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THE STURM-LIOUVILLE PROBLEM WITH NON-LINEAR SPECTRAL PARAMETER IN THE BOUNDARY CONDITIONS

Abstract

The Sturm-Liouville problem containing non-linear spectral parameter λ *in the equation and in the boundary conditions is considered:*

$$-u'' + q(x)u = \lambda^2 u, \quad 0 < x < 1, \tag{1}$$

$$\left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)u(0) + u'(0) = 0, \tag{2}$$

$$\left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right)u(1) + u'(1) = 0. \tag{3}$$

Supposing that $q(x)$ *is any real- valued function from the class* $C[0,1]$ *allocation of eigenvalues is studied, the theorem on number of zeroes of eigenfunctions is proved, the asymptotic formulas for the eigenvalues and eigenfunctions for the boundary value problem (1)-(3) are found.*

Consider the next boundary value problem with a spectral parameter in the equation and in the boundary conditions

$$-u'' + q(x)u = \lambda^2 u, \quad 0 < x < 1, \tag{1}$$

$$\left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)u(0) + u'(0) = 0, \tag{2}$$

$$\left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right)u(1) + u'(1) = 0. \tag{3}$$

Here λ is a spectral parameter, $q(x)$ is a real-valued function from the class $C[0,1]$, α_i and $\beta_i\,(i = 0,1,2)$ are real constants. For the case when $q(x)$ is a non-negative function from the class $C[0,1]$ and $\alpha_0 < 0,\ \alpha_2 > 0,\ \beta_0 > 0,\ \beta_2 < 0,\ |\alpha_1| + |\beta_1| \neq 0$, the problem (1)-(3) was described in paper [1]. In fact that even for a more common case we can describe the spectral properties of the problem (1)-(3) essentially fuller.

In future everywhere we'll suppose, that $q(x)$ is real-valued function from the class $C[0,1]$ and the condition

$$\alpha_2 > 0,\ \beta_2 < 0,\ |\alpha_1| + |\beta_1| \neq 0 \tag{4}$$

is fulfilled.

Lemma 1. *The exists such a number* $R_0 \geq 0$ *that any eigenvalues* λ *the boundary value problem (1)-(3), satisfying inequality* $|\lambda| \geq R_0$, *is real.*

Proof. Let λ be an eigenvalue of the boundary value problem (1)-(3) and $u(x,\lambda)$ is a corresponding eigenfunction. Multiplying the both sides of the equality (1) by the function $\overline{u(x,\lambda)}$ we integrate the obtained identity by x from 0 to 1:

$$-\int_0^1 u''(x,\lambda)\,\overline{u(x,\lambda)}\,dx + \int_0^1 q(x)|u(x,\lambda)|^2\,dx = \lambda^2\int_0^1 |u(x,\lambda)|^2\,dx. \tag{5}$$

Using the formula of integration by parts and the boundary conditions (2) and (3), we get:

$$\int_0^1 u''(x,\lambda)\,\overline{u(x,\lambda)}dx = \left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)|u(0,\lambda)|^2 -$$

$$-\left(\beta_0+\beta_1\lambda+\beta_2\lambda^2\right)\left|u(1,\lambda)\right|^2-\int_0^1\left|u'(x,\lambda)\right|^2dx.$$

From above and from (5) it follows that

$$A(\lambda)\lambda^2+B(\lambda)\lambda+C(\lambda)=0,$$

where

$$A(\lambda)=\int_0^1\left|u(x,\lambda)\right|^2dx+\alpha_2\left|u(0,\lambda)\right|^2-\beta_2\left|u(1,\lambda)\right|^2,$$

$$B(\lambda)=\alpha_1\left|u(0,\lambda)\right|^2-\beta_1\left|u(1,\lambda)\right|^2,$$

$$C(\lambda)=\alpha_0\left|u(0,\lambda)\right|^2-\beta_0\left|u(1,\lambda)\right|^2-\int_0^1 q(x)\left|u(x,\lambda)\right|^2dx-\int_0^1\left|u'(x,\lambda)\right|^2dx.$$

Thus the eigenvalue λ is the root of quadratic equation

$$A(\lambda)z^2+B(\lambda)z+C(\lambda)=0. \tag{6}$$

Let us use the estimations

$$\max_{0\le x\le 1}\left|u(x,\lambda)\right|^2\le c_0\left(1+|\lambda|\right)\int_0^1\left|u(x,\lambda)\right|^2dx, \tag{7}$$

$$\int_0^1\left|u'(x,\lambda)\right|^2dx\ge c_1\left(1+|\lambda|\right)^2\int_0^1\left|u(x,\lambda)\right|^2dx, \tag{8}$$

where c_0 and c_1 are positive constants, not depending on λ. These estimations are obtained in [2].

Let $q_0=\max_{0\le x\le 1}\left|q(x)\right|$. By virtue of (7) and (8) we have

$$C(\lambda)\le\left|\alpha_0\right|\cdot\left|u(0,\lambda)\right|^2+\left|\beta_0\right|\left|u(1,\lambda)\right|^2+q_0\int_0^1\left|u(x,\lambda)\right|^2dx-$$

$$-\int_0^1\left|u'(x,\lambda)\right|^2dx\le\left[c_0\left(\left|\alpha_0\right|+\left|\beta_0\right|\right)\left(1+|\lambda|\right)+q_0-\right.$$

$$\left.-c_1\left(1+|\lambda|\right)^2\right]\int_0^1\left|u(x,\lambda)\right|^2dx.$$

Thus, we proved that for any eigenvalue λ the following inequality is satisfied:

$$C(\lambda)\le-\left(1+|\lambda|\right)^2\left(c_1-\frac{c_2}{1+|\lambda|}\right)\int_0^1\left|u(x,\lambda)\right|^2dx, \tag{9}$$

where $c_2=c_0\left(\left|\alpha_0\right|+\left|\beta_0\right|\right)+q_0$.

Let $R_0=\frac{c_2}{c_1}$. It is easy to show that if $|\lambda|\ge R_0$ then the following inequality is satisfied:

$$c_1-\frac{c_2}{1+|\lambda|}>0.$$

From above and from (9) it follows that when $|\lambda|\ge R_0$ the inequality $C(\lambda)<0$ holds. Besides, by virtue of (4) $A(\lambda)>0$. Therefore when $|\lambda|\ge R_0$ the following inequality is satisfied:

$$B^2(\lambda)-4A(\lambda)C(\lambda)>0.$$

Consequently, the equation (6) when $|\lambda|\geq R_0$ has only real roots. Thus, lemma 1 is proved.

Lemma 2. *The eigenvalues of boundary value problem (1)-(3):*
(a) form at most countable set, not having a finite limit point;
(b) are real and simple excluding a finite number of eigenvalues.

Proof. Similarly to the theorem 1.1. from the [3, p.14] we can prove that there exists a unique solution of the equation (1) satisfying the initial conditions

$$\psi(0,\lambda)=1,\ \psi'(0,\lambda)=-\alpha_0-\alpha_1\lambda-\alpha_2\lambda^2, \tag{10}$$

where at every fixed $x\in[0,1]$ the function $\psi(x,\lambda)$ is a entire function of the argument λ.

The eigenvalues of the boundary value problem (1)-(3) are zeros of the entire function

$$\left(\beta_0+\beta_1\lambda+\beta_2\lambda^2\right)\psi(1,\lambda)+\psi'(1,\lambda).$$

We proved (lemma 1) that this function doesn't turns to zero at non-real λ, satisfying the inequality $|\lambda|\geq R_0$. That is why its zeros form the at most countable set, which hasn't finite limit point.

For the proving of the statement (b) it will be enough to show that the equation

$$\left(\beta_0+\beta_1\lambda+\beta_2\lambda^2\right)\psi(1,\lambda)+\psi'(1,\lambda)=0 \tag{11}$$

outside the circle $\{\lambda:|\lambda|<R_0\}$ has only simple roots.

Really, if $\lambda=\lambda^*$ is the multiple root of the equation (11) and $|\lambda^*|\geq R_0$, as it was proved in [1] the equality

$$\beta_1\psi^2(1,\lambda^*)-\alpha_1=2\lambda^*\left[\int_0^1\psi^2(x,\lambda^*)dx+\alpha_2-\beta_2\psi^2(1,\lambda^*)\right] \tag{12}$$

holds. Besides proving lemma 1 we showed that

$$A(\lambda^*)\lambda^{*2}+B(\lambda^*)\lambda^*+C(\lambda^*)=0, \tag{13}$$

where

$$A(\lambda^*)=\int_0^1\psi^2(x,\lambda^*)dx+\alpha_2-\beta_2\psi^2(1,\lambda^*), \tag{14}$$

$$B(\lambda^*)=\alpha_1-\beta_1\psi^2(1,\lambda^*), \tag{15}$$

$$C(\lambda^*)=\alpha_0-\beta_0\psi^2(1,\lambda^*)-\int_0^1 q(x)\psi^2(x,\lambda^*)dx-\int_0^1\psi'^2(x,\lambda^*)dx.$$

By virtue of (12), (14) and (15) we have $\lambda^*=-\dfrac{B(\lambda^*)}{2A(\lambda^*)}$ (since $A(\lambda^*)>0$). From above and from (13) we get

$$B^2(\lambda^*)=4A(\lambda^*)C(\lambda^*). \tag{16}$$

By proving lemma 1 the inequality $C(\lambda^*)<0$ was shown. The latter contradicts with (16). The lemma 2 is proved.

The following two statements (theorem 1 and theorem 2) are corollaries of lemma 1 and 2 of presented paper and theorem 2.1 and 4.1 of paper [1].

Theorem 1. *The set of eigenvalues of the boundary value problem (1)-(3) consists of the finite number of non-real eigenvalues, of infinitely decreasing sequence of negative eigenvalues* $\{\lambda_{-n}\}_{n=1}^{\infty}$ *and infinitely increasing sequence of positive eigenvalues* $\{\lambda_n\}_{n=1}^{\infty}$:

$$... < \lambda_{-n} < \lambda_{-n+1} < ... < \lambda_{-1} < \lambda_1 < ... < \lambda_{n-1} < \lambda_n < ....$$

Besides, there exist such numbers $n_*, n^* \in N_*$ *and* $k_*, k^* \in N \cup \{0\}$, *that eigenfunctions, corresponding to the eigenvalues* $\lambda_{-n}(n \geq n_*)$ *and* $\lambda_n(n \geq n^*)$, *have correspondingly* $n + k_* - n_*$ *and* $n + k^* - n^*$ *simple zeros in the interval (0,1).*

Assume, that $m \in Z \setminus \{0\}$, $|m| \geq N_0$, where N_0 is a sufficiently great natural number. Let $\vartheta_m(x)$ be the eigenfunction of the boundary value problem (1)-(3), having $|m|$ zeros in the interval (0,1). By the μ_m it is denoted the eigenvalue, corresponding to the eigenfunction $\vartheta_m(x)$. From the oscillation theorem 1 follows, that $\mu_m = \lambda_{m-k^*+n^*}$ for $m > 0$ and $\mu_m = \lambda_{m+k_*-n_*}$ for $m < 0$

Theorem 2. *The next asymptotic formulas are true:*

$$\mu_m = \pi(m - \operatorname{sgn} m) + \frac{1}{\pi m}\left\{\frac{1}{2}\int_0^1 q(x)dx + \frac{1}{\alpha_2} - \frac{1}{\beta_2}\right\} + O\left(|m|^{-1}\omega\left(|m|^{-1}\right)\right), \qquad (17)$$

$$\vartheta_m(x) = \sin\pi(|m| - 1)x + O\left(\frac{1}{m}\right),$$

where $\omega(\delta) = \delta + \omega_1(\delta)$ *and* $\omega_1(\delta)$ *is a modulus of continuity of the function* $q(x)$ *in the segment* $[0,1]$. *Besides, if* $q(x) \neq const$, *then the function* $\omega(\delta)$ *in the formula (17) may be substituted by the function* $\omega_1(\delta)$.

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