frac{|\lambda|}{m!}\int_x^\infty B_\lambda^m(\Lambda)|q(\Lambda)|d\Lambda=\frac{B_\lambda^{m+1}(x)}{m+1!}.$$

From the inequality (3.5) it follows that the series (3.4) is majored in the interval $[0,+\infty)$ convergent by the positive numerical series

$$\sum_{n=0}^\infty\frac{B_\lambda^n(0)}{n!}$$

and therefore by Weierstrass test it converges uniformly by $x\in[0,+\infty)$ and its sum is a unique solution of the equation (3.3). By immediate testing it is easy to establish that this solution is a solution of the input equation (2.7) too.

Fulfilling the corresponding calculations and taking into account the equality (3.2) we can write

$$w=y_0\frac{f(x,-\lambda)}{f(0,-\lambda)}\exp(i\omega t), \tag{3.6}$$

$$\mathcal{P} - p_c = \frac{hE}{R^2} y_0 \frac{f(x,-\lambda)}{f(0,-\lambda)}(1-\zeta)\exp(i\omega t). \qquad (3.7)$$

From the structure of the series (3.4) it follows that the series obtained by the term-by-term differentiation with respect to x, also uniformly converges. Consequently in particular the following equality is true

$$f'(x,-\lambda) = \sum_{k=0}^{\infty} \lambda^k f_k'(x,-\lambda),$$

where

$$f_0'(x,-\lambda) = -i\lambda e^{-i\lambda x}, \ldots, f_k'(x,-\lambda) = -\lambda \int_x^{\infty} \cos\lambda(\Lambda,-x) q(\Lambda) f_k(\Lambda,-\lambda) d\Lambda.$$

Now using the equation (1.2) we can get the expression for deformation of the function $u(x,t)$

$$u(x,t) = \frac{ihE}{\omega\rho R^2} y_0 \frac{f'(x,-\lambda)}{f(0,-\lambda)}(1-\zeta)\exp(i\omega t). \qquad (3.8)$$

Note that the real parts of the constructed solution (3.6)-(3.8) represent physical quantity. By the same taken we complete the considerable part of analysis. The further part is connected with a concrete task of the function $\alpha(x)$ and use of IBM.

In conclusion we determine the quantity y_0. We can take different boundary conditions for $x = 0$. The typical case is a situation for which in initial section the pressure, is changed by the low $p^{\vee}\exp(i\omega t)$ where $p^{\vee}$ is a quantity determined by experiment. Then from the equation (3.7) we have

$$y_0 = \frac{R^2}{hE(1-\zeta)} p^{\vee}.$$

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