MAMEDOV F.I.

POINCARE TYPE WEIGHT INEQUALITIES IN DOMAINS WITH AN ISOPERIMETRIC TYPE CONDITION

Abstract

For the some bounded domains Ω in R^n, $n \geq 2$ with isoperimetrical type conditions $\widetilde{I}_\lambda$, in partial for the domains $\Omega = \{x = (x', x_n) : |x'| < x_n^\beta, 0 < x_n < a\}$, $a > 0$, $\beta \geq 1$ was proved the sufficient conditions on the weights, under which the Poincare's type two weighted inequality holds.

The paper is devoted to investigation the inequality

$$\left(\int_\Omega |u - \overline{u}|^q v dx \right)^{\frac{1}{q}} \leq C \left(\int_\Omega |Du|^p \omega dx \right)^{\frac{1}{p}}, \; 1 \leq p \leq q < \infty \tag{1}$$

of the differentiable functions $u(x)$ for some classes of the bounded domains Ω and the weights v, ω. The sufficient conditions of type A_{pq} are established for pair (v, ω) and isoperimetrical type inequalities between the Lebesque measure of any subsets of domain and $(n-1)$-dimensional of Housdorf measure of the part of boundary for the domains which provide the truthness of the inequality (1).

Here $v, \omega^{1-p'}$ are assumed locally integrable functions, with almost everywhere finite positive values at $1 < p < \infty, \omega^{-1} \in L^{\infty, loc}$ when $p = 1$. Ω-is an open bounded domain in $R^n, n \geq 2$, $\partial\Omega$-is its boundary, $d(\Omega)$-is a diameter of Ω, $\underset{n-1}{mes} \sum (n-1)$-is dimensional Housdorf measure of the set $\sum$ and $\left|\sum\right|$ is its Lebesque measure. $C^1(\Omega)$-are continuously differentiable in Ω functions. By Q denote arbitrary bolls in R^n, $Q_R^x = \{y \in R^n : |y - x| \leq R\}$. $p' = \dfrac{p}{p-1}$ when $1 < p < \infty$, $p' = \infty$-when $p = 1$.

$$\overline{u} = \frac{1}{v(\Omega)} \int_\Omega vudx, \; v(\Omega) = \int_\Omega vdx, \; |Du|^2 = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2.$$

It is known that the inequality

$$\left(\int_\Omega |u - \overline{u}|^q dx \right)^{\frac{1}{q}} \leq C_{n,q} \left(\int_\Omega |Du| dx \right), \; u \in C^1(\Omega), \tag{2}$$

which is got from (1) in the unweighted case when $p = 1, 1 \leq q \leq \dfrac{n}{n-p}$ and the connected domain Ω, is equivalent to the isoperimetrical condition I_λ on Ω

$$\underset{n-1}{mes} \partial g \cap \Omega \geq \theta \min\{|g|, |\Omega \setminus g|\}^\lambda \tag{3}$$

when $\lambda = \dfrac{1}{q}$, where $0 < \theta < \infty$, $g \subseteq \Omega$, see the lemma 3.2.4 from [1].

Unlike the regular domains the inequality of type (1) in domains I_λ have been respectively little studied (see [2] for the regular domains).

From (2) when $\Omega = Q_0$, where Q_0 is a boll we get the inequality.

$$\left(\int_\Omega |u-\bar{u}|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \le C_n \left(\int_\Omega |Du| dx\right),$$

which is equivalent to the inequality

$$\left(\int_{Q_0} |u-\bar{u}|^{kp} dx\right)^{\frac{1}{kp}} \le C_{n,p} |Q_0|^{\frac{1}{n}+\frac{1-k}{kp}} \left(\int_{Q_0} |Du|^p dx\right)^{\frac{1}{p}}$$

for all $p,k: 1 \le p < n,\ 1 \le k \le \dfrac{n}{n-p}$. The last inequality when $k=1$ turns to the Poincare inequality

$$\left(\int_{Q_0} |u-\bar{u}|^p dx\right)^{\frac{1}{p}} \le C_{n,p} |Q_0|^{\frac{1}{n}} \left(\int_{Q_0} |Du|^p dx\right)^{\frac{1}{p}},\ \bar{u} = \int_\Omega u dx.$$

First let's denote the results for the regular domains which are interesting for us in connection with the conditions on weights.

The sufficient conditions on v,ω for the inequality (1) in the case $\Omega = Q_0$ where Q_0-is some boll, have been studied in [3-5]. From the results of papers [3,6,7] it follows that the inequality (1) is true when $q > p$ in the sphere Q_0 if

$$\sup_{Q \subset 8Q_0} \left(\int_Q v dx\right)^{\frac{1}{q}} \left(\int_{8Q_0} \frac{\omega^{1-p'}}{\left|Q_0\right|^{\frac{1}{n}} + \left|x - x_Q\right|^{(n-1)p'}} dx\right) < \infty$$

and it is true when $q = p$ if

$$\sup_{Q \subset 8Q_0} |Q_0|^{\frac{1}{n}} \left(\frac{1}{|Q|}\int_Q v dx\right)^{\frac{1}{rq}} \left(\frac{1}{|Q|}\int_Q \omega^{(1-p')r} dx\right)^{\frac{1}{p'r}} < \infty$$

at some $r > 1$, moreover if $v \in RD$ (It means that it will be found $\varepsilon, \delta \in (0,1)$ such that $v(\delta Q) \le \varepsilon v(Q)$ for any boll $Q \in 8Q_0$) and $q > p$ or if $q = p$ and the both functions $v, \omega^{1-p'}$ belong to the class A_∞^β ($f \in A_\infty^\beta$ means that such $C, \delta > 0$ will be found that for any boll $Q \subset 8Q_0$ and its compact subset E

$$\frac{f(E)}{f(Q)} \le C\left(\frac{\|E\|_{\beta,Q}}{|Q|^{\frac{\beta}{n}}}\right),$$

where $\|E\|_{\beta,Q} = \inf\left\{\sum_i |Q^i|^{\frac{\beta}{n}} : E \subset \bigcup_i Q^i \subset Q\right\}\Big)$ at some $\beta > n-1$ then above integral conditions we can substitute by the condition A_{pq} i.e.

$$\sup_{Q \subset 8Q_0} |Q_0|^{\frac{1}{n}-1} \left(\int_Q v dx\right)^{\frac{1}{q}} \left(\int_Q \omega^{1-p'} dx\right)^{\frac{1}{p'}} < \infty$$

For the weighted results on Poincare inequality in the nonregular domains let's denote [5,8,9].

In the work [10] was proved the imbeding $W_p^1 \subset L_q, 1 \le p \le q < \infty$, $\frac{1}{1+\sigma(n-1)} - \frac{1}{p} + \frac{1}{q} \ge 0$ for the nonregular domains with σ -condition John $(\sigma \ge 1)$.

For the spaces of high smoothness the imbeding theorems was proved in [9].

Let's denote by V the system of bolls

$$\left\{Q : Q = Q_t^x,\, x \in \Omega,\, 0 \le t \le d(\Omega)\right\}$$

for the domain Ω. For the investigation (1) at the domain Ω in paper introduced the condition $\tilde{I}_\lambda$. We'll say that the bounded domain Ω satisfies the condition $\tilde{I}_\lambda\left(\frac{1}{n'} \le \lambda < \infty\right)$, if there is such $0 < \theta < \infty$ that for any boll $Q \in V$ and any compact subsets $A, B,\, A \cap B = \varnothing$ from $\Omega_Q = \Omega \cap Q$ such that

$$|A| > \varepsilon \quad \text{and} \quad |B| > \varepsilon$$

every $C^{0,1}$ surface $\sum$, dividing in Ω_Q A and B, has the following estimation

$$\underset{n-1}{mes} \sum \ge \theta \varepsilon^\lambda .$$

Let's note that for the proving of belongness of concrete domains to the type I_λ, in many examples in monograph [1] was designed the method of suborel mappings (theorem 3.3.2); i.e. the mappings at which $(n-1)$-dimensional measure of boundary of subsets the domains essentially don't increase. For example, from these results follows that domain $\Omega = \left\{x = (x', x_n) : x' \in R^{n-1},\, 0 < x_n < a,\, |x'| < x_n^\beta\right\}$, $\beta \ge 1$ belongs to the class I_λ when $\lambda = \frac{\beta(n-1)}{1+\beta(n-1)}$, the bounded domain which is star with respect to the sphere belongs to $I_{\frac{n-1}{n}}$ (corollary 3.2.1 /1);the bounded domain satisfying the cone condition belongs to the class $I_{\frac{n-1}{n}}$ (corollary 3.1.1/3). The same method can be applied to proof that these domains also belong to the corresponding class $\tilde{I}_\lambda$.

In the theorem 1 we use the $A_\infty(\Omega)$ class: the function v belongs to the class $A_\infty(\Omega)$, $v \in A_\infty(\Omega)$, if there are positive constants M, δ such that

$$\frac{v(E)}{v(Q \cap \Omega)} \le M\left(\frac{|E|}{|Q \cap \Omega|}\right)^\delta,$$

for any measurable subset E of the set $\Omega_Q = \Omega \cap Q,\, Q \in V$;

The main basic results of the paper are the next theorems 1 and 2 (theorem 1 is a simple corollary of theorem 2, by applying lemma 4 given below).

Theorem 1. *Let* $1 \le p \le q < \infty, \frac{1}{n'} \le \lambda \le 1, \Omega$ *belongs to* $\tilde{I}_\lambda$ *class,* $v \in A_\infty(\Omega)$. *If*

[Mamedov F.I.]

$$B_{pq}^{\lambda}=\sup_{Q\in V}|Q\cap\Omega|^{-\lambda}\left(\int_{Q\cap\Omega}vdx\right)^{\frac{1}{q}}\left(\int_{Q\cap\Omega}\omega^{1-p'}dx\right)^{\frac{1}{p'}}<\infty$$

when $1<p<\infty$

$$B_{1q}^{\lambda}=\sup_{Q\in V}|Q\cap\Omega|^{-\lambda}\left(\int_{Q\cap\Omega}vdx\right)^{\frac{1}{q}}\left(\sup_{x\in Q\cap\Omega}\omega^{-1}(x)\right)<\infty \tag{4}$$

when $p=1$ *then for* $\forall u\in C^1(\Omega)$ *the inequality*

$$\left(\int_{\Omega}|u-\overline{u}|^q vdx\right)^{\frac{1}{q}}\leq C_0\frac{B_{p\theta q}^{\lambda}}{\theta}\left(\int_{\Omega}\omega|Du|^p dx\right)^{\frac{1}{p}},$$

is true, where $C_0=C(n,q,M,\delta)>0$ *is some constant.*

Theorem 2. *Let* $1\leq p\leq q<\infty, \frac{1}{n'}\leq\lambda\leq 1, \Omega$ *belongs to* $\tilde{I}_{\lambda}$ *at some* $r>1$.

If

$$A_{pq}^{\lambda}=\sup_{Q\in V}\left(\int_{Q\cap\Omega}vdx\right)^{\frac{1}{q}-r'\lambda}\left(\int_{Q\cap\Omega}v^r dx\right)^{(r'-1)\lambda}\left(\int_{Q\cap\Omega}\omega^{1-p'}dx\right)^{\frac{1}{p'}}<\infty$$

when $1<p<\infty$,

$$A_{1q}^{\lambda}=\sup_{Q\in V}\left(\int_{Q\cap\Omega}vdx\right)^{\frac{1}{q}}\left(\int_{Q\cap\Omega}v^r dx\right)^{(r'-1)\lambda}\left(\sup_{x\in Q\cap\Omega}\omega^{-1}(x)\right)<\infty \tag{5}$$

when $p=1$, *then for* $\forall u\in C^1(\Omega)$

$$\left(\int_{\Omega}|u-\overline{u}|^q vdx\right)^{\frac{1}{q}}\leq C_{q,r}\frac{A_{pq}^{\lambda}}{\theta}\left(\int_{\Omega}\omega|Du|^p dx\right)^{\frac{1}{p}},$$

is true, where $C_{q,r}>0$ *is some constant, depends on* n,q,r,M,δ.

Compare theorem 1 when $q=p>1$, $\Omega=Q_0$-is some boll $\left(\text{i.e. belongs to the } \tilde{I}_{\frac{n-1}{n}}\right)$ with the above given result from paper [3] (theorem 5, the case $q=p$) where for the validity (1) required the condition A_{pp} and $v,\omega^{1-p'}\in A_{\infty}^{\beta}$ at some $\beta>n-1$. At theorem 1 one of the conditions [3] is absent (this is the condition $\omega^{1-p'}\in A_{\infty}^{\beta}$), the other one stronger than [3]. The result of theorem 1 has the intersection with the mentioned result from [3], in the meaning that there exists an example of pair weights (v,ω), satisfying the condition of theorem 1, but not satisfying the condition of [3]. Let's cite this example.

Example. At this example $p=2, Q_0=Q_1^0$ the pair of the weights $(v,\omega)\in A_{pp}$ when $v\in A_{\infty}, \omega^{1-p'}\notin A_{\infty}^{\beta}$ at any $\beta>n-1$.

Let $\Omega = Q_1^0$, $n \geq 3$, a be a sufficiently big number $\geq e^n$, $v = |x|^{n-3} \times \ln\frac{a}{|x|}$, $\omega = |x'|^{n-1}\ln^2\frac{a}{|x'|}$ $(x = (x', x_n) : x' \in R^{n-1}, x_n \in R^1)$. It is easy to see that (5) fulfilled $(p = 2)$ for the pair (v, ω), $v \in A_\infty(Q_1^0)$. Let's show that the condition $\sigma = \omega^{-1} \in A_\infty^\beta$ can't be fulfilled at any $\beta > n-1$.

Let $0 < r < \frac{1}{4}$, $\beta > n-1$, $T_r = \left\{x \in R_n : x = (x', x_n),\ x' \in R^{n-1}, |x'| < r, 0 < x_n < \frac{1}{2}\right\}$. It $\sigma(T_r) < \frac{C_1(n)}{\ln\frac{a}{r}}$ and $\|T_r\|_{\beta, Q_1^0} \leq C_2(n) r^{\beta-1}$. The last estimate follows from the fact that for any $0 < r < \frac{1}{4}$ the set T_r we can cover by N number of bolls with the radius $2r$ lying in Q_1^0 such that $N \sim \frac{1}{r}$. If $\sigma \in A_\infty^\beta$ then will be found $C, \delta > 0$ such that

$$\frac{\sigma(T_r)}{\sigma(Q_1^0)} \leq C\left(\frac{\|T_r\|_{\beta, Q_1^0}}{|Q_1^0|^{\frac{\beta}{n}}}\right)^\delta,$$

at any $0 < r < \frac{1}{4}$, since $T_r \subset Q_1^0$. Then the previous estimations we'll get

$$r^{(\beta-1)\delta}\ln\frac{a}{r} > C_1,$$

where $C_1 > 0$ doesn't depends on r, which can't hold at sufficiently small r. We come to the contradiction that $\sigma \in A_\infty^\beta$, i.e. $\sigma \notin A_\infty^\beta$.

At proving the base results we'll use the following facts.

Lemma 2[11]. *Let A be a bounded set in R^n and let for every $x \in A$ be given a closed bolls $B(x, r(x))$ with the center in x and radius $r(x)$.*

Then from $\{B(x, r(x))\}_{x \in A}$ we can choose the sequence of the bolls $\{B_k\}$ satisfying the following conditions:

i) this sequence covers the set A, i.e. $A \subset \bigcup_k B_k$;

ii) non point from R^n is contained more than in μ_n bolls of the sequence $\{B_k\}$, i.e. for every point $z \in R^n$.

$$\sum_k \chi_{B_k}(z) \leq \mu_n,$$

where μ_n -is a number depending only on n.

Lemma 3 ([1], theorem 1.2.4/1). *Let ϕ be a measurable non-negative function in R^n, $u \in C^{0,1}(\Omega)$, Ω be an open subset of R^n*

Then

$$\int_\Omega \phi(x)|\nabla u| dx = \int_0^\infty dt \left(\int_{E_t} \phi(x) ds(x)\right),$$

where S is $(n-1)$-dimensional Housdorf measure, $E_t=\{x\in\Omega:|u(t)=t|\}$.

Lemma 4 [12]. *Let v be a function from the class $A_\infty(\Omega)$. Then there will be found such $C>0, r>1$ that for any boll $Q\in V$ the "inverse Hölder inequality"*

$$\left(\frac{1}{|Q\cap\Omega|}\int_{Q\cap\Omega} v^r dx\right)^{\frac{1}{r}}\le C\left(\frac{1}{|Q\cap\Omega|}\int_{Q\cap\Omega} v dx\right)$$

is valid.

Proof of theorem 2. There will be found such $a\in R^1$ that

$$|x\in\Omega: u(x)>a|\le\frac{1}{2}|\Omega|\le|x\in\Omega: u(x)\ge a|.$$

Let $\Omega'=\{x\in\Omega: u(x)>a\}, \Omega''=\{x\in\Omega: u(x)<a\}, 0<\alpha<\infty, \Omega_\alpha=\{x\in\Omega: u(x)>a+\alpha\}$.

Then by view of choice a, $|\Omega\setminus\Omega'|\ge\frac{1}{2}|\Omega|$ and $|\Omega\setminus\Omega''|\ge\frac{1}{2}|\Omega|$.

Let $\alpha>0$ be such that $\Omega_{2\alpha}$ isn't empty. If such α doesn't exist then we'll consider the estimation in Ω'' and we'll suppose $\Omega_\alpha=\{x\in\Omega: u(x)<a-\alpha\}$.

For any fixed point x there will be found a boll Q

$$\left|Q^x_{\rho(x,\alpha)}\cap\Omega\setminus\Omega_\alpha\right|^{\frac{1}{r'}}\left(\int_{Q^x_{\rho(x,\alpha)}\cap\Omega} v^r dy\right)^{\frac{1}{r}}=\gamma\left(\int_{Q^x_{\rho(x,\alpha)}\cap\Omega} v\ dy\right), \tag{6}$$

where $0<\gamma<\frac{1}{2^{\frac{1}{r'}}}$ it will be chosen later. The existence of such boll follows from the following concepts. Let's consider the auxiliary function

$$F(t)=\left|Q^x_t\cap\Omega\setminus\Omega_\alpha\right|^{\frac{1}{r'}}\left(\int_{Q^x_t\cap\Omega} v^r dy\right)^{\frac{1}{r}}-\gamma\left(\int_{Q^x_t\cap\Omega} v dy\right),$$

continuous on $[0,\infty)$. $F(t_1)<0$ at sufficient small $t_1>0$. At $t=d(\Omega)$ by view of the Hölder inequality, subject to the value γ and $|\Omega\setminus\Omega'|\ge\frac{1}{2}|\Omega|$ we'll get

$$F(d(\Omega))=\left(\frac{1}{2}|\Omega|\right)^{\frac{1}{r'}}\left(\int_\Omega v^r dy\right)^{\frac{1}{r}}-\gamma\left(\int_\Omega v dy\right)\ge 0.$$

Then by the Cauchy theorem we conclude that there will be found $t=t_2, t_1\le t_2\le d(\Omega)$ for which $F(t_2)=0$, i.e. it holds (6) when $\rho(x,\alpha)=t_2$.

The system of the bolls $\{Q^x_{\rho(x,\alpha)}: x\in\Omega_{2\alpha}\}$ makes the covering for the set $\Omega_{2\alpha}$. By means of lemma 2 we can choose the subset $\{Q^i\}$ $(i=1,2,3,...)$, of finite multiplicity. By view of the choice of bolls, for the every boll Q^i holds

$$\left|Q^i\cap\Omega\setminus\Omega_\alpha\right|^{\frac{1}{r'}}\left(\int_{Q^i\cap\Omega} v^r dy\right)^{\frac{1}{r}}=\gamma\left(\int_{Q^i\cap\Omega} v dy\right), \tag{7}$$

Two variants are possible for every boll Q:

$$\text{a) } \left|\Omega_{2\alpha}\cap Q^i\right|^{\frac{1}{r'}}\left(\int_{Q^i\cap\Omega} v^r dy\right)^{\frac{1}{r}} < \gamma\left(\int_{Q^i\cap\Omega} v dy\right); \text{ b) } \left|\Omega_{2\alpha}\cap Q^i\right|^{\frac{1}{r'}}\left(\int_{Q^i\cap\Omega} v^r dy\right)^{\frac{1}{r}} \geq \gamma\left(\int_{Q^i\cap\Omega} v dy\right).$$

At the first case subject to a), by means of the Hölder inequality we have

$$v\left(\Omega_{2\alpha}\cap Q^i\right)\leq \gamma v\left(Q^i\cap\Omega\right), \tag{8}$$

on the other hand

$$v\left(Q^i\cap\Omega\right)= v\left(Q^i\cap\Omega\setminus\Omega_\alpha\right)+v\left(Q^i\cap\Omega_\alpha\right), \tag{9}$$

by means of the Hölder inequality and subject to (7) in the first additive in (9) we have

$$v\left(Q^i\cap\Omega\right)\leq \gamma v\left(Q^i\cap\Omega\right)+v\left(Q^i\cap\Omega_\alpha\right),$$

i.e.

$$v\left(Q^i\cap\Omega\right)\leq \frac{1}{1-\gamma}v\left(Q^i\cap\Omega_\alpha\right),$$

therefore from (8) we find

$$v\left(\Omega_{2\alpha}\cap Q^i\right)\leq \frac{1}{1-\gamma}v\left(\Omega_\alpha\cap Q^i\right)$$

At the second case by the construction and subject to b) we have

$$\min\left\{\left|\Omega_{2\alpha}\cap Q^i\right|, \left|Q^i\cap\Omega\setminus\Omega_\alpha\right|\right\}\geq\left[\gamma\frac{v\left(Q^i\cap\Omega\right)}{\left(\int_{Q^i\cap\Omega} v^r dy\right)^{\frac{1}{r}}}\right]^{r'}.$$

Then by the condition $\tilde{I}_\lambda$ on Ω and lemma 1 at every t, $\alpha\leq t\leq 2\alpha$ we'll have

$$\operatorname*{mes}_{n-1}\left\{x\in\Omega\cap Q^i : u(x)=t+a\right\}\geq\theta\left[\gamma\frac{v\left(Q^i\cap\Omega\right)}{\left(\int_{Q^i\cap\Omega} v^r dy\right)}\right]^{r'\lambda},$$

and then on the basis of lemma 3

$$\int_{Q^i\cap\Omega_\alpha\setminus\Omega_{2\alpha}}|Du|dy=\int_\alpha^{2\alpha}dt\operatorname*{mes}_{n-1}\left\{x\in\Omega\cap Q^i : u(x)=t+a\right\}\geq\alpha\theta\left[\gamma\frac{v\left(Q^i\cap\Omega\right)}{\int_{Q^i\cap\Omega} v^r dy}\right]^{r'\lambda}.$$

Hence by means of the Hölder inequality

$$1\leq\frac{1}{\alpha^q}\left\{\frac{1}{\theta}\left[\frac{\left(\int_{Q^i\cap\Omega} v^r dy\right)^{\frac{1}{r}}}{\gamma v\left(Q^i\cap\Omega\right)}\right]^{r'\lambda}\left(\int_{Q^i\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\omega^{1-p'}dy\right)^{\frac{1}{p'}}\times\right.$$

$$\times\left(\int_{Q^i\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\omega|Du|^p dy\right)^{\frac{1}{p}}\Bigg\}^q,$$

then

$$v\left(\Omega_{2\alpha}\cap Q^i\right)\le\left\{\frac{1}{\theta\gamma^{r'\lambda}}\left(\int_{Q^i\cap\Omega}v^r dy\right)^{(r'-1)\lambda}v\left(Q^i\cap\Omega\right)^{\frac{1}{q}-r'\lambda}\times\right.$$

$$\left.\times\left(\int_{Q^i\cap\Omega}\omega^{1-p'}dy\right)^{\frac{1}{p'}}\right\}\frac{1}{\alpha^q}\left(\int_{Q^i\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\omega\ |Du|^p dy\right)^{\frac{q}{p}}.$$

Further from the condition on pair of weights (v,ω)

$$v\left(\Omega_{2\alpha}\cap Q^i\right)\le\left(\frac{A_{pq}^\lambda}{\theta\gamma^{r'\lambda}}\right)^q\frac{1}{\alpha^q}\left(\int_{Q^i\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\omega|Du|^p dy\right)^{\frac{q}{p}}.$$

Then in both cases a) and b) we have

$$v\left(\Omega_{2\alpha}\cap Q^i\right)\le\frac{\gamma}{1-\gamma}v\left(\Omega_\alpha\cap Q^i\right)+\left(\frac{A_{pq}^\lambda}{\theta\gamma^{r'\lambda}\alpha}\right)^q\left(\int_{Q^i\cap\Omega_\alpha\setminus\Omega_{2\alpha}}\omega|Du|^p dy\right)^{\frac{q}{p}}.$$

Summing the previous inequality by i, subject to the finite multiplicity $\{Q^i\}$ and $q\ge p$ we'll get

$$v\left(\Omega_{2\alpha}\right)\le\frac{\mu_n\gamma}{1-\gamma}v\left(\Omega_\alpha\right)+\mu_n\left(\frac{A_{pq}^\lambda}{\theta\gamma^{r'\lambda}\alpha}\right)^q\left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}}\omega|Du|^p dy\right)^{\frac{q}{p}}.$$

Integrating the last inequality we'll have

$$\int_0^\infty v\left(\Omega_{2\alpha}\right)d\alpha^q\le\frac{\mu_n\gamma}{1-\gamma}\int_0^\infty v\left(\Omega_\alpha\right)d\alpha^q+\mu_n\left(\frac{A_{pq}^\lambda}{\theta\gamma^{r'\lambda}}\right)^q\int_0^\infty\frac{d\alpha}{\alpha}\left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}}\omega|Du|^p dy\right)^{\frac{q}{p}},$$

hence, by means of the Minkovsky inequality

$$\frac{1}{2^q}\int_{\Omega'}|u-a|^q vdy\le\frac{\mu_n\gamma}{1-\gamma}\int_{\Omega'}|u-a|^q vdy-\mu_n\left(\frac{A_{pq}^\lambda\ln^{1/q}2}{\theta\gamma^{r'\lambda}}\right)^q\left(\int_{\Omega'}\omega|Du|^p dy\right)^{\frac{q}{p}}.$$

Let's choose now $0<\gamma<\frac{1}{2^{\frac{1}{r'}}}$ such that

$$\frac{1}{2^q}-\frac{\mu_n\gamma}{1-\gamma}>0.$$

Then

$$\left(\int\limits_{\Omega'}|u-a|^q v dy\right)^{\frac{1}{q}} \le C_{q,r}\frac{A_{pq}^{\lambda}}{\theta}\left(\int\limits_{\Omega'}\omega|Du|^p dy\right)^{\frac{1}{p}},$$

where $C_{q,r}=\frac{\mu_n^{\frac{1}{q}}}{\gamma^{r'\lambda}}\left(\frac{1}{2^q}-\frac{\mu_n\gamma}{1-\gamma}\right)^{-\frac{1}{q}}$; $C_{q,r}\le 2^{1+\frac{1}{q}}\mu_n^{\frac{1}{q}}\left(2^{q+1}\mu_n+1\right)^{r'\lambda}$ if we choose γ from the $\frac{\mu_n\gamma}{1-\gamma}=\frac{1}{2^{q+1}}$.

The analogous inequality

$$\left(\int\limits_{\Omega''}|u-a|^q v\, dy\right)^{\frac{1}{q}} \le C_{q,r}\frac{A_{pq}^{\lambda}}{\theta}\left(\int\limits_{\Omega''}\omega|Du|^p dy\right)^{\frac{1}{p}}$$

holds in Ω'' too. Then last inequalities give

$$\left(\int\limits_{\Omega}|u-a|^q v dy\right)^{\frac{1}{q}} \le C_{q,r}\frac{A_{pq}^{\lambda}}{\theta}\left(\int\limits_{\Omega}\omega|Du|^p dy\right)^{\frac{1}{p}}. \tag{10}$$

Let's show now that

$$2^q\int\limits_{\Omega}|u-a|^q v\, dy \ge \int\limits_{\Omega}|u-\overline{u}|^q v\, dy. \tag{11}$$

By means of the Minkovsky inequality

$$\left(\int\limits_{\Omega}|u-\overline{u}|^q v dy\right)^{\frac{1}{q}} \le \left(\int\limits_{\Omega}|u-a|^q v\, dy\right)^{\frac{1}{q}} + \left(\int\limits_{\Omega}|a-\overline{u}|^q v\, dy\right)^{\frac{1}{q}}. \tag{12}$$

Subject to the estimation

$$\left(\int\limits_{\Omega}|a-\overline{u}|^q v\, dy\right)^{\frac{1}{q}} \le |a-\overline{u}|v(\Omega)^{\frac{1}{q}} \le v(\Omega)^{\frac{1}{q}-1}\left|\int\limits_{\Omega}(u-a)v\, dy\right| \le \left(\int\limits_{\Omega}|u-a|^q v\, dy\right)^{\frac{1}{q}}$$

on the second sum (12) we'll get (11). Allowing for (11) in (10) we'll get the estimation of theorem 2.

The theorem is proved.

Proof of theorem 1. Let $v\in A_\infty(\Omega)$ and (4) be fulfilled. On the base of lemma 4 it will be found $C>0$, $r>1$ such that

$$\left(\frac{1}{|Q\cap\Omega|}\int\limits_{Q\cap\Omega}v^r dy\right)^{\frac{1}{r}} \le C\left(\frac{1}{|Q\cap\Omega|}\int\limits_{Q\cap\Omega}v\, dy\right)$$

for the any boll $Q\in V$. Let's note that here C,r, depends on M,δ, from the condition $v\in A_\infty(\Omega)$. Subject to this inequality we'll get that the condition (5) of theorem 2 is fulfilled. Therefore the statement of theorem 1 follows from the statement of theorem 2. The theorem is proved.

Remark. Let $\beta\ge 1$, the boll $Q_*=\beta Q$. It the surface Σ in the definition $\bar{I}_\lambda$ dividing the sets A,B there in Ω_Q, divides them in $\Omega\cap Q_*$ also, then the new condition can to change $\tilde{I}_\lambda$ at the theorems 1,2 if v is dubling: $v(\Omega\cap Q_*)\le C_\beta v(\Omega\cap Q)$ with some $C_\beta>1, \forall Q\in V$.

Author expresses his gratitude to prof. Soltanov K.N. and c.ph.m.sc. Akhmedov M.I. for their attention to the work and valuable remarks.

References

[1]. Mazya V.G. Sobolev spaces. Izd. LGU, 1985, 415p. (in Russian)
[2]. Goldshtein V.M., Reshetnyak Yu.G. Introduction to the theory of the functions with the generalized derivatives and quasiconformal mappings. M., Nauka, 1983, 284p. (in Russian)
[3]. Sawyer E. And Wheeden R.L. Weighted inequalities for fractional integrals of Eculidean and homogeneous spaces. Amer. J. Math., 114 (1992), 813-874.
[4]. Franchi B., Gutierres C.E. and Wheeden R.L. Weighted Sobolev-Poincare inequalities for Grushin type operators. Comm. P.D.E. 19(3-4), 1994, 523-604.
[5]. Carofolo N., Nheu D. Isoperimetric and Sobolev inequalities for Garnot-Carateodory spaces spaces and the existence of minimal surfaces// Com. Pure and Appl. Math. 1996, V. XLIX, p.1081-1144.
[6]. Gabidzashvili M. And Kokilashvili V. Two weight weak teak type inequalities for fractional type integrals. Ceskoslovenska Akademie Ved., 45 (1989), 1-11.
[7]. Long R.L., Nie F.S. Weighted Sobolev inequality and eigenvalue estimates of Shrodinger operator, Harmonic Analysis (Tianjin, 1988), Leeture Notes in Math., 1994, Springer, 1991.
[8]. Chua S.K. Weighted Sobolev's inequality on domains satisfying Boman chain condition.// Proc. A.M.S. 117 (1993), p.449-457.
[9]. Besov O.V. The imbedding theorem of Sobolev type fot the nonregular domains // Matem. sbornic, v.192, №3, (2001), p.3-26. (in Russian)
[10]. Kilpelainen T., Maly J. Sobolev ineqialities on sets with irreqular boundaries // Z. Anal. Anwendungen, v.19, №2 (2000), p.369-380.
[11]. Gusman M. Differentiating of integrals in R^n ., "Mir", M., 1978, 200p. (in Russian)
[12]. Coifman R.R., Fefferman C.L. Weighted norm inequalities for maximal functions and singular integrals.// Studia Math., 1974, v.51, p.241-150.

Farman I. Mamedov
Azerbaijan Technical University, Depart. Mathematical.
25, H.Javid av., 370073, Baku, Azerbaijan.

Received March 22, 2000; Revised April 12, 2001.
Translated Mamedova V.I.

MAMEDOV I.T., SALMANOVA Sh.Yu.

THE A.D. ALEKSANDROV TYPE INEQUALITY FOR A CLASS OF SECOND ORDER EQUATIONS WITH NON-NEGATIVE CHARACTERISTIC FORM

Abstract

The analogue of the classical A.D. Aleksandrov inequality is proved for a class degenerating on boundary of domain of second order elliptic-parabolic equations of non-divergent structure with generally speaking discontinuous coefficients.

Let $\mathbf{R}_{n+1}$ be an $(n+1)$ dimensional Euclidean space of the points $(x,t)=(x_1,...,x_n,t)$, $Q_T=\Omega\times(0,T)$ be a cylindrical domain in $\mathbf{R}_{n+1}$, where Ω is a bounded n-dimensional domain with the boundary $\partial\Omega$, and $T\in(0,\infty)$. Let further $Q_0=\{(x,t):x\in\Omega,\,t=0\}$, and $\Gamma(Q_T)=Q_0\cup(\partial\Omega\times[0,T])$ be a parabolic boundary of Q_T. Consider the following second order degenerate elliptic-parabolic operator in Q_T

$$L=\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)+w(x,t)\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial t}$$

in assumption that $\|a_{ij}(x,t)\|$ is a real symmetric matrix where for all $(x,t)\in Q_T$ and any n-dimensional vector ξ

$$\gamma|\xi|\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\le\gamma^{-1}|\xi|^2,\ \gamma\in(0,1]=const. \tag{1}$$

We determine the function $w(x,t)$ by the equality $w(x,t)=\psi_1(\rho)\psi_2(t)\varphi(T-t)$, where $\rho=dist(x,\partial\Omega)$, ψ_1,ψ_2 and φ are continuous, non-negative and non-decreasing functions of themselves arguments, where

$$\int_0^T\left(\frac{\varphi(v)}{v^2}\right)^{n+1}dv<\infty. \tag{2}$$

Besides we'll assume that all coefficients of the operator L are measurable in Q_T functions.

Denote by $W_w^{2,2}(Q_T)$ a Banach space of the functions $u(x,t)$ given on Q_T with the finite norm

$$\|u\|_{W_w^{2,2}(Q_T)}=\|u\|_{C(Q_T)}+\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_{L_{n+1}(Q_T)}+$$
$$+\sum_{i,j=1}^{n}\left\|\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|_{L_{n+1}(Q_T)}+\left\|\frac{\partial u}{\partial t_i}\right\|_{L_{n+1}(Q_T)}+\left\|w\frac{\partial^2 u}{\partial t^2}\right\|_{L_{n+1}(Q_T)},$$

and let $\dot{W}_w^{2,2}(Q_T)$ be a subspace of $W_w^{2,2}(Q_T)$, dense set in which is a set of all functions from $C^\infty(\overline{Q}_T)$ vanishing on $\Gamma(Q_T)$.

The aim of the present note is determination of conditions on the coefficients $b_1(x,t),...,b_n(x,t)$ and $c(x,t)$, for fulfillment of which for arbitrary functions $u\in\dot{W}_w^{2,2}(Q_T)$ the estimation

$$\|u\|_{C(Q_T)}\le C_1\|Lu\|_{L_{n+1}(Q_T)} \tag{3}$$