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SOLUTION OF A CLASS OF INVERSE PROBLEMS FOR THE DIRAC OPERATOR

Abstract

A complete solution of a class of inverse problems in spectral analysis for the Dirac operator is presented. That is,

- *Necessary and sufficient conditions are found for two sequences of real numbers to be spectra of boundary-value problems generated on a finite interval by a Dirac equation and certain non-separated self-conjugate boundary conditions. A procedure for recovery of all such problems is given.*
- *Additional spectral characteristics are found which together with the spectra uniquely define the Dirac operator.*

Introduction.

We denote by $(Q(x),\omega,\alpha)$ the self-conjugate boundary-value problem generated on the interval $[0,\pi]$ by canonical Dirac equation

$$By'(x)+Q(x)y(x)=\lambda y(x) \tag{1}$$

and boundary conditions

$$A_0 y(0)+A_1 y(\pi)=0,$$

where $B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, $Q(x)=\begin{pmatrix}p(x) & q(x)\\ q(x) & -p(x)\end{pmatrix}$, $A_0=\begin{pmatrix}1 & 0\\ 0 & \overline{\omega}\end{pmatrix}$, $A_1=\begin{pmatrix}\omega & 0\\ \alpha & 1\end{pmatrix}$, $y(x)=\begin{pmatrix}y_1(x)\\ y_2(x)\end{pmatrix}$,

$p(x)$, $q(x)$ are real functions in $L_2[0,\pi]$, ω and α are arbitrary parameters, complex and real respectively.

Direct and inverse problems for the Dirac operator with separated boundary conditions ($\omega=0$) are studied enough (see [1]). The case of non-separated boundary conditions ($\omega\neq 0$) makes essential alterations in the analysis of inverse problems. Such inverse problems have not been studied before excluding the case of periodic $(\omega=-1,\alpha=0)$ and antiperiodic $(\omega=1,\alpha=0)$ boundary conditions [2].

The following inverse spectral problem is solved in the present paper: to find necessary and sufficient conditions which should be satisfied by two sequences of real numbers in order to the sequences be the spectrums of boundary-value problems (Q,ω,α_1), (Q,ω,α_2) when $|\omega|=1$, $\alpha_1\neq\alpha_2$, as well as a procedure for recovery of all such problems (the case $|\omega|\neq 1$ has its own peculiarities and will be studied in the other paper). Besides in the present paper the uniqueness theorem is proved for the solution of inverse problem. Similar problem for the Sturm-Liouville operator was studied in [3].

On a structure of the paper. In § 1 multiplicity criterion for the eigenvalues of the problem (Q,ω,α) is given, asymptotic formulae for the spectrum of problems $(Q,-1,\alpha)$, $(Q,1,\alpha)$, (Q,ω,α) at $|\omega|=1$, $\omega\neq\pm 1$ are derived and the theorem on mutual arrangement of eigenvalues for problems (Q,ω,α_1), (Q,ω,α_2) at $\alpha_1<\alpha_2$ is proved. The statement on representation of a class of entire functions with given zeroes is proved in § 2. Lastly in § 3 above-stated inverse problem is solved (where cases $\omega=\pm 1$ and $|\omega|=1$ ($\omega\neq\pm 1$) are considered apart).

Everywhere further we will assume that k takes integer values and j - the values 1 and 2.

§ 1. Some properties of the eigenvalues of the boundary-value problem (Q,ω,α).

General solution of the equation (1) has a form $y(\lambda,x)=e(\lambda,x)M$, where $e(\lambda,x)=\begin{pmatrix} c_1(\lambda,x) & s_1(\lambda,x) \\ c_2(\lambda,x) & s_2(\lambda,x)\end{pmatrix}$, $M=\begin{pmatrix} M_1 \\ M_2\end{pmatrix}$, M_1 and M_2 are arbitrary constants, $c(\lambda,x)=\begin{pmatrix} c_1(\lambda,x) \\ c_2(\lambda,x)\end{pmatrix}$ and $s(\lambda,x)=\begin{pmatrix} s_1(\lambda,x) \\ s_2(\lambda,x)\end{pmatrix}$ are solutions of the equation (1) satisfying initial conditions $c(\lambda,0)=\begin{pmatrix} 1 \\ 0\end{pmatrix}$, $s(\lambda,0)=\begin{pmatrix} 0 \\ 1\end{pmatrix}$. That's why $\Delta(\lambda)=\det(A_0+A_1e(\lambda,\pi))$ will be characteristic function of a problem (Q,ω,α). Expanding this determinant and taking into consideration the following identify

$$\det e(\lambda,x)=c_1(\lambda,x)s_2(\lambda,x)-c_2(\lambda,x)s_1(\lambda,x)\equiv 1, \tag{2}$$

we find

$$\Delta(\lambda)=2\operatorname{Re}\omega+|\omega|^2c_1(\lambda,\pi)+s_2(\lambda,\pi)+\alpha s_1(\lambda,\pi). \tag{3}$$

Boundary-value problem (Q,ω,α) is self-conjugate, consequently its eigenvalues are real, i.e. zeroes of function $\Delta(\lambda)$ are real. Eigenvalues of this problem can be double.

Theorem 1. *The number λ_0 is double eigenvalue of boundary-value problem (Q,ω,α) if and only if the number ω is real, nonzero and the following equalities hold:*

$$s_1(\lambda_0,\pi)=\alpha c_1(\lambda_0,\pi)+c_2(\lambda_0,\pi)=0.$$

We prove the theorem in similar fashion as multiplicity criterion for eigenvalues of periodic problem ([4, p. 256]).

Theorem 2. *The eigenvalues* $\ldots\le a_k^-\le a_k^+\le a_{k+1}^-\le a_{k+1}^+\le\ldots$, $\ldots\le b_k^-\le b_k^+\le b_{k+1}^-\le b_{k+1}^+\le\ldots$, $\ldots\le c_k^-\le c_k^+\le c_{k+1}^-\le c_{k+1}^+\le\ldots$ *of boundary-value problems* $(Q,-1,\alpha)$, $(Q,1,\alpha)$, (Q,ω,α) ($|\omega|=1$, $\omega\neq\pm 1$) *respectively satisfy at* $|k|\to\infty$ *the following asymptotic formulae*

$$a_k^\pm=2k-a^\pm+\varepsilon_k^\pm, \tag{4}$$

$$b_k^\pm=2k+1-a^\pm+\beta_k^\pm, \tag{5}$$

$$c_k^\pm=2k-c^\pm+\gamma_k^\pm, \tag{6}$$

where $a^\pm=\dfrac{1}{\pi}(1\mp\operatorname{sgn}\alpha)\arctan\dfrac{\alpha}{2}$, $c^\pm=\dfrac{2}{\pi}\arctan\dfrac{\alpha\mp\sqrt{\alpha^2+4(1-c^2)}}{2(1-c)}$, $c=\operatorname{Re}\omega$

and $\sum\limits_{k=-\infty}^{\infty}\left[(\varepsilon_k^\pm)^2+(\beta_k^\pm)^2+(\gamma_k^\pm)^2\right]<\infty$.

Proof. By formula (3) the characteristic equation of a problem $(Q,-1,\alpha)$ has a form:

$$c_1(\lambda,\pi)+\alpha s_1(\lambda,\pi)+s_2(\lambda,\pi)-2=0. \qquad (7)$$

It is known [2] that $c_1(\lambda,\pi)=\cos\lambda\pi+f_1(\lambda)$, $s_1(\lambda,\pi)=-\sin\lambda\pi+f_2(\lambda)$, $s_2(\lambda,\pi)=\cos\lambda\pi+f_3(\lambda)$, where $f_m(\lambda)=\int_{-\pi}^{\pi}\widetilde{f}_m(t)e^{i\lambda t}dt$, $\widetilde{f}_m(t)\in L_2[-\pi,\pi)$, $m=1,2,3$. So the equation (7) can be reduced to

$$2(\cos\lambda\pi-1)-\alpha\sin\lambda\pi+f(\lambda)=0, \qquad (8)$$

where $f(\lambda)=f_1(\lambda)+\alpha f_2(\lambda)+f_3(\lambda)$.

Using the estimate $f(\lambda)=o\left(e^{|\operatorname{Im}\lambda\pi|}\right)$ (at $|\lambda|\to\infty$) and Roushet theorem we get that the roots of the equation (8) (taking into account that they are nondecreasing) form a sequence

$$a_k^{\pm}=2k-a^{\pm}+\varepsilon_k^{\pm}, \qquad (9)$$

where $\varepsilon_k^{\pm}=o(1)$ at $|k|\to\infty$. Substituting the right-hand side of the equality (9) into the equation (8) and using the relationship $f\left(a_k^{\pm}\right)=f_k^{+}+\varepsilon_k^{\pm}g_k^{\pm}$, $\sum_{k=-\infty}^{\infty}\left[\left(f_k^{\pm}\right)^2+\left(g_k^{\pm}\right)^2\right]<\infty$ ([5, p. 67]), we say more precisely that $\sum_{k=-\infty}^{\infty}\left(\varepsilon_k^{\pm}\right)^2<\infty$. So formula (4) has been proved. Asymptotic formula (5) and (6) can be proved in similar fashion. Theorem has been proved.

Denote the eigenvalues of the boundary-value problem (Q,ω,α_j) by $u_{j,k}^{\pm}$ (here $\omega\neq0$ are arbitrary complex numbers).

Theorem 3. *The eigenvalues* $u_{1,k}^{\pm}$, $u_{2,k}^{\pm}$ *of the boundary-value problems* (Q,ω,α_1), (Q,ω,α_2) *(where* $\alpha_1<\alpha_2$*) are intermitted at* $\operatorname{Im}\omega\neq0$, *i.e.*

$$\ldots<u_{2,k}^{-}<u_{1,k}^{-}<u_{2,k}^{+}<u_{1,k}^{+}<u_{2,k+1}^{-}<u_{1,k+1}^{-}<\ldots, \qquad (10)$$

and satisfy the following inequalities at $\operatorname{Im}\omega=0$:

$$\ldots\le u_{2,k}^{-}\le u_{1,k}^{-}\le u_{2,k}^{+}\le u_{1,k}^{+}\le u_{2,k+1}^{-}\le u_{1,k+1}^{-}\le\ldots, \qquad (11)$$

and the double eigenvalue of one of these problems is simple eigenvalue of the other one.

Proof. It's clear that

$$z(\lambda,x)\equiv\begin{pmatrix}z_1(\lambda,x)\\ z_2(\lambda,x)\end{pmatrix}=\left[1+\omega c_1(\lambda,\pi)\right]s(\lambda,x)-\omega s_1(\lambda,\pi)c(\lambda,x) \qquad (12)$$

is a solution of the equation (1) satisfying conditions

$$\begin{aligned}&z_1(\lambda,0)=-\omega s_1(\lambda,\pi), z_2(\lambda,0)=1+\omega c_1(\lambda,\pi),\\ &z_1(\lambda,\pi)=s_1(\lambda,\pi), z_2(\lambda,\pi)=\omega+s_2(\lambda,\pi).\end{aligned} \qquad (13)$$

Multiplying from the left the following equalities

$$\begin{aligned}&Bz'(\lambda,x)+Q(x)z(\lambda,x)=\lambda z(\lambda,x),\\ &B\overline{z'(\lambda,x)}+Q(x)\overline{z(\lambda,x)}=\overline{\lambda}\,\overline{z(\lambda,x)}\end{aligned} \qquad (14)$$

by $\left(\overline{z_1(\lambda,x)},\overline{z_2(\lambda,x)}\right)$ and $\left(z_1(\lambda,x),z_2(\lambda,x)\right)$ respectively, subtracting and integrating with respect to x within $[0,\pi]$ we get by (13):

$$2i\,\mathrm{Im}\,\lambda\int_0^{\pi}\left[\left|z_1(\lambda,x)\right|^2+\left|z_2(\lambda,x)\right|^2\right]dx=\int_0^{\pi}\left[z_2(\lambda,x)\overline{z_1(\lambda,x)}-\overline{z_2(\lambda,x)}z_1(\lambda,x)\right]dx=$$

$$=\overline{s_1(\lambda,\pi)}r(\lambda)-s_1(\lambda,\pi)\overline{r(\lambda)},\ \text{where}\ r(\lambda)=2\,\mathrm{Re}\,\omega+|\omega|^2c_1(\lambda,\pi)+s_2(\lambda,\pi).$$

It is easy to see that

$$s_1(\lambda,\pi)=\frac{\Delta_2(\lambda)-\Delta_1(\lambda)}{\alpha_2-\alpha_1}, \tag{15}$$

$$r(\lambda)=\frac{\alpha_2\Delta_1(\lambda)-\alpha_1\Delta_2(\lambda)}{\alpha_2-\alpha_1},$$

where

$$\Delta_j(\lambda)=2\,\mathrm{Re}\,\omega+|\omega|^2c_1(\lambda,\pi)+s_2(\lambda,\pi)+\alpha_js_1(\lambda,\pi) \tag{16}$$

is characteristic function of a problem (Q,ω,α_j) (see (3)). So

$$2i\,\mathrm{Im}\,\lambda\int_0^{\pi}\left[\left|z_1(\lambda,x)\right|^2+\left|z_2(\lambda,x)\right|^2\right]dx=\frac{\Delta_1(\lambda)\overline{\Delta_2(\lambda)}-\overline{\Delta_1(\lambda)}\Delta_2(\lambda)}{\alpha_2-\alpha_1}.$$

Consequently,

$$\mathrm{Im}\,\lambda\int_0^{\pi}\frac{\left|z_1(\lambda,x)\right|^2+\left|z_2(\lambda,x)\right|^2}{\left|\Delta_2(\lambda)\right|^2}dx=\frac{1}{\alpha_2-\alpha_1}\mathrm{Im}\frac{\Delta_1(\lambda)}{\Delta_2(\lambda)}.$$

So $\dfrac{\Delta_1(\lambda)}{\Delta_2(\lambda)}$ is meromorphic function, mapping an upper half-plane into itself. Then by the known theorem [6, p. 398] noncoincident zeroes of functions $\Delta_2(\lambda)$ and $\Delta_1(\lambda)$ are intermitted.

Reasoning in similar fashion for the equalities (14) and

$$B\overline{\dot z'(\lambda,x)}+Q(x)\overline{\dot z(\lambda,x)}=\overline{\lambda}\,\overline{\dot z(\lambda,x)}+\overline{z(\lambda,x)},$$

(where we denote differentiation with respect to the parameter λ by the dot over the function) we get at $\mathrm{Im}\,\lambda=0$:

$$\int_0^{\pi}\left[\left|z_1(\lambda,x)\right|^2+\left|z_2(\lambda,x)\right|^2\right]dx=\frac{\dot\Delta_1(\lambda)\Delta_2(\lambda)-\Delta_1(\lambda)\dot\Delta_2(\lambda)}{\alpha_2-\alpha_1}.$$

Let $\mathrm{Im}\,\omega\neq 0$. Then it's easy to see that the left-hand side of the last equality is not equal to zero. Hence it follows that the functions $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ have only simple roots and haven't common zeroes. So at $\mathrm{Im}\,\omega\neq 0$ the eigenvalues of problems (Q,ω,α_1) and (Q,ω,α_2) satisfy the system of inequalities (10).

Now let ω be a real number. Then by (12) and (15) problems (Q,ω,α_1) and (Q,ω,α_2) can have finite number of coincident as well as double eigenvalues (as at $\mathrm{Im}\,\omega=0$ $\quad 1+\omega c_1(u_{j,k}^{\pm},\pi)=s_1(u_{j,k}^{\pm},\pi)=0$ can take place, and, consequently, $z(u_{j,k}^{\pm},x)\equiv 0$ for some values k). As we indicated above the noncoincident eigenvalues of said boundary-value problems are intermitted. Due to the theorem 1 double zero u_0 of the function $\Delta_j(\lambda)$ is a root of the equation $s_1(\lambda,\pi)=0$, i.e. it's an eigenvalue of a problem generated by the equation (1) and separated boundary conditions

$$y_1(0)=y_1(\pi)=0. \tag{17}$$

Then according to the formula (15) we have $\Delta_{3-j}(u_0)=0$. But u_o is a simple root of the function $\Delta_{3-j}(\lambda)$, because if it isn't so u_o would be a double zero of the function $s_1(\lambda,\pi)$ (by (15)) which is impossible (the eigenvalues of the problem (1), (17) are simple [4, p. 241]). So at $\operatorname{Im}\omega=0$ the inequalities (11) take place and the double eigenvalues of one of the problems (Q,ω,α_1), (Q,ω,α_2) is the simple eigenvalue of the other one. Theorem has been proved.

§ 2. On the representation of a class entire functions

The present paragraph is devoted to proving of a lemma on the representation of a class of entire functions of exponential type with the known zeroes. It plays an important role when solving inverse problems for systems of differential equations of the first order.

Lemma. *In order to the functions* $d_p(z)$, $p=1,2,3$ *accept representations*

$$d_1(z)=2(\cos\pi z-1)-\alpha\sin\pi z+v_1(z), \tag{18}$$

$$d_2(z)=2(\cos\pi z+1)-\alpha\sin\pi z+v_2(z), \tag{19}$$

$$d_3(z)=2(\cos\pi z+c)-\alpha\sin\pi z+v_3(z),\ |c|<1,$$

where $v_p(z)=\int_{-\pi}^{\pi}\tilde v_p(t)e^{izt}dt, \tilde v_p(t)\in L_2[-\pi,\pi]$, *it's necessary and sufficient that*

$$d_1(z)=-\frac{\pi\alpha}{2a}(a_0^- -z)(a_0^+ -z)\prod_{\substack{k=-\infty\\k\neq0}}^{\infty}\frac{(a_k^- -z)(a_k^+ -z)}{4k(k-a)}, \tag{20}$$

$$d_2(z)=4\prod_{k=-\infty}^{\infty}\frac{(b_k^- -z)(b_k^+ -z)}{(2k+1)(2k+1-2a)},$$

$$d_3(z)=2(c+1)\prod_{k=-\infty}^{\infty}\frac{(c_k^- -z)(c_k^+ -z)}{4(k-c^-)(k-c^+)},$$

where $a_k^\pm$, $b_k^\pm$, $c_k^\pm$ *satisfy asymptotic formulae of the form* (4)-(6), $a=\frac{1}{\pi}\arctan\frac{\alpha}{2}$.

Proof. Let's limit ourselves with giving the proof for the function $d_1(z)$. The proof of the lemma for the functions $d_2(z)$, $d_3(z)$ is constructed in similar fashion.

Necessity. In § 1 we found that zeroes of the function $d_1(z)$ representable in the form (18) are subjected to asymptotic (4). Besides $d_1(z)$ is an entire function of exponential type that's why it can be expanded into infinite product

$$d_1(z)=A(a_0^- -z)(a_0^+ -z)\prod_{\substack{k=-\infty\\k\neq0}}^{\infty}\frac{(a_k -z)(a_k^+ -z)}{4k(k-a)},$$

where A is some constant. To define it one can use the equality $\lim_{y\to\infty}\frac{d_1(iy)}{d_0(iy)}=1$, where

$d_0(z)=2(\cos\pi z-1)-\alpha\sin\pi z$. From here it follows that $A=-\frac{\pi\alpha}{2a}$.

Sufficiency. Let the function $d_1(z)$ be representable as (20). Assume that $\alpha > 0$ (the case $\alpha < 0$ is considered analogously). Then $a_k^- = 2k - 2a + \varepsilon_k^-$, $a_k^+ = 2k + \varepsilon_k^+$. It's evident that

$$d_1(z) = \pi\alpha\varphi^-(z)\varphi^+(z), \qquad (20')$$

where $\varphi^-(z) = \prod_{k=-\infty}^{\infty} \frac{a_k^- - z}{2(k-a)}$, $\varphi^+(z) = (a_0^+ - z)\prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{a_k^+ - z}{2k}$.

By lemma 4 of the paper [2] the function $g(z)$ accept representation $g(z) = \sin\pi z + g_1(z)$, where $g_1(z) = \int_{-\pi}^{\pi} \tilde{g}_1(t)e^{izt}dt, \tilde{g}(t) \in L_2[-\pi,\pi]$ if and only if $g(z) = \pi(\lambda_0 - z)\prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\lambda_k - z}{k}$, $\lambda_k = k + \varepsilon_k$, $\sum_{k=-\infty}^{\infty} \varepsilon_k^2 < \infty$.

Using this fact and the known formula $\sin\pi z = \pi z \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left(1 - \frac{z}{k}\right)$ we have

$$\varphi^-(z) = -\frac{a_0^- - z}{2a}\prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{k + \frac{\varepsilon_k^-}{2} - \frac{z+2a}{2}}{k} \cdot \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{1-\frac{a}{k}} = -\frac{1}{\sin\pi a}\left[\sin\frac{z+2a}{2}\pi + g^-\left(\frac{z+2a}{2}\right)\right],$$

$$\varphi^+(z) = (a_0^+ - z)\prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{k + \frac{\varepsilon_k^+}{2} - \frac{z}{2}}{k} = \frac{2}{\pi}\left[\sin\frac{\pi z}{2} + g^+\left(\frac{z}{2}\right)\right],$$

where $g^{\pm}(z) = \int_{-\pi}^{\pi} \tilde{g}^{\pm}(t)e^{izt}dt, \tilde{g}^{\pm}(t) \in L_2[-\pi,\pi]$.

Substituting these values of the functions $\varphi^{\pm}(z)$ into $(20')$ we get for the function $d_1(z)$ desired representation (18) and $v_1(z)$ has the property indicated above according to the Plancherel [7, p. 439] and Paley-Wiener [8, p. 47] theorems. Lemma has been proved.

§ 3. Solution of the inverse problem.

In this paragraph the inverse problem of spectral analysis formulated in the introduction is solved entirely. First we consider the case $\omega = -1$.

Theorem 4. *In order to sequences of real numbers* $\{a_{1,k}^{\pm}\}$ *and* $\{a_{2,k}^{\pm}\}$ *be the spectrums of the boundary-value problems of the type* $(Q,-1,\alpha_1)$ *and* $(Q,-1,\alpha_2)$ $(\alpha_1 < \alpha_2)$ *it's necessary and sufficient, that the following conditions hold:*

1) the asymptotic formulae

$$a_{j,k}^{\pm} = 2k - (1 \mp \operatorname{sgn} a_j)a_j + \varepsilon_{j,k}^{\pm}, \sum_{k=-\infty}^{\infty}\left(\varepsilon_{j,k}^{\pm}\right)^2 < \infty \qquad (21)$$

take place;

2) $..\le a_{2,k}^- \le a_{1,k} \le a_{2,k}^+ \le a_{1,k}^+ \le a_{2,k+1}^- \le a_{1,k+1}^- \le ..$, *and if two successive members of the sequence* $\{a_{2,k}^{\pm}\}$ *(or* $\{a_{1,k}^{\pm}\}$*) coincide, then the member from* $\{a_{1,k}^{\pm}\}$ *(or* $\{a_{2,k}^{\pm}\}$*) coincident with these two members differs from the other members of the sequence* $\{a_{1,k}^{\pm}\}$ *(or* $\{a_{2,k}^{\pm}\}$*);*

3) the following inequalities hold:

$$\left|\Delta_j(\lambda_k)+2\right| \ge 2; \tag{22}$$

4)

$$\sum_{k=-\infty}^{\infty}\Delta_j(\lambda_k)\left[\Delta_j(\lambda_k)+4\right] < \infty, \tag{23}$$

where

$$\Delta_j(z) = -\frac{\pi\tan\pi a_j}{a_j}(a_{j,0}^- - z)(a_{j,0}^+ - z)\prod_{\substack{k=-\infty \\ k\ne 0}}^{\infty}\frac{(a_{j,k}^- - z)(a_{j,k}^+ - z)}{4k(k-a_j)}, \tag{24}$$

$-\dfrac{1}{2} < a_j < \dfrac{1}{2}, a_1 < a_2, \lambda_k$ *are zeroes of the function* $\Delta_1(z)-\Delta_2(z)$.

Proof. *Necessity.* Let the sequence $\{a_{j,k}^{\pm}\}$ be the spectrum of the boundary-value problem $(Q,-1,\alpha_j)$. Necessity of conditions 1) and 2) of the theorem follows from the theorems 2 and 3. By the formula (4) the number a_j is defined by the relationship $a_j = \dfrac{1}{\pi}\arctan\dfrac{\alpha_j}{2}$. That's why $-\dfrac{1}{2} < a_j < \dfrac{1}{2}$ and as $\alpha_1 < \alpha_2$, then $a_1 < a_2$.

As we saw in § 1 the characteristic function $\Delta_j(\lambda)$ of the boundary-value problem $(Q,-1,\alpha_j)$ is the entire function of exponential type not higher than π. That's why it can be represented as an infinite product (24). As $\Delta_1(\lambda_k)-\Delta_2(\lambda_k)=0$ then by (15) $s_1(\lambda_k,\pi)=0$. Then from the identify (2) it follows that

$$c_1(\lambda_k,\pi)s_2(\lambda_k,\pi)=1. \tag{25}$$

Consider the function $u_+(\lambda)=c_1(\lambda,\pi)+s_2(\lambda,\pi)$. By (25)

$$u_+(\lambda_k)=\frac{1}{s_2(\lambda_k,\pi)}+s_2(\lambda_k,\pi).$$

So $\left|u_+(\lambda_k)+2\right| \ge 2$. As $\Delta_j(\lambda)=u_+(\lambda)+\alpha_j s_1(\lambda,\pi)-2$ (see *(16)*), *then by the last* inequality we have

$$\left|\Delta_j(\lambda_k)+2\right| = \left|u_+(\lambda_k)\right| \ge 2.$$

So the inequality (22) is fulfiled.

Now we show that the inequality (23) is true. Introduce notation

$$u_-(\lambda)=c_1(\lambda,\pi)-s_2(\lambda,\pi).$$

It's easy to get that

$$u_-^2(\lambda_k)=u_+^2(\lambda_k)-4. \tag{26}$$

It follows from the representation for the functions $c_1(\lambda,\pi), s_2(\lambda,\pi)$ and the known asymptotic formula $\lambda_k = k+o(1)(|k|\to\infty)$ that $\sum_{k=-\infty}^{\infty}u_-^2(\lambda_k)<\infty$. Consequently

$$\sum_{k=-\infty}^{\infty}\Delta_j(\lambda_k)\left[\Delta_j(\lambda_k)+4\right]=\sum_{k=-\infty}^{\infty}\left[(\Delta_j(\lambda_k)+2)^2-4\right]=\sum_{k=-\infty}^{\infty}\left[u_+^2(\lambda_k)-4\right]=\sum_{k=-\infty}^{\infty}u_-^2(\lambda_k)<\infty .$$

Sufficiency. Let the conditions 1)-4) hold. By lemma in § 2 the function $\Delta_j(z)$ constructed in accordance with the given sequence $\{a_{j,k}^{\pm}\}$ by means of the formula (24) accepts representation

$$\Delta_j(z)=2(\cos\pi z-1)-\alpha_j\sin\pi z+m_j(z), \tag{27}$$

where $\alpha_j=2\tan\pi a_j$, $m_j(z)=\int_{-\pi}^{\pi}\widetilde{m}_j(t)e^{izt}dt$, $\widetilde{m}_j(t)\in L_2[-\pi,\pi]$.

Consider the function

$$s_1(z)=\frac{\Delta_1(z)-\Delta_2(z)}{\alpha_1-\alpha_2}. \tag{28}$$

By the formula (27) the following representation is true for that function

$$s_1(z)=-\sin\pi z+\frac{m_1(z)-m_2(z)}{\alpha_1-\alpha_2}.$$

Then by the paper [2] zeroes λ_k of the function $s_1(z)$ satisfy asymptotic formulae

$$\lambda_k=k+\delta_k,\ \sum_{k=-\infty}^{\infty}\delta_k^2<\infty . \tag{29}$$

It follows from the second condition of the theorem and the relationships (21), (22), (24), (28) that the arrangement of members of the sequences $\{a_{j,k}^{+}\}$, $\{\lambda_k\}$ is defined by inequalities

$$...\le a_{2,k}^{-}\le a_{1,k}^{-}\le\lambda_{2k}\le a_{2,k}^{+}\le a_{1,k}^{+}\le\lambda_{2k+1}\le..., \tag{30}$$

and $\lambda_n<\lambda_{n+1}, n=0,\pm1,\pm2,...$.

Let's construct the function

$$u_1(z)=\frac{\alpha_2\Delta_1(z)-\alpha_1\Delta_2(z)}{\alpha_2-\alpha_1}+2 . \tag{31}$$

Hence by (27) we have

$$u_1(z)=2\cos\pi z+\frac{\alpha_2 m_1(z)-\alpha_1 m_2(z)}{\alpha_2-\alpha_1}. \tag{32}$$

As $\Delta_1(\lambda_k)=\Delta_2(\lambda_k)$ then from the formula (31) it follows that

$$u_1(\lambda_k)=\Delta_j(\lambda_k)+2 . \tag{33}$$

That's why by (22) the inequalities $|u_1(\lambda_k)|\ge 2$ take place, i.e. $u_1(\lambda_k)\le -2$ or $u_1(\lambda_k)\ge 2$. Taking into account the inequalities (30) we get more precisely that signs of the members of the sequence $\{u_1(\lambda_k)\}$ are alternating. Then there exists such θ_k that

$$u_1(\lambda_k)=2(-1)^k\sigma\cosh\theta_k , \tag{34}$$

where the number σ is equal to 1 or -1. Now consider the function $u_2(z)$ such that

$$|u_2(\lambda_k)|=\sqrt{u_1^2(\lambda_k)-4}. \tag{35}$$

Hence taking into account (34) we obtain

$$|u_2(\lambda_k)|=2|\sinh\theta_k|. \tag{36}$$

If we claim the equality

$$u_2(\lambda_k) = 2\sinh\theta_k \tag{37}$$

to be true, then we define not only $|u_2(\lambda_k)|$ but also $\operatorname{sgn} u_2(\lambda_k) = \operatorname{sgn}\theta_k$.

Now let's define $u_2(z)$ by the equality

$$u_2(z) = 2s_1(z)\sum_{k=-\infty}^{\infty}\frac{\sinh\theta_k}{(z-\lambda_k)s'(\lambda_k)}.$$

It follows from the fourth condition of the theorem and the equalities (33), (35), (36) that

$$4\sum_{k=-\infty}^{\infty}\sinh^2\theta_k = \sum_{k=-\infty}^{\infty}\left[u_1^2(\lambda_k)-4\right] = \sum_{k=-\infty}^{\infty}\Delta_j(\lambda_k)\left[\Delta_j(\lambda_k)+4\right] < \infty.$$

Then by the theorem 28 of the paper [8] the function $u_2(z)$ accepts representation

$$u_2(z) = \int_{-\pi}^{\pi} h(t)e^{izt}dt, h(t)\in L_2[-\pi,\pi]. \tag{38}$$

As the function

$$s_2(z) = \frac{1}{2}\left[u_1(z) - u_2(z)\right] \tag{39}$$

due to the formulae (32) and (38) has a form

$$s_2(z) = \cos\pi z + \psi(z)\left(\psi(z) = \int_{-\pi}^{\pi}\widetilde{\psi}(t)e^{izt}dt, \widetilde{\psi}(t)\in L_2[-\pi,\pi]\right),$$

then by lemma 4 of the paper [2] its zeroes v_k satisfy asymptotic formula

$$v_k = k - \frac{1}{2} + \xi_k,\ \sum_{k=-\infty}^{\infty}\xi_k^2 < \infty. \tag{40}$$

Putting in (39) $z=\lambda_k$ and taking into account (34), (37) we get

$$s_1(\lambda_k) = \frac{1}{2}\left[u_1(\lambda_k) - u_2(\lambda_k)\right] = (-1)^k\sigma\cosh\theta_k - \sinh\theta_k =$$
$$= (-1)^k\sigma\cosh\theta_k\left[1-(-1)^k\sigma\tanh\theta_k\right].$$

From here one can see that $\operatorname{sgn} s_1(\lambda_k) = (-1)^k\sigma$ because $\tanh\theta_k < 1$. That's why on each interval $(\lambda_k, \lambda_{k+1})$ there is at least one root v_{k+1} of the function $s_2(z)$ and due to the asymptotic formula (40) it can't have another roots. Consequently, zeroes of the function $s_1(z)$ are intermitted with zeroes of the function $s_2(z)$.

So the sequences $\{\lambda_k\}$ and $\{v_k\}$ *satisfy conditions of the theorem 2 of the paper* [2], in according to which there exists the matrix-function $Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x)\end{pmatrix}$ ($p(x)$, $q(x)$ are real functions from $L_2[0,\pi]$), such that $s_1(z)$ and $s_2(z)$ are characteristic functions of the boundary-value problems generated by the equation (1) with this coefficient $Q(x)$ and the boundary conditions (17) and

$$y_1(0) = y_2(\pi) = 0. \tag{41}$$

Consequently $s_1(z) = s_1(z,\pi)$ and $s_2(z) = s_2(z,\pi)$ (for constructing Dirac equation). It's easy to get that the spectrums of constructed boundary-value problems $(Q,-1,\alpha_1)$ and $(Q,-1,\alpha_2)$ coincide with the sequences $\{a_{1,k}^{\pm}\}$ and $\{a_{2,k}^{\pm}\}$ respectively. Theorem has been proved.

As we see the present proof includes the method of recovery of considered boundary-value problems. Note that one can't in an one-to-one way recover the problems of the type $(Q,-1,\alpha_1)$, $(Q,-1,\alpha_2)$ (i.e. nor the matrix $Q(x)$ neither the parameters α_1,α_2) by their spectrums. Let's find out which additional information about these problems one should have to recover them in an one-to-one way.

We indicated above that one can recover the characteristic functions $\Delta_1(\lambda)$, $\Delta_2(\lambda)$ of the boundary-value problems $(Q,-1,\alpha_1)$, $(Q,-1,\alpha_2)$ by the sequences $\{a_{1,k}^{\pm}\}$, $\{a_{2,k}^{\pm}\}$ of the eigenvalues of these problems in the form of infinite product. Besides knowing $a_{j,k}^{\pm}$ one can define the parameter α_j as by the formula (4)

$$\alpha_j = -2\lim_{k\to\infty}\tan\frac{\pi}{2}\left(a_{j,k}^{-}+a_{j,k}^{+}\right).$$

The function $s_1(\lambda,\pi)$ can be recovered by the formula (15). Hence we find the sequence $\{\lambda_k\}$ of the eigenvalues of the problem (1), (17). By means of $\Delta_j(\lambda)$, α_j one can recover the function

$$u_+(\lambda)=\frac{\alpha_1\Delta_2(\lambda)-\alpha_2\Delta_1(\lambda)}{\alpha_1-\alpha_2}+2\,.$$

By formula (26)

$$u_-(\lambda_k)=\sigma_k\sqrt{u_+^2(\lambda_k)-4}\,,$$

where $\sigma_k=\operatorname{sgn}u_-(\lambda_k)$. This equality and the estimate $u_-(\lambda)=o\left(e^{|\operatorname{Im}\lambda\pi|}\right)$ show that knowing the function $u_+(\lambda)$ and the sequences $\{\lambda_k\}$, $\{\sigma_k\}$ we can recover the function $u_-(\lambda)$ by means of the interpolation formula

$$u_-(\lambda)=s_1(\lambda,\pi)\sum_{k=-\infty}^{\infty}\frac{\sigma_k\sqrt{u_+^2(\lambda_k)-4}}{\dot{s}_1(\lambda_k,\pi)(\lambda-\lambda_k)}\,.$$

We find the function $s_2(\lambda,\pi)$ by the formula

$$s_2(\lambda,\pi)=\frac{1}{2}\left[u_+(\lambda)-u_-(\lambda)\right].$$

Zeroes v_k of this function are the eigenvalues of the boundary-value problem (1), (41). It's known ([2]) that the sequences $\{\lambda_k\}$, $\{v_k\}$ define the matrix $Q(x)$ in an one-to-one way.

So in order to recover the boundary-value problems $(Q,-1,\alpha_1)$, $(Q,-1,\alpha_2)$ in an one-to-one way it's sufficient to know the sequences $\{a_{1,k}^{\pm}\}$, $\{a_{2,k}^{\pm}\}$ and $\{\sigma_k\}$. Thus the following theorem on uniqueness of recovery for considered boundary-value problems is true.

Theorem 5. *The boundary-value problems $(Q,-1,\alpha_1)$, $(Q,-1,\alpha_2)$ can be recovered in an one-to-one way if their spectrums and the sequence of signs $\sigma_k=\operatorname{sgn}\left[c_1(\lambda_k,\pi)-s_2(\lambda_k,\pi)\right]$ are known.*

Note that the results concerning the case $\omega=1$ are obtained quite analogously. One should only use the fact that the eigenvalues of the problem $(Q,1,\alpha)$ are subjected to the asymptotic (5) and the representation (19) takes place for the characteristic function of this problem.

Consider now the case $|\omega|=1,\ \omega\neq\pm1$. Proofs of the main theorem for inverse problem and the uniqueness theorem really differ a little in this case from those stated above. That's why we limit ourselves with the formulation of results.

Theorem 6. *In order to two sequences of real numbers* $\{c_{1,k}^{\pm}\}$ *and* $\{c_{2,k}^{\pm}\}$ *be the spectrums of the boundary-value problems of the type* (Q,ω,α_1) *and* (Q,ω,α_2) *respectively* ($|\omega|=1,\ \omega\neq\pm1,\ \alpha_1<\alpha_2$) *it's necessary and sufficient that the following conditions hold:*

1) the asymptotic formulae

$$c_{j,k}^{+}=2k-c_j^{\pm}+\gamma_{j,k}^{\pm},\ \sum_{k=-\infty}^{\infty}\left(\gamma_{j,k}^{\pm}\right)^2<\infty$$

take place;

2) $c_{1,k}^{\pm}$ *and* $c_{2,k}^{\pm}$ *are intermitted, i.e.*

$$\ldots<c_{2,k}^{-}<c_{1,k}^{-}<c_{2,k}^{+}<c_{1,k}^{+}<c_{2,k+1}^{-}<c_{1,k+1}^{-}<\ldots;$$

3) the inequalities

$$\left|\Delta_j(\lambda_k)-2c\right|\geq 2\ \textit{hold;}$$

4) $\sum_{k=-\infty}^{\infty}\left[\left(\Delta_j(\lambda_k)-2c\right)^2-4\right]<\infty$,

where $\Delta_j(z)=2(c+1)\prod_{k=-\infty}^{\infty}\dfrac{(c_{j,k}^{-}-z)(c_{j,k}^{+}-z)}{4(k-c_j^{-})(k-c_j^{+})}$, $c_j^{\pm}=\dfrac{2}{\pi}\arctan\left(B_j\mp\sqrt{B_j^2+\dfrac{1+c}{1-c}}\right)$, c, B_j *are real numbers, and* $|c|<1$, $B_1<B_2$, λ_k *are zeroes of the function* $\Delta_1(z)-\Delta_2(z)$.

Theorem 7. *The boundary-value problems* (Q,ω,α_1), (Q,ω,α_2) *are recovered in an one-to-one way (with the accuracy to the sign of* $\operatorname{Im}\omega$*) if their spectrums and the sequence of signs* $\sigma_k=\operatorname{sgn}\left[c_1(\lambda_k,\pi)-s_2(\lambda_k,\pi)\right]$ *are known.*

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