IBIKLI E.

APPROXIMATION OF BOREL DERIVATIVES OF FUNCTIONS BY SINGULAR INTEGRALS

Abstract

The definition of right and left Borel derivatives are given and the theorems on approximation of functions, having Borel derivatives, by the sequences of linear integral operators with positive kernels are established.

Key words. Borel derivatives, singular integrals, approximation problem.

1. Approximation of functions and its derivatives by the sequences of integral operators with positive kernels (so called singular integrals) or, in general, by linear positive operators wave investigated by many authors. Many results in this direction may be found in books [1]-[4]. We also refer to papers [5]-[8].

This paper is devoted to a problem of approximation of functions, having a Borel derivative. The function f has a Borel derivative $B'f(x_0)\neq\pm\infty$ at x_0 if

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\frac{f(x_0+t)-f(x_0)}{t}dt=B'f(x_0)$$

and has a symmetrical Borel derivative $B'_s f(x_0)\neq\pm\infty$ at x_0 if

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\frac{f(x_0+t)-f(x_0-t)}{2t}dt=B'_s f(x_0)$$

where $\int_0^h=\lim_{\varepsilon\to 0}\int_\varepsilon^h$ (see, for example, [5]).

Obviously, if the ordinary first derivative $f'(x_0)$ exists, so does Borel derivatives and $B'f(x_0)=B'_s f(x_0)=f'(x_0)$. The converse is not hold.

As usual may be given a definition of right and left Borel derivatives $B'_+ f(x_0)$ and $B'_- f(x_0)$. Namely, we shall say that the function f has a right and left Borel derivatives at x_0 if

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\frac{f(x_0+t)-f(x_0)}{t}dt=B'_+ f(x_0),$$

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\frac{f(x_0)-f(x_0-t)}{t}dt=B'_- f(x_0).$$

Clearly, if $B'_+ f$ and $B'_- f$ there exists then there exists also

$$B'_s f(x_0)=\frac{1}{2}\left(B'_+ f(x_0)+B'_- f(x_0)\right).$$

2. Let $(-R,R)$, $R>0$ is finite or infinite interval and $n=1,2,\ldots$. Consider a sequence of integral operators

$$L_n(f;x)=\int_{-R}^{R}f(t)K_n(t-x)dt, \tag{1}$$

where the kernel $K_n(t)$ satisfies the conditions:

a) $K_n(t)$ is positive, even, infinitely differentiable function, $K_n(t)$ decreasing on $(0,R)$ and

$$\int_{-R}^{R} K_n(t)dt = 1, \ \forall n = 1,2,...;$$

b) for any fixed $\delta > 0$

$$\lim_{n\to\infty} K_n(\delta) = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{t\geq\delta}\left(t\left|K_n'(t)\right|\right) = 0; \qquad (2)$$

c) if $R = \infty$, then for any fixed $\delta > 0$

$$\lim_{n\to\infty} \int_{\delta}^{\infty} K_n(t)dt = 0 .$$

Note that in the case of $R < \infty$ we additionally assume that f and K_n has a period $2R$.

Theorem 1. *Let the kernel $K_n(t)$ of operator (1) satisfy a)-c) and $f \in L_1(-R,R)$. If there exists Borel derivatives $B_+' f$ and $B_-' f$ at x_0, then*

$$\lim_{n\to\infty} L_n'(f;x_0) = \frac{B_+' f(x_0) + B_-' f(x_0)}{2}, \qquad (3)$$

where $L_n'(f;x_0) = \frac{d}{dx} L_n(f;x)\Big|_{x=x_0}$.

Proof. We have

$$L_n'(f;x) = \int_{-R}^{R} f(t)\frac{d}{dx}K_n(t-x) = -\int_{-R}^{R} f(t)\frac{d}{dt}K_n(t-x) = -\int_{-R}^{R} f(x+t)K_n'(t)dt$$

and therefore by a)

$$L_n'(f;x_0) = -\int_0^R \left[f(x_0+t) - f(x_0-t)\right]K_n'(t)dt . \qquad (4)$$

Let

$$F(t) = \int_0^t A(x_0,\xi)d\xi ,$$

where

$$A(x_0,\xi) = \frac{f(x_0+\xi) - f(x_0-\xi)}{\xi} - \left(B_+' f(x_0) + B_-' f(x_0)\right). \qquad (5)$$

Then by the definition of right and left Borel derivatives

$$\lim_{h\to 0} \frac{1}{h}\int_0^h A(x_0,t)dt = 0$$

and therefore

$$|F(t)| \leq \varepsilon t \quad \text{if} \quad t < \delta = \delta(\varepsilon). \qquad (6)$$

Fixed this number δ, we can write

$$L_n'(f;x_0) = -\int_0^{\delta} A(x_0,t)tK_n'(t)dt - \left(B_+' f(x_0) + B_-' f(x_0)\right)\int_0^{\delta} tK_n'(t)dt -$$
$$- \int_{\delta}^{\infty}\left[f(x_0+t) - f(x_0-t)\right]K_n'(t)dt = I_n' + I_n'' + I_n''' \qquad (7)$$

$$K_n(t)=K_\rho(t)=\frac{1-\rho}{1+\rho^2-2\rho\cos t}.$$

Operator (1) with the Kernel $K_\rho(t)$ is Poisson integral $P_\rho(f;x)$. Since

$$\left|K'_\rho(t)\right|=\left|\frac{2\rho\sin t\left(1-\rho^2\right)}{\left(1+\rho^2-2\rho\cos t\right)^2}\right|\le\frac{1-\rho}{2\rho\sin^4\frac{t}{2}},$$

then for $\delta\le t\le\pi$

$$\sup\left(t\left|K'_\rho(t)\right|\right)\le\frac{\pi}{2\rho}\frac{1-\rho}{\sin^4\frac{\delta}{2}}\to 0 \quad \text{as} \quad \rho\to 1^{\cdot}.$$

All other conditions also holds and we have from theorem 1.

Theorem 3. *If the function $f\in L_1(-\pi,\pi)$ has a finite right and left Borel derivatives then for Poisson integral*

$$\lim_{\rho\to 1}P'_\rho(f;x_0)=\frac{B'_+f(x_0)-B'_-f(x_0)}{2}$$

holds.

Corollary. *If the function f has a Borel derivative $B'f(x_0)$ then*

$$\lim_{\rho\to 1}P'_\rho(f;x_0)=B'f(x_0).$$

This result were proved early in [5].

Application of theorem 2 to the integral operators (1) with kernels

$$K_n(t)=\frac{n}{\sqrt{\pi}}e^{-n^2t^2},\quad K_n(t)=\frac{n}{\sqrt{\pi}}\frac{1}{1+n^2t^2}$$

gives the theorems on approximation of Borel derivatives of functions by singular integrals of Gauss-Weierstrasse and Abel-Poisson correspondingly.

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Ertan Ibikli
Ankara University, Faculty of Science, Department of Mathematics.
06100, Beselver, Ankara, Turkey.

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Translated by author.

Ertan Ibikli
Ankara University, Faculty of Science, Department of Mathematics.
06100, Beselver, Ankara, Turkey.

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Translated by author.

ISKENDEROV N.Sh., JABBAROVA K.A.

THE SCATTERING PROBLEM FOR A HYPERBOLIC SYSTEM OF FIVE FIRST ORDER EQUATIONS ON SEMI-AXIS WITH TWO GIVEN INCIDENT WAVE

Abstract

At the paper the direct scattering problem is considered for a hyperbolic system of five equations of the first order on semi-axis with two given incident waves.

At the given paper the direct scattering problem on the semi-axis $x \geq 0$ is solved for the system of the following five equations of the first order:

$$\xi_i \frac{\partial u_i(x,t)}{\partial t} - \frac{\partial u_i(x,t)}{\partial x} = \sum_{j=1}^{5} C_{ij}(x,t) u_j(x,t) \quad (i = 1,...,5; -\infty < t < +\infty). \qquad (1)$$

The direct scattering problem on a semi-axis is studied paper [1] for the system of five equations of the form (1) in case when $\xi_1 > \xi_2 > \xi_3 > 0 > \xi_4 > \xi_5$.

At the system (1) the coefficients $C_{ij}(x,t)$ are supposed as complex valued, measurable by x and t functions which satisfy the following estimations:

$$\left|C_{ij}(x,t)\right| \leq \frac{C}{(1+|x|)^{1+\varepsilon}(1+|t|)^{1+\varepsilon}}, \; C > 0, \; \varepsilon > 0 \qquad (2)$$

and the condition $C_{ii}(x,t) = 0, \; i = 1,...,5$.

Under the solution of the system (1) we'll understand such local-integral vector-function $u(x,t) = \{u_1(x,t),...,u_5(x,t)\}$, which in general case satisfies the system (1).

Let in the system (1) $C_{ij}(x,t) = 0 \; (i,j = 1,...,5)$. Then we get five independent equations:

$$\xi_i \frac{\partial u_i(x,t)}{\partial t} - \frac{\partial u_i(x,t)}{\partial x} = 0 \; (i = 1,...,5). \qquad (1')$$

It is known that every bounded solution of this system has the following form:

$$u(x,t) = \{f_1(t+\xi_1 x), f_2(t+\xi_2 x),..., f_5(t+\xi_5 x)\},$$

where $f_i(s)$ is an arbitrary function from the class $L_\infty(\mathbf{R}) \; (i = 1,...,5)$.

Every essentially bounded solution of the system (1) when $x \to +\infty$ asymptotically approximates to the solution of the equations (1'). Exactly the following theorem is true.

Theorem 1. *Every essentially bounded solution* $u(x,t) = \{u_1(x,t),...,u_5(x,t)\}$ *the system of the equations (1) with the coefficients* $C_{ij}(x,t)$ *satisfying to the conditions (2) admit on the semi-axis* $x \geq 0$ *the following asymptotics representations:*

$$\begin{cases} u_1(x,t) = a_1(t+\xi_1 x) + 0(1) \\ u_2(x,t) = a_2(t+\xi_2 x) + 0(1) \\ u_3(x,t) = b_3(t+\xi_3 x) + 0(1) \\ u_4(x,t) = b_4(t+\xi_4 x) + 0(1) \\ u_5(x,t) = b_5(t+\xi_5 x) + 0(1) \end{cases} \qquad (3)$$