MECHANICS

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LONGITUDINAL OSCILLATIONS OF THE DAMAGED NONLINEAR-ELASTIC BAR

Abstract

The problem about the longitudinal forced oscillations of the nonlinear-elastic damaged bar of finite length has been solved. One end-wall of the bar is subjected to the oscillations of the given amplitude and frequency and other and-wall is free from efforts. The nonlinear function of the instant deformation containing small parameter on nonlinear term is taken by Cauderery. The deformation low in the presence of damage and healing of defects is taken according to Souvorov. The solution is sought by the method of small parameter. Zero and first approximation expressed by the number of cycles of load have been written. The dependence of critical number of cycles before destruction from the nominal tension has been obtained from the destruction criterion for the zero approximation. The corresponding curves of this dependence. Veler curves - for the different values of parameter of damage have been constructed.

Longitudinal oscillations of the damaged nonlinear elastic bar have been treated in [1]. In the oscillation process the bar is subjected to the multiple cyclic repeated loading and emptying. The alternate states of tension and compression of the bar correspond to this. As known [2] this process is accompanied by the process of the formation and accumulation of damages with their possible partial healing in fluence of this damage process to the longitudinal oscillations of damaged nonlinear-elastic bar is studied and the attempt to define the long strength estimation of working resource is made.

As the base model of the damaging one dimensional body we'll take the one given in [3].

The deformation equation has the form:

$$\varphi(\varepsilon) = \sigma + L^{*}\sigma + M^{*}\sigma \qquad (1)$$

and the criterion of destruction:

$$\sigma + M^{*}\sigma = \sigma_{*}. \qquad (2)$$

Where ε and σ are the longitudinal deformation and tension, σ is the limit of strength of t he defectless material $\varphi(\varepsilon)$ is the function of the instantaneous deformation for according to [1] we take

$$\varphi(\varepsilon) = E\left(1 + \lambda a \varepsilon^{2}\right)\varepsilon. \qquad (3)$$

$$a = \frac{2}{3}\frac{k}{3k + G}\left(\frac{E}{G}\right)^{2},$$

λ is the dimensionless small parameter, E, G and K are Young's modulus of shift and the volumetric compression, correspondingly.

In its neighborhood L^{*} is the continuous operator of the viscous flow which is not accounted in our case, and M^{*} is the discrete operator of the damage having the form:

$$M^{*}\sigma=\sum_{k=1}^{n}f_k\int_{t_k^-}^{t_k^+}M(t_k^+-\tau)\sigma(\tau)d(\tau)+\int_{t_{n+1}^-}^{t}M(t-\tau)\sigma(\tau)d(\tau). \tag{4}$$

Here $M(t-\tau)$ is the kernel of the damage operator, (t_k^-,t_k^+) are intervals of accumulation of damages determined in the code of solution of problem, f_k is the coefficient of heading of defects, generally speaking depending on the level of the accumulated damage in the corresponding stage of loading. It is natural that $0\le f_k\le 1$ holds, moreover $f_k=0$ corresponds to the full healing of defects, and $f_k=1$ corresponds to absence of this phenomenon.

The motion equation has the form:

$$\rho\ddot{u}=\sigma', \tag{5}$$

where the point above the function means the differentiation in time, and the differentiation prime with respect to the axis coordinate x.

Allowing for (1) in (5) subject to the relation $\varepsilon=u'$, we'll obtain:

$$\frac{1}{\rho}\varphi'_{\varepsilon}u''(x,t)=\ddot{u}(x,t)+\sum_{k=1}^{n}f_k\int_{t_k^-}^{t_k^+}M(t_k^+-\tau)\ddot{u}(x,\tau)d\tau+\int_{t_{n+1}^-}^{t}M(t-\tau)\ddot{u}(x,\tau)d\tau, \tag{6}$$

where

$$\frac{1}{\rho}\varphi'_{\varepsilon}=c_0^2\left(1+3\lambda a u'^2(x,t)\right);\ c_0^2=E/\rho. \tag{7}$$

Let the bar of the length l is subjected to oscillations with given amplitude a_0 and frequency ω in one end-wall, and the another end-wall is free from effort. The corresponding form of the boundary conditions will be

$$u(0,t)=a_0\cos\omega t;\ u'(l,t)=0. \tag{8}$$

We search the solution (6) in the form of series in powers of the small parameter λ:

$$u(x,t)=\sum_{m=0}^{\infty}\lambda^m u_m(x,t). \tag{9}$$

Putting (9) in (6) and (8) we obtain the recurrent system of mathematical problems to the definition of functions $u(x,t)$. Let's cite similar ones for zero and first approximations

$$c_0^2u_0''(x,t)=\ddot{u}_0(x,t)+\sum_{k=1}^{n}f_k\int_{t_k}^{t_k^+}M(t_k^+-\tau)\ddot{u}_0(x,\tau)d\tau+\int_{t_{n+1}}^{t}M(t-\tau)\ddot{u}_0(x,\tau)d\tau, \tag{10}$$

$$u_0(0,t)=a_0\cos\omega t;\ u_0'(l,t)=0, \tag{11}$$

$$\begin{aligned}c_0^2u_1''(x,t)=\ddot{u}_1(x,t)+\sum_{k=1}^{n}f_k\int_{t_k^-}^{t_k^+}M(t_k^+-\tau)\ddot{u}_1(x,\tau)d\tau+\\+\int_{t_{n+1}^-}^{t}M(t-\tau)\ddot{u}_1(x,\tau)d\tau-3ac_0^2u'^2_0(x,t)u_0''(x,t),\end{aligned} \tag{12}$$

$$u_1(0,t)=0;\ \ u_1'(l,t)=0. \tag{13}$$

Analytically, the solution of the equations (10) and (12) are problematic, since besides the functions $u_0(x,t)$ and $u_1(x,t)$ the values t_k^-, t_k^+ that naturally depend on the solution, are also unknown. Therefore here we prefer the numerical step-type method of solution and to define the quantities t_k^-, t_k^+ from the condition of the active loading and emptying. But it is possible to obtain some average approximate estimations to the solutions admissible from the engineering point of view and permitting to judge about the qualitative side of interdependency of the characteristic parameters of a process, in particular to obtain the analytical dependence of the limit cycle of load, basing on the destruction criterion.

For zero approximation the solution (10) search the solution (10) in the form

$$u_0(0,t) = U_0(x)\cos\omega t \,. \tag{14}$$

For the conditions of the effect of damage accumulation process we admit the followings:

$$u_0(0,t) > 0; \quad \dot{u}_0(0,t) > 0 \,. \tag{15}$$

Subject to the representation (14) they permit to choose the areas (t_k^-, t_k^+). Then putting (14) to the motion equation (10) and taking the results $M(t-\tau) = \beta = const$ due to simplicity and visibility, we will obtain:

$$c_0^2 U_0'' \cos\omega t + \left[\omega^2 \cos\omega t + \beta\omega\left(1 + \sin\omega t + \sum_{k=1}^{n} f_k\right)\right]U_0 = 0 \,. \tag{16}$$

Multiplying (16) by $\cos\omega t$ and integrating by the last $(n+1)$-th interval of the active loading from $(1{,}5 + 2n)\pi/\omega$ to $2\pi(n+1)/\omega$ we'll get

$$U_0''(x) + \left(\frac{\omega\chi}{c_0}\right)^2 U_0 = 0 \,, \tag{17}$$

where the notation

$$\chi^2 = 1 + \frac{4\beta}{\pi\omega}\left(\frac{1}{2} + \sum_{k=1}^{n} f_k\right) \tag{18}$$

was taken.

The solution (17) has the form

$$U_0(x) = b_0 \cos\left(\frac{\omega\chi}{c_0}x + \alpha\right). \tag{19}$$

From the boundary conditions (11) we have:

$$b_0 = \frac{a_0}{\cos\dfrac{\omega\chi l}{c_0}}; \quad \alpha = -\frac{\omega\chi}{c_0}l \,. \tag{20}$$

Then the final solution of the problem (10), (11) will be

$$u_0 = a_0 \frac{\cos\left[\dfrac{\omega\chi}{c_0}(l-x)\right]}{\cos\dfrac{\omega\chi}{c_0}l}\cos\omega t \,. \tag{21}$$

Let's pass to the determination of first approximation. The solution (12) is representable in the form:

$$u_1(x,t) = \bar{u}_1(x,t) + \tilde{u}_1(x,t), \tag{22}$$

where $\bar{u}_1(x,t)$ is a general solution of the homogeneous part of the equation (12) that has the form analogous to (21):

$$\bar{u}_1(x,t)=b_1\cos\left(\frac{\omega\chi}{c_0}x+\alpha_1\right)\cos\omega t\ , \tag{23}$$

where b_1 and α_1 are constants to be defined from the boundary conditions (13) and $\tilde{u}_1(x,t)$ is the particular solution of the equation (12), for which subject to the formula (21) we have the following equation

$$\begin{gathered} c_0^2\tilde{u}_1''(x,t)=\ddot{\tilde{u}}_1(x,t)+\beta\sum_{k=1}^{n} f_k\left[\dot{\tilde{u}}_1\left(x,t_k^+\right)-\dot{\tilde{u}}_1\left(x,t_k^-\right)\right]+ \\ +\beta\left[\dot{\tilde{u}}_1(x,t)-\dot{\tilde{u}}_1\left(x,t_{n+1}^-\right)\right]=\tilde{F}(x)(\cos 3\omega t+3\cos\omega t)\ , \end{gathered} \tag{24}$$

where

$$\tilde{F}=F\left\{\cos\left[\frac{\omega\chi}{c_0}(l-x)\right]-\cos\left[\frac{3\omega\chi}{c_0}(l-x)\right]\right\}, \tag{25}$$

$$F=\frac{aa_0^3\chi^4\omega^4}{2c_0^2\cos^3\dfrac{\omega\chi l}{c_0}}. \tag{26}$$

We'll search the solution (24) in the form:

$$\tilde{u}_1(x,t)=U_1(x)\cos\omega t+U_2(x)\cos 3\omega t\ . \tag{27}$$

Allowing for the representation (27) in the equation (25) and then multiplying by turns the obtained one by $\cos\omega t$ and $\cos 3\omega t$ and integrating them correspondingly from $(1{,}5+2n)\pi/\omega$ to $2\pi(n+1)/\omega$ and from $(1{,}5+2n)\pi/3\omega$ to $2\pi(n+1)/3\omega$ representing the last intervals of the active loading we'll obtain the following system of differential equations with respect to the functions $U_1(x)$ and $U_2(x)$

$$\begin{cases} U_1''+a_{11}U_1+a_{12}U_2=F_1\left\{\cos\left[\dfrac{\omega\chi}{c_0}(l-x)\right]-\cos\left[\dfrac{3\omega\chi}{c_0}(l-x)\right]\right\} \\ b_1U_1''+b_2U_2''+b_{11}U_1+b_{12}U_2=Q_1\left\{\cos\left[\dfrac{\omega\chi}{c_0}(l-x)\right]-\cos\left[\dfrac{3\omega\chi}{c_0}(l-x)\right]\right\} \end{cases} \tag{28}$$

Here $F_1=3F/c_0^2$; $Q_1=F/4c_0^2$

$$a_{11}=\left(\frac{\omega\chi}{c_0}\right)^2;\ a_{12}=3\left(\frac{\omega}{c_0}\right)^2\left(\chi^2-1\right);\ b_1=\frac{\sqrt{3}\eta_{2,n}-2\eta_{1,n}}{8};\ b_2=\frac{\pi}{12};$$

$$b_{11}=\left(\frac{\omega}{c_0}\right)^2\left\{\frac{\sqrt{3}\eta_{2,n}-2\eta_{1,n}}{8}+\frac{\beta}{\omega}\left(\frac{2\eta_{2,n}-\sqrt{3}\eta_{1,n}}{8}+\frac{1}{6}+\frac{\left(\chi^2-1\right)\pi\omega}{12\beta}\right)\right\};$$

$$b_{12}=\frac{3\pi}{4}\left(\frac{\omega}{c_0}\right)^2\left\{1+3\left(\chi^2-1\right)\right\}$$

in its turn

$$\eta_{1,n}=\begin{cases}0; & for\ \ n=3m-2\\ -\dfrac{\sqrt{3}}{2}; & for\ \ n=3m-1\\ \dfrac{\sqrt{3}}{2}; & for\ \ n=3m\end{cases} \tag{29}$$

$$\eta_{2,n}=\begin{cases}-1; & for\ \ n=3m-2\\ \dfrac{1}{2}; & for\ \ n=3m-1\ \ and\ \ n=3m,\ \ m=1,2,3,...,\end{cases}$$

Let's represent the solution of the equation (28) in the form

$$U_k(x)=c_k\cos\left[\frac{\omega\chi}{c_0}(l-x)\right]+d_k\cos\left[\frac{3\omega\chi}{c_0}(l-x)\right]. \tag{30}$$

Allowing for these representations in (28) and comparing coefficients of $\cos\left[\frac{\omega\chi}{c_0}(l-x)\right]$ and $\cos\left[\frac{3\omega\chi}{c_0}(l-x)\right]$, we'll obtain the system of linear algebraic equations with respect to coefficients c_k and d_k for which we'll find

$$c_1=\left\{F_1\left[b_{12}-\left(\frac{\omega\chi}{c_0}\right)^2 b_2\right]-a_{12}Q_1\right\}\Big/\Delta^{(1)};$$

$$c_2=\left\{Q_1\left[a_{11}-\left(\frac{\omega\chi}{c_0}\right)^2\right]-F_1\left[b_{11}-\left(\frac{3\omega\chi}{c_0}\right)^2 b_1\right]\right\}\Big/\Delta^{(1)};$$

$$d_1=\left\{Q_1a_{12}-F_1\left[b_{12}-\left(\frac{3\omega\chi}{c_0}\right)^2 b_2\right]\right\}\Big/\Delta^{(2)}; \tag{31}$$

$$d_2=\left\{F_1\left[b_{11}-\left(\frac{3\omega\chi}{c_0}\right)^2 b_1\right]-Q_1\left[a_{11}-\left(\frac{3\omega\chi}{c_0}\right)^2\right]\right\}\Big/\Delta^{(2)}.$$

$$\Delta^{(1)}=\left[a_{11}-\left(\frac{\omega\chi}{c_0}\right)^2\right]\left[b_{12}-\left(\frac{\omega\chi}{c_0}\right)^2 b_2\right]-b_2\left[b_{11}-\left(\frac{\omega\chi}{c_0}\right)^2 b_1\right],$$

$$\Delta^{(1)}=\left[a_{11}-\left(\frac{3\omega\chi}{c_0}\right)^2\right]\left[b_{12}-\left(\frac{3\omega\chi}{c_0}\right)^2 b_2\right]-a_{12}\left[b_{11}-\left(\frac{3\omega\chi}{c_0}\right)^2 b_1\right].$$

Allowing for (31) in (30) and the latter in (27) subject to (23) from (22) we will obtain the representation for the function of the first approximation $u_1(x,t)$. It will some amend to the estimation of durability derived by the zero approximation function.

Let's bring this estimation founding on the destruction criterion (2). Allowing for (21) in the expansion of Hook law and substituting the obtained expression of zero approximation of tension

$$\sigma_0(x,t) = \frac{E\omega a_0 \chi \sin\left[\frac{\omega\chi(l-x)}{c_0}\right]}{\cos\frac{\omega\chi l}{c_0}}\cos\omega t \tag{32}$$

to the destruction criterion (2) subject to its maximal value, multiplying the obtained to $\cos\omega t$ and integrating the obtained expression by the last $(n+1)$-th interval of the active loading we'll get:

$$\frac{\pi E a_0 \omega \chi^3}{4c_0 \cos\frac{\omega\chi l}{c_0}} = \sigma_*. \tag{33}$$

From (33) allowing for (18) one can obtain the formula for the number of cycles of loading up to destruction. Let's demonstrate the above said for the partial case of small frequencies ω assuming that $\cos\frac{\omega\chi l}{c_0} \approx 1$. Let's introduce the following notations for the nominal tension

$$\sigma_H = \frac{\pi E a_0 \omega}{4c_0}. \tag{34}$$

Then from (33) we obtain

$$\chi = \chi_{kp} = \left(\frac{\sigma_*}{\sigma_H}\right)^{1/3} \tag{35}$$

or on the base of (18)

$$1 + \frac{4\beta}{\pi\omega}\left(\frac{1}{2} + \sum_{k=1}^{n_{kp}} f_k\right) = \left(\frac{\sigma_*}{\sigma_H}\right)^{2/3} \tag{36}$$

in the absence of damage, $\beta = 0$ from (36) we have the classical durability (strength) condition

$$\sigma_H^* = \sigma_*. \tag{37}$$

In the presence of damage but in the hall healing of defects $f_k = 0$, we'll obtain the analogue of the durability criterion (37) with correction on the volume of damage accumulated in the last interval of the active loading

$$\sigma_H^{**} = \left(1 + \frac{2\beta}{\pi\omega}\right)^{-2/3}. \tag{38}$$

For the case of absence of the phenomena of heal of defects, $f_k = 0$, from (35) we'll obtain the following expression for the limiting value of loading cycles up to destruction

$$n_{kp} = \frac{\pi\omega}{4\beta}\left[\left(\frac{\sigma_*}{\sigma_H}\right)^{2/3} - 1\right] - \frac{1}{2}. \tag{39}$$

In (39) we have the entire part of its right hand side. In addition it is necessary that the condition $\sigma_H^* < \sigma_H^{**}$ be fulfilled. Otherwise the destruction will arise already in the first cycle of loading

The qualitative curves of the fatigue destruction (Veler curves) depending on the parameter of damage β are reduced in fig.1

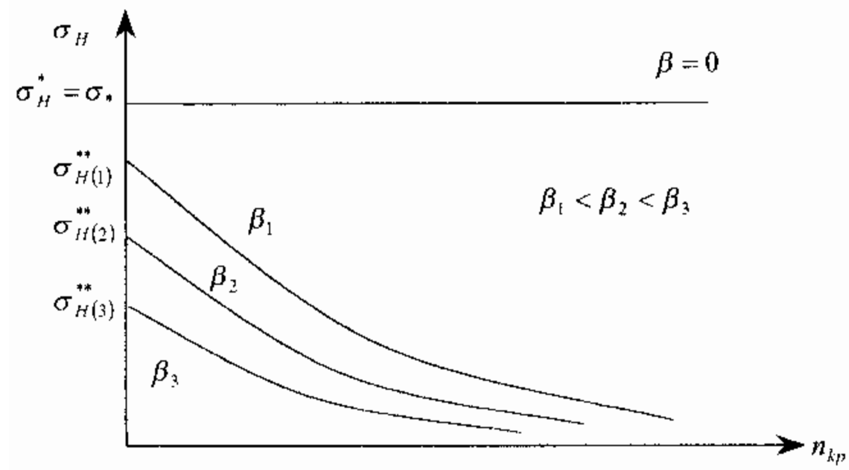

Fig.1

As follows from the cite of graphics the presence of the damage reduces to the natural decrease of fatigue durability and the corresponding conditional limit of the endurance.

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