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THE SCATTERING PROBLEM FOR A HYPERBOLIC SYSTEM OF FIVE FIRST ORDER EQUATIONS ON SEMI-AXIS WITH TWO GIVEN INCIDENT WAVE

Abstract

At the paper the direct scattering problem is considered for a hyperbolic system of five equations of the first order on semi-axis with two given incident waves.

At the given paper the direct scattering problem on the semi-axis $x \geq 0$ is solved for the system of the following five equations of the first order:

$$\xi_i \frac{\partial u_i(x,t)}{\partial t} - \frac{\partial u_i(x,t)}{\partial x} = \sum_{j=1}^{5} C_{ij}(x,t) u_j(x,t) \quad (i = 1,...,5; -\infty < t < +\infty). \tag{1}$$

The direct scattering problem on a semi-axis is studied paper [1] for the system of five equations of the form (1) in case when $\xi_1 > \xi_2 > \xi_3 > 0 > \xi_4 > \xi_5$.

At the system (1) the coefficients $C_{ij}(x,t)$ are supposed as complex valued, measurable by x and t functions which satisfy the following estimations:

$$\left|C_{ij}(x,t)\right| \leq \frac{C}{(1+|x|)^{1+\varepsilon}(1+|t|)^{1+\varepsilon}}, \quad C > 0, \ \varepsilon > 0 \tag{2}$$

and the condition $C_{ii}(x,t) = 0, \ i = 1,...,5$.

Under the solution of the system (1) we'll understand such local-integral vector-function $u(x,t) = \{u_1(x,t),...,u_5(x,t)\}$, which in general case satisfies the system (1).

Let in the system (1) $C_{ij}(x,t) = 0 \ (i,j = 1,...,5)$. Then we get five independent equations:

$$\xi_i \frac{\partial u_i(x,t)}{\partial t} - \frac{\partial u_i(x,t)}{\partial x} = 0 \ (i = 1,...,5). \tag{1'}$$

It is known that every bounded solution of this system has the following form:

$$u(x,t) = \{f_1(t+\xi_1 x), f_2(t+\xi_2 x),..., f_5(t+\xi_5 x)\},$$

where $f_i(s)$ is an arbitrary function from the class $L_\infty(\mathbf{R}) \ (i = 1,...,5)$.

Every essentially bounded solution of the system (1) when $x \to +\infty$ asymptotically approximates to the solution of the equations (1'). Exactly the following theorem is true.

Theorem 1. *Every essentially bounded solution $u(x,t) = \{u_1(x,t),...,u_5(x,t)\}$ the system of the equations (1) with the coefficients $C_{ij}(x,t)$ satisfying to the conditions (2) admit on the semi-axis $x \geq 0$ the following asymptotics representations:*

$$\begin{cases} u_1(x,t) = a_1(t+\xi_1 x) + 0(1) \\ u_2(x,t) = a_2(t+\xi_2 x) + 0(1) \\ u_3(x,t) = b_3(t+\xi_3 x) + 0(1) \\ u_4(x,t) = b_4(t+\xi_4 x) + 0(1) \\ u_5(x,t) = b_5(t+\xi_5 x) + 0(1) \end{cases} \tag{3}$$

in the space $L_\infty(\mathbf{R}^2, \mathbf{C})$. *Here* $a_1(s), a_2(s) \in L_\infty(\mathbf{R})$ *determine the incident waves, and* $b_3(s), b_4(s), b_5(s) \in L_\infty(\mathbf{R})$ *the scattering waves.*

Proof. By the condition the solution $u(x,t)$ is essentially bounded, i.e. almost everywhere $|u_i(x,t)| \le M \quad (i=1,\dots,5)$. Let's consider the function:

$$\vartheta_i(x,t) = u_i(x,t) - \int_x^{+\infty} \sum_{j=1}^{5} C_{ij}(y, t+\xi_i(x-y)) u_j(y, t+\xi_i(x-y)) dy,$$
$$(i=1,\dots,5). \tag{4}$$

Let's estimate (4) allowing for the condition (2)

$$|\vartheta_i(x,t)| \le |u_i(x,t)| + \int_x^{+\infty} \sum_{j=1}^{5} \left| \frac{1}{(1+|y|)^{1+\varepsilon}(1+|t+\xi_i(x-y)|)^{1+\varepsilon}} \right| \cdot |u_j(y, t+\xi_i(x-y))| dy \le$$
$$\le M + M \sum_{j=1}^{5} \int_0^{+\infty} \frac{1}{(1+|y|)^{1+\varepsilon}} dy = M + M \cdot \frac{1}{\varepsilon} \le N = const.$$

Consequently the function $\vartheta_i(x,t)$ belong to the class $L_\infty(\mathbf{R}^2, \mathbf{C})$.

Besides there are solutions of the free equations (1')

$$\xi_i \frac{\partial \vartheta_i}{\partial t} - \frac{\partial \vartheta_i}{\partial x} = 0, \quad i=1,\dots,5.$$

So, there exist the functions $a_i(s)\,(i=1,2)$ and $b_3(s), b_4(s), b_5(s) \in L_\infty(\mathbf{R})$ such

$$\vartheta_i(x,t) = a_i(t+\xi_i x) \quad (i=1,2),$$
$$\vartheta_i(x,t) = b_i(t+\xi_i x) \quad (i=3,4,5).$$

Substituting this in (4) we get that

$$u_i(x,t) = a_i(t+\xi_i x) + \int_x^{+\infty} \sum_{j=1}^{5} (C_{ij}u_j)(y, t+\xi_i(x-y)) \quad (i=1,2),$$
$$u_i(x,t) = b_i(t+\xi_i x) + \int_x^{+\infty} \sum_{j=1}^{5} (C_{ij}u_j)(y, t+\xi_i(x-y)) \quad (i=3,4,5). \tag{5}$$

Substituting allowing for the estimations (2) we get (3)

The theorem is proved.

Let's consider jointly three problems. The first problem is in the finding the solution the system of the equations (1) satisfying the boundary conditions:

$$\begin{cases} u_5(0,t) = u_1(0,t) \\ u_4(0,t) = u_2(0,t) \\ u_3(0,t) = 0 \end{cases} \tag{6}$$

by the given incident waves $a_1, a_2 \in L_\infty(\mathbf{R})$ determine when $x \to +\infty$ the asymptotics of the solutions $u_1(x,t), u_2(x,t)$ form (3).

The second problem is in the finding the solution of the system (1) satisfying the boundary conditions

$$\begin{cases} u_5(0,t) = u_2(0,t) \\ u_3(0,t) = u_1(0,t) \\ u_4(0,t) = 0 \end{cases} \tag{7}$$

by the given waves $a_1, a_2 \in L_\infty(\mathbf{R})$.

The third problem is in the finding solution of the system (1) satisfying the conditions:

$$\begin{cases} u_4(0,t)=u_1(0,t) \\ u_3(0,t)=u_2(0,t) \\ u_5(0,t)=0 \end{cases} \tag{8}$$

by the same given waves a_1, a_2.

The joint consideration of these problems we'll call the scattering problem for the system (1) on a semi-axis.

Theorem 2. *Let the coefficients of the system (1) $C_{ij}(x,t)$ satisfy the conditions (2). The solution of the scattering problem on the semi-axis $x \geq 0$ for the system (1) with arbitrary given incident waves $a_1, a_2 \in L_\infty(\mathbf{R})$ exists and is unique.*

Proof. The scattering problem for k-th problem is equivalent to the following system of integral equations:

$$\begin{cases} u_1^k(x,t)=a_1(t+\xi_1 x)+\int\limits_x^{+\infty}\sum\limits_{j=1}^{5}C_{1j}(y,t+\xi_1(x-y))u_j^k(y,t+\xi_1(x-y))dy, \\ u_2^k(x,t)=a_2(t+\xi_2 x)+\int\limits_x^{+\infty}\sum\limits_{j=1}^{5}C_{2j}(y,t+\xi_2(x-y))u_j^k(y,t+\xi_2(x-y))dy, \\ u_3^k(x,t)=b_3(t+\xi_3 x)+\int\limits_x^{+\infty}\sum\limits_{j=1}^{5}C_{3j}(y,t+\xi_3(x-y))u_j^k(y,t+\xi_3(x-y))dy, \\ u_4^k(x,t)=b_4(t+\xi_4 x)+\int\limits_x^{+\infty}\sum\limits_{j=1}^{5}C_{4j}(y,t+\xi_4(x-y))u_j^k(y,t+\xi_4(x-y))dy, \\ u_5^k(x,t)=b_5(t+\xi_5 x)+\int\limits_x^{+\infty}\sum\limits_{j=1}^{5}C_{5j}(y,t+\xi_5(x-y))u_j^k(y,t+\xi_5(x-y))dy, \ (k=1,2,3) \end{cases} \tag{9}$$

From here allowing for the boundary conditions (6), (7) and (8) we can find the functions b_3^k, b_4^k, b_5^k:

$$\begin{cases} b_3^1(t)=-\int\limits_0^{+\infty}\sum\limits_{j=1}^{5}C_{3j}(y,t-\xi_3 y)u_j^1(y,t-\xi_3 y)dy, \\ b_4^1(t)=a_2(t)+\int\limits_0^{+\infty}\sum\limits_{j=1}^{5}\left[C_{2j}(y,t-\xi_2 y)u_j^1(y,t-\xi_2 y)-C_{4j}(y,t-\xi_4 y)u_j^1(y,t-\xi_4 y)\right]dy, \\ b_5^1(t)=a_1(t)+\int\limits_0^{+\infty}\sum\limits_{j=1}^{5}\left[C_{1j}(y,t-\xi_1 y)u_j^1(y,t-\xi_1 y)-C_{5j}(y,t-\xi_5 y)u_j^1(y,t-\xi_5 y)\right]dy, \end{cases} \tag{9.1}$$

$$\begin{cases} b_3^2(t)=a_1(t)+\int\limits_0^{+\infty}\sum\limits_{j=1}^{5}\left[C_{1j}(y,t-\xi_1 y)u_j^2(y,t-\xi_1 y)-C_{3j}(y,t-\xi_3 y)u_j^2(y,t-\xi_3 y)\right]dy, \\ b_4^2(t)=-\int\limits_0^{+\infty}\sum\limits_{j=1}^{5}C_{4j}(y,t-\xi_4 y)u_j^2(y,t-\xi_4 y)dy, \\ b_5^2(t)=a_2(t)+\int\limits_0^{+\infty}\sum\limits_{j=1}^{5}\left[C_{2j}(y,t-\xi_2 y)u_j^2(y,t-\xi_2 y)-C_{5j}(y,t-\xi_5 y)u_j^2(y,t-\xi_5 y)\right]dy, \end{cases} \tag{9.2}$$

$$\begin{cases} b_3^3(t)=a_2(t)+\int\limits_0^{+\infty}\sum\limits_{j=1}^{5}\left[C_{2j}(y,t-\xi_2 y)u_j^3(y,t-\xi_2 y)-C_{3j}(y,t-\xi_3 y)u_j^3(y,t-\xi_3 y)\right]dy, \\ b_4^3(t)=a_1(t)+\int\limits_0^{+\infty}\sum\limits_{j=1}^{5}\left[C_{1j}(y,t-\xi_1 y)u_j^3(y,t-\xi_1 y)-C_{4j}(y,t-\xi_4 y)u_j^3(y,t-\xi_4 y)\right]dy, \\ b_5^3(t)=-\int\limits_0^{+\infty}\sum\limits_{j=1}^{5}C_{5j}(y,t-\xi_5 y)u_j^3(y,t-\xi_5 y)dy. \end{cases} \quad (9.3)$$

Let's the system of the equations (5) write in operator form

$$u(x,t)=h(x,t)+(Au)(x,t), \quad (10)$$

where

$$u(x,t)=\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \\ u_4(x,t) \\ u_5(x,t) \end{pmatrix}, \quad h(x,t)=\begin{pmatrix} a_1(t+\xi_1 x) \\ a_2(t+\xi_2 x) \\ b_3(t+\xi_3 x) \\ b_4(t+\xi_4 x) \\ b_5(t+\xi_5 x) \end{pmatrix}, \quad Au(x,t)=\begin{pmatrix} A_1u(x,t) \\ A_2u(x,t) \\ A_3u(x,t) \\ A_4u(x,t) \\ A_5u(x,t) \end{pmatrix},$$

$$A_k u(x,t)=\int\limits_x^{+\infty}\sum\limits_{j=1}^{5}C_{kj}(y,t+\xi_k(x-y))u_j(y,t+\xi_k(x-y))dy \quad (k=1,\dots,5).$$

Let's use the following lemma for proving the existance uniqueness of the solution of the equation (10).

Lemma [2]. *Let in the Banach space B the following equation*

$$u=h+Au$$

be given, where A is a linear operator. If relative to the monotone by $T\in(-\infty;+\infty)$ the class of the semi-norm $\|u\|_T$ $\left(\|u\|_{+\infty}\equiv\|u\|_B\right)$ the inequality

$$\|Au\|_T\le\int\limits_T^{+\infty}\alpha(\tau)\|u\|_\tau\,d\tau \quad (10.1)$$

is fulfilled, where $\alpha(\tau)$ is an integrable function

$$\int\limits_{-\infty}^{+\infty}\alpha(\tau)d\tau<+\infty, \quad (10.2)$$

then at any $h\in B$ there exists and is unique the solution of the equation (10) and

$$\|u\|_B\le\|h\|_B\exp\int\limits_{-\infty}^{+\infty}\alpha(\tau)d\tau.$$

Using the estimation (2) let's prove that the operator A is a Volterra type in direction $\beta=(1;0)$ with intergable majorant $\alpha(y)=k(1+|y|)^{-1-\varepsilon}$.

Really

$$\|Au(x,t)\|_T=vrai\sup_{\substack{x\ge T\\-\infty<t<+\infty}}\|Au(x,t)\|_{C^5}=vrai\sup_{\substack{x\ge T\\-\infty<t<+\infty}}\max_{k=1,\dots,5}|A_k u(x,t)|=$$

$$=vrai\sup_{\substack{x\ge T\\-\infty<t<+\infty}}\max_{k=1,\dots,5}\left|\int\limits_x^{+\infty}\sum\limits_{j=1}^{5}C_{kj}(y,t+\xi_k(x-y))u_j(y,t+\xi_k(x-y))dy\right|\le$$

$$= vrai \sup_{\substack{x\geq T\\ -\infty<t<+\infty}} \max_{k=1,\dots,5} \int_x^{+\infty} \sum_{j=1}^{5} \left|C_{kj}\left(y,t+\xi_k(x-y)\right)\right| \cdot \left|u_j\left(y,t+\xi_k(x-y)\right)\right| dy \leq$$

$$\leq vrai \sup_{\substack{x\geq T\\ -\infty<t<+\infty}} \max_{k=1,\dots,5} \int_x^{+\infty} C(1+|y|)^{-1-\varepsilon}\left(1+\left|t+\xi_k(x-y)\right|\right)^{-1-\varepsilon} \sum_{j=1}^{5} \left|u_j\left(y,t+\xi_k(x-y)\right)\right| dy \leq$$

$$\leq vrai \sup_{\substack{x\geq T\\ -\infty<t<+\infty}} \max_{k=1,\dots,5} \int_x^{+\infty} C(1+|y|)^{-1-\varepsilon} \sum_{j=1}^{5} \left|u_j\left(y,t+\xi_k(x-y)\right)\right| dy \leq$$

$$= C\, vrai \sup_{\substack{x\geq T\\ -\infty<t<+\infty}} \int_x^{+\infty} 5(1+|y|)^{-1-\varepsilon}\, vrai \sup_{\substack{x'\geq y\\ -\infty<t'<+\infty}} \max_{j=1,\dots,5} \left|u_j(x',t')\right| dy \leq 5C \int_T^{+\infty} (1+|y|)^{-1-\varepsilon} \|u\|_y\, dy .$$

Thus

$$\|Au(x,t)\|_T \leq \int_T^{+\infty} \alpha(y)\|u\|_y\, dy ,$$

where

$$\alpha(y) = \frac{5C}{(1+|y|)^{1+\varepsilon}} .$$

Applying the lemma to the equation (10) concludes the proof of the theorem 2.

So, we'll construct that if the coefficients $C_{ij}(x,t)$ satisfy the condition (2) then the problem for the system (1) on the semi-axis $x \geq 0$ has a unique solution. It means that every vector-function $a(s) = (a_1(s), a_2(s)) \in L_\infty(\mathbf{R})$ giving the incident waves corresponds the vector $b(s) = \left(b_3^1(s), b_4^1(s), b_5^1(s), b_3^2(s), b_4^2(s), b_5^2(s), b_3^3(s), b_4^3(s), b_5^3(s)\right)$ are solutions of three problems for the system (1) with the boundary conditions (4), (5), (6) and with the given asymptotic (3).

The vector in (5) we'll find in the following form:

$$\begin{array}{lll} b_3^1 = S_{11}a_1 + S_{12}a_2 & b_3^2 = S_{21}a_1 + S_{22}a_2 & b_3^3 = S_{31}a_1 + S_{32}a_2 \\ b_4^1 = S_{13}a_1 + S_{14}a_2 & b_4^2 = S_{23}a_1 + S_{24}a_2 & b_4^3 = S_{33}a_1 + S_{34}a_2 \\ b_5^1 = S_{15}a_1 + S_{16}a_2 & b_5^2 = S_{25}a_1 + S_{26}a_2 & b_5^3 = S_{35}a_1 + S_{36}a_2 \end{array}$$

where S_{kj} $(k = 1,2,3;\ j = 1,\dots,6)$ are expressed by the coefficients of a system equations (1).

Thus in the space $L_\infty(\mathbf{R}, \mathbf{C}^3)$ we'll determine the operator S_k $(k = 1,2,3)$, which transfers $a(s)$ to $b(s)$:

$$S_k \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \begin{pmatrix} b_3^k(t) \\ b_4^k(t) \\ b_5^k(t) \end{pmatrix}, \text{ where } S_k = \begin{pmatrix} S_{k1} & S_{k2} \\ S_{k3} & S_{k4} \\ S_{k5} & S_{k6} \end{pmatrix} \quad (k = 1,2,3).$$

The operator $S = (S_1, S_2, S_3)$ is called the scattering operator for the system (1) on the semi-axis $x \geq 0$.

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UNIQUE STRONG SOLVABILITY OF THE FIRST BOUNDARY VALUE PROBLEM FOR PARABOLIC GILBARG-SERRIN EQUATION IN THE PARABOLOID TYPE DOMAINS

Abstract

In the paper the first boundary value problem for parabolic Gilbarg-Serrin equation is considered in paraboloid type domains. Unique strong solvability of this problem in weighted Sobolev space is proved.

Let $\mathbf{R}_{n+1}$ and $\mathbf{E}_n$ be $(n+1)$-dimensional and n-dimensional Euclidean spaces of the points $(x,t)=(x_1,...,x_n,t)$ and $x=(x_1,...,x_n)$ respectively. Let's call the domain G, situated in half-space $t<0$, P-domain, if its intersection with every hyperplane $t=-\tau\,(\tau>0)$ has the form $\left\{x:\frac{x}{2\sqrt{-\tau}}\in D\right\}$, where D is some bounded domain in $\mathbf{E}_n$. The domain D is called generating for the domain G. Let further $G_T=G\cap\{(x;t):t>-T\}$, where $T\in(0,+\infty)$. Let's consider in G_T the first boundary value problem

$$\mathcal{L}u=\Delta u+\lambda\sum_{i,j=1}^{n}\frac{x_i x_j}{4(-t)}\frac{\partial^2 u}{\partial x_i\partial x_j}-\frac{\partial u}{\partial t}=f(x,t), \tag{1}$$

$$u\big|_{\partial G_T}=0, \tag{2}$$

where the numerical parameter λ satisfies the following condition

$$-\frac{1}{d^2}<\lambda<\infty,\ d=\sup_{\xi\in D}|\xi|. \tag{3}$$

The aim of the given paper is to prove the unique strong (almost everywhere) solvability of the first boundary value problem (1)-(2) in corresponding weighted Sobolev spaces. The equation (1) is called the parabolic Gilbarg-Serrin equation. We know [1-2], that the first boundary value problem for parabolic equations of the second order of non-divergence structure is uniquely strongly solvable in the space $W_p^{2,1}\,(1<p<\infty)$ in an arbitrary boundary cylinder domain, if the coefficients of equation are uniformly continuous, the right hand side belongs to the space L_p, and the domain of cylinder foundation is the double smooth surface. In case $p=2$ the analogous fact holds also for some class of equations with discontinuous coefficients in particular satisfying the parabolic Cordes condition [3-6]. As to parabolic equations given in non-cylindrical domains then even for equations with smooth coefficients the unique solvability of the first boundary value problem holds only in weighted Sobolev spaces [7]. It is easy to see that the equation (1) satisfy the parabolic Cordes condition not at all the values of the parameter λ. Nevertheless as it is shown at the given paper the unique strong solvability of the first boundary value problem (1)-(2) in weighted Sobolev spaces holds at any value of the parameter λ, satisfying the condition (3).