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UNIQUE STRONG SOLVABILITY OF THE FIRST BOUNDARY VALUE PROBLEM FOR PARABOLIC GILBARG-SERRIN EQUATION IN THE PARABOLOID TYPE DOMAINS

Abstract

In the paper the first boundary value problem for parabolic Gilbarg-Serrin equation is considered in paraboloid type domains. Unique strong solvability of this problem in weighted Sobolev space is proved.

Let $\mathbf{R}_{n+1}$ and $\mathbf{E}_n$ be $(n+1)$-dimensional and n-dimensional Euclidean spaces of the points $(x,t)=(x_1,...,x_n,t)$ and $x=(x_1,...,x_n)$ respectively. Let's call the domain G, situated in half-space $t<0$, P-domain, if its intersection with every hyperplane $t=-\tau\,(\tau>0)$ has the form $\left\{x:\frac{x}{2\sqrt{-\tau}}\in D\right\}$, where D is some bounded domain in $\mathbf{E}_n$. The domain D is called generating for the domain G. Let further $G_T=G\cap\{(x;t):t>-T\}$, where $T\in(0,+\infty)$. Let's consider in G_T the first boundary value problem

$$\mathcal{L}u=\Delta u+\lambda\sum_{i,j=1}^{n}\frac{x_i x_j}{4(-t)}\frac{\partial^2 u}{\partial x_i\partial x_j}-\frac{\partial u}{\partial t}=f(x,t), \qquad (1)$$

$$u\big|_{\partial G_T}=0, \qquad (2)$$

where the numerical parameter λ satisfies the following condition

$$-\frac{1}{d^2}<\lambda<\infty,\ d=\sup_{\xi\in D}|\xi|. \qquad (3)$$

The aim of the given paper is to prove the unique strong (almost everywhere) solvability of the first boundary value problem (1)-(2) in corresponding weighted Sobolev spaces. The equation (1) is called the parabolic Gilbarg-Serrin equation. We know [1-2], that the first boundary value problem for parabolic equations of the second order of non-divergence structure is uniquely strongly solvable in the space $W_p^{2,1}\,(1<p<\infty)$ in an arbitrary boundary cylinder domain, if the coefficients of equation are uniformly continuous, the right hand side belongs to the space L_p, and the domain of cylinder foundation is the double smooth surface. In case $p=2$ the analogous fact holds also for some class of equations with discontinuous coefficients in particular satisfying the parabolic Cordes condition [3-6]. As to parabolic equations given in non-cylindrical domains then even for equations with smooth coefficients the unique solvability of the first boundary value problem holds only in weighted Sobolev spaces [7]. It is easy to see that the equation (1) satisfy the parabolic Cordes condition not at all the values of the parameter λ. Nevertheless as it is shown at the given paper the unique strong solvability of the first boundary value problem (1)-(2) in weighted Sobolev spaces holds at any value of the parameter λ, satisfying the condition (3).

Let's agree now in some notations. By u_i and u_{ij} we'll denote the derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial u}{\partial x_i \partial x_j}$ respectively, $u_{xx} = (u_{ij});\ i,j = \overline{1,n};\ u_x^2 = \sum_{i=1}^{n} u_i^2;\ u_{xx}^2 = \sum_{i,j=1}^{n} u_{ij}^2$. We'll suppose that the numerical parameter γ satisfies the condition

$$\gamma \in \left(\frac{n^2\left(\lambda - \frac{1}{d^2}\right) + 2\lambda n}{8}\ ,\ +\infty \right). \qquad (4)$$

Let $C_0^\infty(G_T)$ be the space of all infinitely differentiable functions with compact support in G_T, $A_0^\infty(G_T)$ is a space of all infinitely differentiable in G_T functions for which the integral $\int\limits_{G_T} (-t)^\gamma u^2 dxdt$ is finite.

Let's denote by $L_{2,\gamma}(Q_T)$ and $W_{2,\gamma}^{2,1}(Q_T)$ the Banach spaces of the functions $u(x,t)$, given on G_T with the finite norms

$$\|u\|_{L_{2,\gamma}(G_T)} = \left(\int\limits_{G_T} (-t)^\gamma u^2 dxdt \right)^{1/2}$$

and

$$\|u\|_{W_{2,\gamma}^{2,1}(G_T)} = \left(\int\limits_{G_T} \left((-t)^\gamma u^2 + (-t)^{\gamma+1} u_x^2 + (-t)^{\gamma+2} u_{xx}^2 + (-t)^{\gamma+2} u_t^2 \right) dxdt \right)^{1/2}$$

respectively. Let finally the $\dot{W}_{2,\gamma}^{2,1}(G_T)$ is the subspace of $W_{2,\gamma}^{2,1}(G_T)$ dense set in which there is a totality of all functions from $A_0^\infty(G_T)$ vanishing on ∂G_T. Let's denote by S_T and D_T the set $\partial G \cap \{(x,t): -T \le t \le 0\}$ and $G \cap \{(x,t): t = -T\}$ respectively. We'll use two statements which are proved at the paper of the author [8].

Lemma 1. *For any function $\vartheta \in \dot{W}_{2,\gamma}^{2,1}(G_T)$ the inequality*

$$\varepsilon \int\limits_{G_T} (-t)^\gamma \vartheta_x^2 dxdt \le - \int\limits_{G_T} (-t)^\gamma \vartheta \mathcal{L} \vartheta dxdt \qquad (5)$$

is true, where the positive constant ε depends only on λ, n, d.

Lemma 2. *For any function $v(x,t) \in A_0^\infty(G_T)$ equal to zero on D_T and near S_T, the estimation*

$$\int\limits_{G_T} (-t)^\gamma \frac{u^2}{4(-t)} dxdt \le \frac{4}{n^2} \int\limits_{G_T} (-t)^\gamma \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j dxdt \qquad (6)$$

is true.

We'll denote by $\mathcal{A}_0^\infty(G_T)$ the class of functions given in lemma 2. Let's ingage in obtaining the basic coercive estimations for the operator $\mathcal{L}$. To this aim let's denote by K_R the intersection of P- domain K with the set $\{(x,t): t > -T\}$ such that generating domain for K is the ball of the radius R strongly interior with respect to the D.

Let's finally $K_{R,\sigma} = K_R \cap \{(x,t): t > -\sigma\}$, where σ is sufficiently small positive constant.

Lemma 3. *For any function* $u(x,t)\in \mathcal{A}_0^\infty(K_R)$ *the estimation*

$$\int_{K_{R,\sigma}}(-t)^{\gamma+2}u_{xx}^2dxdt \le C_1 \int_{K_{R,\sigma}}(-t)^{\gamma+2}(\mathcal{L}u)^2dxdt \tag{7}$$

is true.

Here and further by C we'll denote the positive constants which depends only on γ, λ and d.

Proof. Let's fix an arbitrary k, $1\le k\le n$, and, allowing for that

$$\mathcal{L}u=\Delta u+\lambda\sum_{i,j=1}^n\left(\frac{x_ix_j}{4(-t)}u_j\right)_i-\lambda(n+1)\sum_{i=1}^n\frac{x_iu_i}{4(-t)}-u_t, \tag{8}$$

in the estimation (5) we put the function ϑ instead of the function $-tu_k$. We'll get

$$\varepsilon\int_{K_{R,\sigma}}(-t)^{\gamma+2}\sum_{i=1}^n u_{ki}^2dxdt\le-\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_k\mathcal{L}((-t)u_k)dxdt, \tag{9}$$

where σ is an arbitrary fixed sufficiently small positive number.

But on the other hand

$$\mathcal{L}((-t)u_k)=\Delta((-t)u_k)+\lambda\sum_{i,j=1}^n\left(\frac{x_ix_j}{4(-t)}(-t)u_{kj}\right)_i-\lambda(n+1)\sum_{i=1}^n\frac{x_iu_i}{4(-t)}(-t)u_{ki}-$$
$$-u_{kt}+u_k=(-t)\mathcal{L}(u_k)+u_k. \tag{10}$$

Therefore

$$J_1=-\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_k\left((-t)\mathcal{L}u_k+u_k\right)dxdt=-\int_{K_{R,\sigma}}(-t)^{\gamma+2}u_k\mathcal{L}u_kdxdt-$$
$$-\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_k^2dxdt. \tag{11}$$

On the other hand

$$-\int_{K_{R,\sigma}}(-t)^{\gamma+2}\left(u_k\mathcal{L}u_k-u_k(\mathcal{L}u)_k\right)dxdt=\int_{K_{R,\sigma}}(-t)^{\gamma+2}u_{kk}\mathcal{L}udxdt-$$
$$-\int_{K_{R,\sigma}}(-t)^{\gamma+2}\frac{\lambda}{2(-t)}\sum_{i=1}^n x_iu_{ki}u_kdxdt=\int_{K_{R,\sigma}}(-t)^{\gamma+2}u_{kk}\mathcal{L}udxdt-$$
$$-\frac{\lambda}{4}\int_{K_{R,\sigma}}(-t)^{\gamma+1}\sum_{i=1}^n x_i\left(u_k^2\right)_idxdt. \tag{12}$$

Allowing for (12) in (11) we get

$$J_1=\int_{K_{R,\sigma}}(-t)^{\gamma+2}u_{kk}\mathcal{L}udxdt-\frac{\lambda n-4}{4}\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_k^2dxdt,$$

that together with (9)-(11) gives

$$\varepsilon\int_{K_{R,\sigma}}(-t)^{\gamma+2}\sum_{i=1}^n u_{ki}^2dxdt\le\int_{K_{R,\sigma}}(-t)^{\gamma+2}u_{kk}\mathcal{L}udxdt+\frac{\lambda n-4}{4}\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_k^2dxdt. \tag{13}$$

From (13) for any $\varepsilon_1>0$ we have

$$\varepsilon\int_{K_{R,\sigma}}(-t)^{\gamma+2}\sum_{i=1}^n u_{ki}^2dxdt\le\frac{\varepsilon_1}{2}\int_{K_{R,\sigma}}(-t)^{\gamma+2}u_{kk}^2dxdt+\frac{1}{2\varepsilon_1}\int_{K_{R,\sigma}}(-t)^{\gamma+2}(\mathcal{L}u)^2dxdt+$$

$$+\frac{|\lambda n-4|}{4}\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_k^2\,dxdt\quad. \tag{14}$$

Summing by k from 1 to n, from the last estimation we get

$$\varepsilon\int_{K_{R,\sigma}}(-t)^{\gamma+2}u_{xx}^2dxdt<\frac{\varepsilon_1}{2}\int_{K_{R,\sigma}}(-t)^{\gamma+2}u_{xx}^2dxdt+\frac{1}{2\varepsilon_1}\int_{K_{R,\sigma}}(-t)^{\gamma+2}(\mathcal{L}u)^2dxdt+$$

$$+\frac{|\lambda n-4|}{4}\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_x^2dxdt. \tag{15}$$

It is easy to see that the inequality (5) holds if we replace the exponent γ by $\gamma+1$. Therefore for any $\varepsilon_2>0$ we get

$$M_2=\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_x^2dxdt\le C_2\int_{K_{R,\sigma}}(-t)^{\gamma+1}u\mathcal{L}udxdt\le\frac{C_2\varepsilon_2}{2}\int_{K_{R,\sigma}}(-t)^{\gamma}u^2dxdt+$$

$$+\frac{C_2}{2\varepsilon_2}\int_{K_{R,\sigma}}(-t)^{\gamma+2}(\mathcal{L}u)^2dxdt. \tag{16}$$

On the other hand according to (6)

$$\int_{K_{R,\sigma}}(-t)^{\gamma+1}\frac{u^2}{4(-t)}dxdt\le\frac{4}{n^2}\int_{K_{R,\sigma}}(-t)^{\gamma+1}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_jdxdt=$$

$$=\frac{4}{n^2}\int_{K_{R,\sigma}}(-t)^{\gamma+1}\left(\sum_{i=1}^{n}\frac{x_i}{2\sqrt{-t}}u_i\right)^2dxdt\le\frac{4d^2}{n^2}\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_x^2dxdt.$$

Therefore

$$\int_{K_{R,\sigma}}(-t)^{\gamma}u^2dxdt\le\frac{16d^2}{n^2}\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_x^2dxdt. \tag{17}$$

Allowing for (17) in (16) we conclude

$$\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_x^2dxdt\le C_3\varepsilon_2\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_x^2dxdt+\frac{C_3}{\varepsilon_2}\int_{K_{R,\sigma}}(-t)^{\gamma+2}(\mathcal{L}u)^2dxdt.$$

Now choosing $\varepsilon_2=\frac{1}{2C_3}$ from the last inequality we get

$$\int_{K_{R,\sigma}}(-t)^{\gamma+1}u_x^2dxdt\le C_4\int_{K_{R,\sigma}}(-t)^{\gamma+2}(\mathcal{L}u)^2dxdt. \tag{18}$$

Allowing for (18) in (15) we arrive at the estimation

$$\varepsilon\int_{K_{R,\sigma}}(-t)^{\gamma+2}u_{xx}^2dxdt-\frac{\varepsilon_1}{2}\int_{K_{R,\sigma}}(-t)^{\gamma+2}u_{xx}^2dxdt\le\frac{1}{2\varepsilon_1}\int_{K_{R,\sigma}}(-t)^{\gamma+2}(\mathcal{L}u)^2dxdt+$$

$$+C_5\frac{1}{2\varepsilon_1}\int_{K_{R,\sigma}}(-t)^{\gamma+2}(\mathcal{L}u)^2dxdt. \tag{19}$$

Now choosing $\varepsilon_1=\varepsilon$ and tending σ to zero from (19) we get the required estimation (7). The lemma is proved.

Lemma 4. *If the conditions of previous lemma are fulfilled, then any function* $u(x,t)\in\mathcal{A}_0^\infty(K_R)$ *the estimation*

$$\|u\|_{W_{2,\gamma}^{2,1}(K_R)}\le C_6\|\mathcal{L}u\|_{L_{2,\gamma+2}(K_R)} \tag{20}$$

is true.

By virtue of lemma 2 the inequality (18) and the statement of lemma 3 to prove the estimation (20) it is sufficient to prove that

$$\int_{K_R}(-t)^{\gamma+2}u_t^2\,dxdt \le C_7 \int_{K_R}(-t)^{\gamma+2}(\mathcal{L}u)^2\,dxdt. \tag{21}$$

Let δ_{ij} is Cronecker's symbol. We have

$$u_t^2 = \left[\sum_{i,j=1}^{n}\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}\right)u_{ij}-\mathcal{L}u\right]^2 \le 2\left(\sum_{i,j=1}^{n}\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}\right)u_{ij}\right)^2+2(\mathcal{L}u)^2 \le$$

$$\le C_8\sum_{i,j=1}^{n}u_{ij}+2(\mathcal{L}u)^2. \tag{22}$$

Now it is sufficient to apply lemma 3, and from (22) follows the required inequality (21). The lemma is proved.

Now let's make substitute of variables $\xi=\frac{x}{2\sqrt{-t}}$, $\eta=\ln\frac{1}{-t}$. It is easy to see that Jacobian of this transformation equals to $2^n e^{-\frac{\eta(n+2)}{2}}$. At such transformation our domain $K_{R,\sigma}$ will be mapped into the cylinder $\mathbf{C}^{\sigma}$ whose foundations are n-dimensional sphere of the radius R and arranged on hypersurfaces $\eta=\ln\frac{1}{T}$ and $\eta=\ln\frac{1}{\sigma}$ (without loss of generality we suppose $T<1$). Let $\widetilde{\mathcal{L}}\widetilde{u}$ be a product e^{η} and the image of $\mathcal{L}u$.

Then from lemma 4 follows that

$$\int_{\mathbf{C}^{\sigma}}e^{-\gamma\eta}e^{-\frac{\eta(n+2)}{2}}\left[\widetilde{u}^2+\widetilde{u}_{\xi}^2+\widetilde{u}_{\xi\xi}^2+\left(\sum_{i=1}^{n}\xi_i\widetilde{u}_{\xi_i}+2\widetilde{u}_{\eta}\right)^2\right]d\xi d\eta \le$$

$$\le C_9\int_{\mathbf{C}^{\sigma}}e^{-\gamma\eta}e^{-\frac{\eta(n+2)}{2}}\left(\widetilde{\mathcal{L}}\widetilde{u}\right)^2 d\xi d\eta, \tag{23}$$

where $\widetilde{u}$ is an image of the function u and the constant C_9 doesn't depend on σ.

Let's denote by $\mathbf{C}_1^{\sigma}$ a cylinder that is coaxial and equihigh to $\mathbf{C}^{\sigma}$ whose bases are the balls of radius $2R$.

Lemma 5. *Let $u(x,t)$ be infinitely differentiable function from $W_{2,\gamma}^{2,1}(K_{2R,\sigma})$, vanishing for $t=-T$. Then there exist the constant C_{10}, depending only on λ, n, d and R such that*

$$\int_{\mathbf{C}^{\sigma}}e^{-\gamma\eta}e^{-\frac{\eta(n+2)}{2}}\left[\widetilde{u}^2+\widetilde{u}_{\xi}^2+\widetilde{u}_{\xi\xi}^2+\left(\sum_{i=1}^{n}\xi_i\widetilde{u}_{\xi_i}+2\widetilde{u}_{\eta}\right)^2\right]d\xi d\eta \le$$

$$\le C_{10}\left(\int_{\mathbf{C}_1^{\sigma}}e^{-\gamma\eta}e^{-\frac{\eta(n+2)}{2}}\left[\left(\widetilde{\mathcal{L}}\widetilde{u}\right)^2+\widetilde{u}^2+\widetilde{u}_{\xi}^2\right]d\xi d\eta\right). \tag{24}$$

Proof. Let's consider the following function $z(\xi)$: $z(\xi)=1$ if $\xi\in B_1$, $z(\xi)$ out of B_2, $z(\xi)\in C_0^{\infty}(B_2)$, $0\le z(\xi)\le 1$ and there exists the constant C_{11} depending only on n, such that

$$|z_i| \le \frac{C_{11}}{R};|z_{ij}| \le \frac{C_{11}}{R^2} \quad (i,j=\overline{1,n}). \tag{25}$$

Here B_1 and B_2 are foundations of the cylinder $\mathbf{C}^\sigma$ and $\mathbf{C}_1^\sigma$ respectively.

Now we apply the inequality (23) to prove the estimation (24) to the function $\vartheta = \widetilde{u}\cdot z$ in cylinder $\mathbf{C}_1^\sigma$ and take into account the estimations (25) and the equality $\vartheta = \widetilde{u}$ for $(\xi,\eta)\in \mathbf{C}^\sigma$.

Let's denote the left hand side of the inequality (23) by $\|\widetilde{u}\|^2_{W^{2,1}_{2,\gamma}(\mathbf{C}^\sigma)}$, and $\int\limits_{\mathbf{C}^\sigma} e^{-\gamma\eta} e^{-\frac{\eta(n+2)}{2}}\widetilde{u}^2 d\xi d\eta$ by $\|\widetilde{u}\|_{L_{2,\gamma+2}(\mathbf{C}^\sigma)}$.

By virtue of interpolation inequality we arrive at following corollary from lemma 5.

Corollary. *In conditions of the lemma for any* $\nu>0$ *there exist the constants* C_{12} *and* C_{13} *such that*

$$\|\widetilde{u}\|_{W^{2,1}_{2,\gamma}(\mathbf{C}^\sigma)} \le C_{12}\left\|\widetilde{\mathcal{L}u}\right\|_{L_{2,\gamma+2}(\mathbf{C}_1^\sigma)} + \nu\|\widetilde{u}\|_{W^{2,1}_{2,\gamma}(\mathbf{C}_1^\sigma)} + C_{13}\|\widetilde{u}\|_{L_{2,\gamma+2}(\mathbf{C}_1^\sigma)}.$$

At this the constant C_{12} *depends on* λ, n, d *and* R, *and the constant* C_{13} *depends also on* ν.

Let's denote for $\rho>0$ the set $\{\xi : dist(\xi,\partial D)>\rho\}$ by D_ρ, and let $\widetilde{G}_{\rho,\sigma} = D_\rho \times \left(\ln\frac{1}{T},\ln\frac{1}{\sigma}\right)$, $\widetilde{G}_\sigma = D\times\left(\ln\frac{1}{T},\ln\frac{1}{\sigma}\right)$.

Lemma 6. *Let* $\widetilde{u}(\xi,\eta)$ *be infinitely differentiable function in closure of* $\widetilde{G}_\sigma$ *vanishing for* $\eta=\ln\frac{1}{T}$. *Then for any* $\rho>0$ *and* $\nu>0$ *there exist the constants* C_{14} *and* C_{15} *such that*

$$\|\widetilde{u}\|_{W^{2,1}_{2,\gamma}(\widetilde{G}_{\rho,\sigma})} \le C_{14}\left\|\widetilde{\mathcal{L}u}\right\|_{L_{2,\gamma+2}(\widetilde{G}_\sigma)} + \nu\|\widetilde{u}\|_{W^{2,1}_{2,\gamma}(\widetilde{G}_\sigma)} + C_{15}\|\widetilde{u}\|_{L_{2,\gamma+2}(\widetilde{G}_\sigma)}.$$

At this the constant C_{14} *depends only on* λ, n, d, *domain* D *and* ρ, *and the constant* C_{15} *- also on* ν.

To prove this it is enough to cover $\widetilde{G}_{\rho,\sigma}$ by the cylinders of form $\mathbf{C}^\sigma$ when $R=\frac{\rho}{2}$ and apply the corollary from the previous lemma.

Let's denote by $H_{\rho,\sigma}$ the set $\widetilde{G}_\sigma \setminus \widetilde{G}_{\rho,\sigma}$.

Lemma 7. *Let* $\widetilde{u}(\xi,\eta)$ *be infinitely differentiable in closure* $\widetilde{G}_\sigma$ *function vanishing on the parabolic domain* $\widetilde{G}_\sigma$. *Then if* $\partial D\in C^2$ *and* $\gamma\in(\gamma_0,\infty)$, *then for any* $\rho>0$ *and* $\nu>0$ *there exist the constants* C_{16} *and* C_{17} *such that*

$$\|\widetilde{u}\|_{W^{2,1}_{2,\gamma}(H_{\rho,\sigma})} \le C_{16}\left\|\widetilde{\mathcal{L}u}\right\|_{L_{2,\gamma+2}(\widetilde{G}_\sigma)} + \nu\|\widetilde{u}\|_{W^{2,1}_{2,\gamma}(\widetilde{G}_\sigma)} + C_{17}\|\widetilde{u}\|_{L_{2,\gamma+2}(\widetilde{G}_\sigma)}.$$

At this the constant C_{16} depends only on λ, n, d, domain D and ρ, and the constant C_{17} also on ν. Here $\gamma_0 = \max\left\{-2-\frac{n^2}{16d^2}, \frac{n^2\left(\lambda-\frac{1}{d^2}\right)+2\lambda n}{8}\right\}$.

Proof. Let's fix an arbitrary point $\xi^0 \in \partial D$. Then there exists a neighborhood of the point ξ^0 and non-degenerating transformation $\xi \to \zeta$ such that the intersection of the images ∂D and shown neighborhood is given by the equation $\zeta_n = 0$, and for the points of intersection of the image D and this neighborhood the inequality $\zeta_n > 0$ is true. The operator $\widetilde{\mathcal{L}}$ passes at such transformation to some uniform parabolic operator $\widetilde{M}$ of the second order with continuous coefficients.

Let $w(\zeta,\eta)$ be image of the function $\widetilde{u}(\xi,\eta)$ at such transformation. Let's consider again the cylinder $\mathbf{C}^\sigma$, whose foundations are n- dimensional balls of sufficiently small radius R with the center in the point ξ^0 and situated on hyperplanes $\eta = \ln\frac{1}{T}$ and $\eta = \ln\frac{1}{\sigma}$. Let further $\mathbf{C}_+^\sigma = \mathbf{C}^\sigma \cap \{(\zeta,\eta) : \zeta_n > 0\}$. Let's continue the function $w(\zeta,\eta)$ by odd image over the hyperplane $\zeta_n = 0$ in $\mathbf{C}^\sigma \setminus \mathbf{C}_+^\sigma$ and let's denote the continued function again by $w(\zeta,\eta)$.

It is easy to see that the statement of the lemma follows from the following fact.

Let $\widetilde{M}_0 = \sum_{i=1}^n \frac{\partial^2}{\partial \zeta^2} - 2\sum_{i=1}^n \zeta_i \frac{\partial}{\partial \zeta_i} - 4\frac{\partial}{\partial \eta}$ be a heat operator in coordinates (ζ,η), and the function $w(\zeta,\eta)$ vanishes near the lateral surface of the cylinder $\mathbf{C}^\sigma$ and at $\eta = \ln\frac{1}{T}$. Then holds the estimation

$$\int_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta} w_{\zeta\zeta}^2 d\zeta d\eta \le C_{18} \int_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta} \left(\widetilde{M}_0 w\right)^2 d\zeta d\eta \tag{26}$$

with the constant C_{18}, depending only on γ, n, λ and d. Let's denote the integral in the right hand side of the inequality by I. We have

$$I = \int_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta}\left(\sum_{i=1}^n w_{ii}\right) d\zeta d\eta + 4\int_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta}\left(\sum_{i=1}^n \zeta_i w_i\right)^2 +$$

$$+16\int_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta} w_\zeta^2 d\zeta d\eta - 4\int_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta}\sum_{i=1}^n w_{ii}\sum_{i=1}^n \zeta_i w_i d\zeta d\eta -$$

$$-8\int_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta}\sum_{i=1}^n w_{ii} w_\eta d\zeta d\eta + 16\int_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta}\sum_{i=1}^n \zeta_i w_i w_\eta d\zeta d\eta =$$

$$= i_1 + i_2 + i_3 + i_4 + i_5 + i_6. \tag{27}$$

On the other hand

$$i_1 = \int_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta} w_{\zeta\zeta}^2 d\zeta d\eta, \tag{28}$$

$$i_4 = (-2n+4)\int\limits_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta}\sum_{i=1}^{n} w_i^2 d\zeta d\eta\,, \tag{29}$$

$$i_5 \geq 4\left(\gamma+\frac{n+2}{2}\right)\int\limits_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta}\sum_{i=1}^{n} w_i^2 d\zeta d\eta\,, \tag{30}$$

$$i_6 \geq -4\int\limits_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta}\left(\sum_{i=1}^{n}\zeta_i w_i\right)^2 d\zeta d\eta - 16\int\limits_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta} w_\zeta^2 d\zeta d\eta\,. \tag{31}$$

Allowing for (28)-(31) in (27) we get

$$I \geq \int\limits_{\mathbf{C}^\sigma} e^{\left(-\gamma\frac{n+2}{2}\right)\eta} w_{\zeta\zeta}^2 d\zeta d\eta + 4(\gamma+2)\int\limits_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta} w_\zeta^2 d\zeta d\eta\,. \tag{32}$$

From (32) follows that if $\gamma \geq -2$, then the required estimation (26) is true. Let $\gamma < -2$. Then using (6) from (32) we conclude

$$I \geq \left(1+\frac{16d^2(\gamma+2)}{n^2}\right)\int\limits_{\mathbf{C}^\sigma} e^{\left(-\gamma-\frac{n+2}{2}\right)\eta} w_{\zeta\zeta}^2 d\zeta d\eta\,. \tag{33}$$

But if $\gamma > \gamma_0$, then $1+\frac{16d^2(\gamma+2)}{n^2} > 0$. Now from (33) follows the required estimation (26). The lemma is proved.

Theorem 1. *If* $\gamma \in (\gamma_0, \infty)$ *and* $\partial D \in C^2$, *then for every function* $u(x,t) \in \dot{W}_{2,\gamma}^{2,1}(G_T)$ *the estimation*

$$\|u\|_{W_{2,\gamma}^{2,1}(G_T)} \leq C_{19}\left(\|\mathcal{L}u\|_{L_{2,\gamma+2}(G_T)} + \|u\|_{L_{2,\gamma+2}(G_T)}\right) \tag{34}$$

is true. At this the constant C_{19} *depends only on* γ, λ, n, d *and the domain* D.

Proof. Let $G_{T,\sigma} = G_T \cap \{(x,t): t > -\sigma\}$. It is evident that it is sufficient to prove the estimation (34) for smooth functions from $\dot{W}_{2,\gamma}^{2,1}(G_T)$. From lemmas 6 and 7 follows that for any $\nu > 0$ the estimation

$$\|\tilde{u}\|_{W_{2,\gamma}^{2,1}(\tilde{G}_\sigma)} \leq C_{20}\left\|\tilde{\mathcal{L}}\tilde{u}\right\|_{L_{2,\gamma+2}(\tilde{G}_\sigma)} + \nu\|\tilde{u}\|_{W_{2,\gamma}^{2,1}(\tilde{G}_\sigma)} + C_{21}\|\tilde{u}\|_{L_{2,\gamma+1}(\tilde{G}_\sigma)} \tag{35}$$

is true, where the constant C_{20} depends only on γ, λ, n, d and the domain D, and the constant C_{21} - on ν too. Fixing $\nu = \frac{1}{2}$ turning to the variables (x,t) and tending σ to zero from (35) we arrive at the required estimation (34). The theorem is proved.

Corollary. *If the conditions of the theorem are fulfilled then there exists the positive number* T_0, *which depends only on* γ, λ, n, d *and the domain* D *such that if* $T \leq T_0$, *then*

$$\|u\|_{W_{2,\gamma}^{2,1}(G_T)} \leq C_{21}\|\mathcal{L}u\|_{L_{2,\gamma+2}(G_T)}. \tag{36}$$

Here $C_{21} = 2C_{19}$.

For the proof it is sufficient to note that according to (6)

$$\int\limits_{G_T}(-t)^{\gamma+2}u^2 dxdt \leq T_0\int\limits_{Q_T}(-t)^{\gamma}u^2 dxdt \leq \frac{16T_0^2}{n^2}d^2\int\limits_{G_T}(-t)^{\gamma+1}u_x^2 dxdt\,.$$

Thus

$$\|u\|_{L_{2,\gamma+2}(G_T)} \le \frac{4T_0}{n} d \|u\|_{W^{2,1}_{2,\gamma}(G_T)}. \tag{37}$$

Now it is enough to choose $T_0 = \frac{n}{C_{19}d}$ and from (37) and (34) follows the required estimation (36).

Theorem 2. *Let $\partial D \in C^2$, $T \le T_0$, $\gamma \in (\gamma, \infty)$. Then the first boundary value problem (1)-(2) is uniquely strongly solvable in the space $\dot{W}^{2,1}_{2,\gamma}(G_T)$ for any $f(x,t) \in L_{2,\gamma+2}(G_T)$. At this for solution $u(x,t)$ the estimation*

$$\|u\|_{W^{2,1}_{2,\gamma}(G_T)} \le C_{21} \|f\|_{L_{2,\gamma+2}(G_T)}$$

is true.

Proof. It is sufficient to prove the theorem for $f(x,t) \in C^\infty(\overline{G}_T)$. Let's fix an arbitrary $\sigma > 0$. Let for sufficiently large natural m $u^m(x,t)$ is the classical solution of the first boundary value problem

$$\mathcal{L}u^m = f(x,t),\ (x,t) \in G_{T,\frac{1}{m}};\ u^m\Big|_{\Gamma\left(G_{T,\frac{1}{m}}\right)},$$

where $\Gamma\left(G_{T,\frac{1}{m}}\right)$ is a parabolic boundary of the domain $G_{T,\frac{1}{m}}$. Since at every m the coefficients of the operator $\mathcal{L}$ are infinitely differentiable in $G_{T,\frac{1}{m}}$, then the solution $u^m(x,t)$ exists. At this $u^m \in C^2(\overline{G}_{T,\frac{1}{m}})$. According to the corollary from theorem 1

$$\|u^m\|_{W^{2,1}_{2,\gamma}(G_{T,\sigma})} \le C_{21} \|f\|_{L_{2,\gamma+2}(G_T)}.$$

Thus the sequence $\{u^m(x,t)\}_{m=1}^\infty$ is weakly compact in $\dot{W}^{2,1}_{2,\gamma}(G_{T,\sigma})$. From here follows that there exists the sequence $m_k \to \infty$ when $k \to \infty$ and the function $u(x,t) \in \dot{W}^{2,1}_{2,\gamma}(G_{T,\sigma})$ such that $(\mathcal{L}u^{m_k}, \varphi) \to (\mathcal{L}u, \varphi)$ when $k \to \infty$ for every function $\varphi(x,t) \in C^\infty(\overline{G}_{T,\sigma})$. Here $(\vartheta, \varphi) = \int_{G_{T,\sigma}} (-t)^{\gamma+2} \vartheta \varphi dx dt$. It is easy to see that

$$(\mathcal{L}u, \varphi) = (f, \varphi). \tag{38}$$

From (38) we conclude that $\mathcal{L}u = f$ almost everywhere in $G_{T,\sigma}$. Now it is sufficient to take into account the arbitrariness of σ and the existence of strong solution of the problem (1)-(2) is proved.

The uniqueness of solution and the inequality (37) follows from the coercive estimation (36). The theorem is proved.

Remark. We can show that unique strong solvability of the problem (1)-(2) in the space $\dot{W}^{2,1}_{2,\gamma}(G_T)$ holds at any $T \in (0, \infty)$.

The author expresses his deep gratitude to his supervisor corresponding member of NASA I.T. Mamedov for his constant attention to the work.

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Received February 20, 2001; Revised July 23, 2001.
Translated by Mamedova V.I.

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POINCARE TYPE WEIGHT INEQUALITIES IN DOMAINS WITH AN ISOPERIMETRIC TYPE CONDITION

Abstract

For the some bounded domains Ω in R^n, $n \geq 2$ with isoperimetrical type conditions $\widetilde{I}_\lambda$, in partial for the domains $\Omega = \{x = (x', x_n) : |x'| < x_n^\beta, 0 < x_n < a\}$, $a > 0$, $\beta \geq 1$ was proved the sufficient conditions on the weights, under which the Poincare's type two weighted inequality holds.

The paper is devoted to investigation the inequality

$$\left(\int_\Omega |u - \bar{u}|^q v dx \right)^{\frac{1}{q}} \leq C \left(\int_\Omega |Du|^p \omega dx \right)^{\frac{1}{p}}, \ 1 \leq p \leq q < \infty \tag{1}$$

of the differentiable functions $u(x)$ for some classes of the bounded domains Ω and the weights v, ω. The sufficient conditions of type A_{pq} are established for pair (v, ω) and isoperimetrical type inequalities between the Lebesque measure of any subsets of domain and $(n-1)$-dimensional of Housdorf measure of the part of boundary for the domains which provide the truthness of the inequality (1).

Here $v, \omega^{1-p'}$ are assumed locally integrable functions, with almost everywhere finite positive values at $1 < p < \infty, \omega^{-1} \in L^{\infty, loc}$ when $p = 1$. Ω-is an open bounded domain in $R^n, n \geq 2$, $\partial\Omega$-is its boundary, $d(\Omega)$-is a diameter of Ω, $\underset{n-1}{mes} \sum (n-1)$-is dimensional Housdorf measure of the set $\sum$ and $\left|\sum\right|$ is its Lebesque measure. $C^1(\Omega)$-are continuously differentiable in Ω functions. By Q denote arbitrary bolls in R^n, $Q_R^x = \{y \in R^n : |y - x| \leq R\}$. $p' = \dfrac{p}{p-1}$ when $1 < p < \infty$, $p' = \infty$-when $p = 1$.

$\bar{u} = \dfrac{1}{v(\Omega)} \int_\Omega vudx$, $v(\Omega) = \int_\Omega vdx$, $|Du|^2 = \sum_{i=1}^n \left(\dfrac{\partial u}{\partial x_i} \right)^2$.

It is known that the inequality

$$\left(\int_\Omega |u - \bar{u}|^q dx \right)^{\frac{1}{q}} \leq C_{n,q} \left(\int_\Omega |Du| dx \right), \ u \in C^1(\Omega), \tag{2}$$

which is got from (1) in the unweighted case when $p = 1, 1 \leq q \leq \dfrac{n}{n-p}$ and the connected domain Ω, is equivalent to the isoperimetrical condition I_λ on Ω

$$\underset{n-1}{mes} \partial g \cap \Omega \geq \theta \min\{|g|, |\Omega \setminus g|\}^\lambda \tag{3}$$

when $\lambda = \dfrac{1}{q}$, where $0 < \theta < \infty$, $g \subseteq \Omega$, see the lemma 3.2.4 from [1].

Unlike the regular domains the inequality of type (1) in domains I_λ have been respectively little studied (see [2] for the regular domains).