GULIYEV V.S., MUSTAFAYEV R.Ch.

ON GENERALIZED FRACTIONAL INTEGRALS

Abstract

The properties of generalized fractional integrals are studied.

It is known that the fractional integral

$$I_\alpha f(x)=\int_{R^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy, \quad 0<\alpha<n,$$

is bounded from $L_p(R^n)$ to $L_q(R^n)$, when $1<p<n/\alpha$ and $1/q=1/p-\alpha/n$ as the Hardy-Littlewood-Sobolev theorem. We consider generalized fractional integrals and prove the Hardy-Littlewood-Sobolev theorem for them. Related questions were studied in works of Y.Mizuta, E.Nakai, A.Gadjiev and etc.

It is also known that the modified fractional integral

$$\widetilde{I}_\alpha f(x)=\int_{R^n}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|y|^{n-\alpha}}\right)\chi_{B_1^*}(y)f(y)dy$$

is bounded from $L_{n/\alpha}(R^n)$ to $BMO(R^n)$. Here B_1^* - exterior of unit ball $B_1=B(0,1)$, that is $B_1^*=R^n\setminus B_1$, χ_A - the characteristic function of set A. We also investigate the boundedness of modified generalized fractional integrals from $L_p(R^n)$ to $BMO(R^n)$.

The $L_p(R^n)$, $1\le p\le\infty$ spaces are defined as the set of all measurable functions $f(x)$, $x\in R^n$ on R^n with finite norm

$$\|f\|_{L_p}(R^n)=\left(\int_{R^n}|f(x)|^p dx\right)^{1/p}, \quad 1\le p<\infty.$$

At $p=\infty$ the spaces $L_\infty(R^n)$ are defined by means of usual modification

$$\|f\|_{L_\infty(R^n)}=\operatorname*{esssup}_{x\in R^n}|f(x)|.$$

A measurable function $f(x), x\in R^n$ belongs to $WL_p(R^n), 0<p<\infty$ (weak L_p space) by definition if the norm

$$\|f\|_{WL_p(R^n)}=\sup_{t>0} t\left|\{x\in R^n : |f(x)|>t\}\right|^{1/p}, \ 0<p<\infty$$

is finite. At $p=\infty$ $WL_\infty(R^n)\equiv L_\infty(R^n)$ and

$$\|f\|_{WL_\infty(R^n)}=:\|f\|_{L_\infty(R^n)}.$$

A locally integrable function f will be said to belong to BMO if the norm

$$\|f\|_{BMO(R^n)}=\sup_{x\in R^n, r>0}|B(x,r)|^{-1}\int_{B(x,r)}|f(y)-f_{B(x,r)}|dy$$

is finite; here

$$f_{B(x,r)} = |B(x,r)|^{-1} \int\limits_{B(x,r)} f(y)dy\ , \quad f \in L_1^{loc}(R^n)$$

denotes the mean value of f over the ball $B(x,r)$.

We denote by M the Hardy-Littlewood maximal operator on R^n

$$(Mf)(x) = \sup_{t>0} |B(x,t)|^{-1} \int\limits_{B(x,t)} |f(y)|dy\ .$$

The following theorem is valid:

Theorem 1. **[4]** *Let $f \in L_1(R^n)$, then for $\alpha > 0$*

$$\left|\left\{x \in R^n : Mf(x) > \alpha\right\}\right| \le \frac{C}{\alpha} \int\limits_{R^n} |f(x)|dx\ , \tag{1}$$

where C is independent of f.

Let $f \in L_p(R^n)$, $1 < p \le \infty$, then $Mf(x) \in L_p(R^n)$ and

$$\|Mf\|_p \le C_p \|f\|_p, \tag{2}$$

where C_p - is dependent only of p and n.

For a function $K:(0,+\infty) \to (0,+\infty)$, let

$$T_k f(x) = \int\limits_{R^n} K(|x-y|) f(y)\, dy\ .$$

If $K(t) = t^{\alpha-n}$, $0 < \alpha < n$, then T_k is the fractional integral or the Riesz potential denoted by I_α.

A function $\theta:(0,\infty) \to (0,\infty)$ is said to be almost decreasing if there exists a constant $C > 0$ such that

$$\theta(r) \ge C\theta(s)$$

for $r \le s$.

We consider the following conditions on K:

(K_1) $0 \le K(t)$ is almost decreasing on $(0,\infty)$, $\lim\limits_{t\to 0} K(t) = \infty$;

(K_2) $\exists C_1 > 0, \exists \sigma > 0 \quad \forall R > 0 \quad \int\limits_0^R K(t) t^{n-1} dt \le B_1 R^\sigma$;

(K_3) $\exists C_2 > 0, \exists \gamma(p) > 0 \quad \forall R > 0 \quad \left(\int\limits_R^\infty K^{p'}(t) t^{n-1} dt\right)^{1/p'} \le B_2 R^{-\gamma(p)}$;

(K_4) $\exists C_3 > 0, |K(\tau) - K(s)| \le C_3 |r-s| \frac{K(r)}{r}$; $\frac{1}{2} \le \frac{s}{r} \le 2$;

(K_5) $\exists C_4 > 0, \forall R > 0 \quad \left(\int\limits_R^\infty K^{p'}(t) t^{n-p'-1} dt\right)^{1/p'} \le C_4 R^{-1}$.

It is valid

Theorem 2. *a) Let $f \in L_p(R^n)$, $1 \le p < \infty$ and the kernel K satisfies (K_1), (K_2) and (K_3). Then the initial $T_k f(x)$ absolutely converges a.e. in R^n.*

b) Let $1<p<\infty, \frac{1}{p}-\frac{1}{q}=\frac{\sigma}{p(\gamma(p)+\sigma)}$ *and the kernel* K *satisfies* $(K_1),(K_2)$ *and* (K_3). *Then* $T_k f\in L_q(R^n)$ *and*

$$\|T_k f\|_{L_q(R^n)}\le C\|f\|_{L_p(R^n)}.$$

c) Let $1-\frac{1}{q}=\frac{\sigma}{n}$ *the kernel* K *satisfies* $(K_1),(K_2)$.

Then

$$\|T_k f\|_{WL_q(R^n)}\le C\|f\|_{L_1(R^n)}.$$

Proof. Note that the part a) theorem 2 was proved in [1].

Let's prove b).

Fixing any $t>0$ we have

$$|T_K f(x)|\le \int_{B(0,t)} K(|y|)|f(x-y)|dy+\int_{R^n\setminus B(0,t)} K(|y|)|f(x-y)|dy=A(x,t)+C(x,t).$$

Let's estimate $A(x,t)$. Taking into account (K_1) and (K_2) we obtain

$$A(x,t)=\sum_{k=-\infty}^{-1}\int_{2^k t\le|y|\le 2^{k+1}t} K(|y|)|f(x-y)|dy\le C\sum_{k=-\infty}^{-1}K(2^k t)\int_{2^k t\le|y|\le 2^{k+1}t}|f(x-y)|dy\le$$

$$\le CMf(x)\sum_{k=-\infty}^{-1}K(2^k t)(2^k t)^n\le CMf(x)\int_0^t K(\tau)\tau^{n-1}d\tau\le Ct^{\sigma}Mf(x).$$

There by

$$A(x,t)\le Ct^{\sigma}Mf(x), \tag{3}$$

where C does not depend of f,x and t.

On the other hand applying the Hölder's inequality and using (K_3) we get

$$C(x,t)\le\left(\int_{R^n\setminus B(0,t)}|f(x-y)|^p dy\right)^{1/p}\left(\int_{R^n\setminus B(0,t)}K(|y|)^{p'}dy\right)^{1/p'}\le$$

$$\le C\|f\|_{L_p(R^n)}\left(\int_t^{\infty}K(\tau)^{p'}\tau^{n-1}d\tau\right)^{1/p'}\le Ct^{-\gamma(p)}\|f\|_{L_p(R^n)}.$$

Consequently

$$C(x,t)\le Ct^{-\gamma(p)}\|f\|_{L_p(R^n)}. \tag{4}$$

Thus

$$|T_K(x)|\le C\left(t^{\sigma}Mf(x)+t^{-\gamma(p)}\|f\|_{L_p(R^n)}\right). \tag{5}$$

Minimizing on t at $\left(\frac{\gamma(p)\|f\|_{L_p(R^n)}}{\sigma Mf(x)}\right)^{\frac{1}{\sigma+\gamma(p)}}$ we have

$$|T_K f(x)|\le C(\sigma,\gamma(p))Mf(x)^{\frac{\gamma(p)}{\gamma(p)+\sigma}}\|f\|_{L_p(R^n)}^{\frac{\sigma}{\gamma(p)+\sigma}}. \tag{6}$$

Therefore by (2)

$$\|T_K f\|_{L_p(R^n)} \le C\|f\|_{L_p(R^n)}^{\frac{\sigma}{\gamma(p)+\sigma}} \left(\int_{R^n} (Mf(x))^{\frac{\gamma(p)q}{\gamma(p)+\sigma}} dx \right)^{1/q} =$$

$$= C\|f\|_{L_p(R^n)}^{\frac{\sigma}{\gamma(p)+\sigma}} \left(\int_{R^n} (Mf(x))^p dx \right)^{1/q} \le C\|f\|_{L_p(R^n)}^{\frac{\sigma}{\gamma(p)+\sigma}} \cdot \|f\|_{L_p(R^n)}^{\frac{p}{q}} = C\|f\|_{L_p(R^n)}.$$

The proof of b) is completed.

Proof of c)

Let $f \in L_1(R^n)$. From (K_1) and (K_2) it follows that

$$K(t) \le Ct^{-n} \int_0^t K(\tau)\tau^{n-1} d\tau \le Ct^{-n+\sigma}.$$

Thus

$$\left|\left\{x \in R^n : |T_K f(x)| > \beta\right\}\right| \le C\left|\left\{x \in R^n : |I_\sigma f(x)| > \beta\right\}\right| \le C\beta^{-q}\|f\|_{L_1(R^n)}^q.$$

c) has been proved. ■

We will consider the modified integral operator

$$\widetilde{T}_K f(x) = \int_{R^n} \left[K(|x-y|) - K(|y|)\chi_{B_1^*}(y) \right] f(y) dy.$$

Theorem 3. *Let* $1 < p < \infty$, $\sigma = \dfrac{n}{p}$ *and the kernel* K *satisfies* $(K_1),(K_2),(K_4)$ *and* (K_5). *Then* $\widetilde{T}_K \in BMO(R^n)$ *and*

$$\left\|\widetilde{T}_K\right\|_{BMO(R^n)} \le C\|f\|_{L_p(R^n)}.$$

Proof. Let $f \in L_p(R^n)$. For any fixed $t > 0$ we put

$$f_1(x) = f(x)\chi_{B(0,2t)}(x),\ f_2(x) = f(x) - f_1(x).$$

Then

$$\widetilde{T}_K f(x) = \widetilde{T}_K f_1(x) + \widetilde{T}_K f_2(x) = F_1(x) + F_2(x),$$

where

$$F_1(x) = \int_{B(0,2t)} \left[K(|x-y|) - K(|y|)\chi_{B_1^*}(y) \right] f(y) dy,$$

$$F_2(x) = \int_{R^n \setminus B(0,2t)} \left[K(|x-y|) - K(|y|)\chi_{B_1^*}(y) \right] f(y) dy.$$

Note that the function f_1 has compact support and that's why 0

$$a_1 = -\int_{B(0,2t) \setminus B(0,\min 1,2t)} K(|y|) f(y) dy$$

is finite. Note also that

$$F_1(x) - a_1 = T_K f_1(x).$$

Then

$$|F_1(x) - a_1| \le \int_{\{y : |x-y| \le 2t\}} K(|y|)|f(x-y)| dy.$$

Since $|x|<t$, $|x-y|<2t$ implies $|y|<3t$ then 0

$$|F_1(x)-a_1|=\int_{B(0,3t)}K(|y|)|f(x-y)|dy,\ \ x\in B(0,t).$$

Using (3) at $\sigma=\frac{n}{p}$ we obtain

$$\frac{1}{|B(0,t)|}\int_{B(0,t)}|F_1(x+y)-a_1|dy\le\frac{1}{|B(0,t)|}\int_{B(0,t)}\int_{B(0,3t)}K(|z|)|f(x+y-z)|dzdy\le$$

$$\le Ct^{-n+\sigma}\int_{B(0,t)}Mf(x+y)dy\le Ct^{-n+\sigma+\frac{n}{p'}}\left(\int_{B(0,t)}Mf(x+y)^p\,dy\right)^{1/p}\le$$

$$\le Ct^{\sigma-\frac{n}{p}}\|f\|_{L_p(R^n)}=C\|f\|_{L_p(R^n)}.$$

Denote by

$$a_2=\int_{B(0,\max 1,2t)\setminus B(0,2t)}K(|y|)f(y)dy.$$

Let's estimate $|F_2(x)-a_2|$.

$$|F_2(x)-a_2|\le\int_{R^n\setminus B(0,2t)}|K(|x-y|)-K(|y|)||f(y)|dy.$$

Taking into account (K_5) and applying the Hölder's inequality we get

$$|F_2(x)-a_2|\le C\int_{R^n\setminus B(0,2t)}\left||x-y|-|y|\right|\frac{K(|y|)}{|y|}|f(y)|dy\le$$

$$\le C|x|\|f\|_{L_p(R^n)}\left(\int_t^{\infty}K(\tau)^{p'}\tau^{n-p'-1}d\tau\right)^{1/p'}\le C\|f\|_{L_p(R^n)}.$$

Denote by

$$a_f=a_1+a_2=\int_{B(0,\max\{1,2t\})}K(|y|)f(y)dy.$$

Finally,

$$\sup_{x,t}\frac{1}{|B(0,t)|}\int_{B(0,t)}|\widetilde{T}_K f(x+y)-a_f|dy\le C\|f\|_{L_p(R^n)}.$$

■

Hence

$$\|\widetilde{T}_K\|_{BMO(R^n)}\le C\|f\|_{L_p(R^n)}.$$

Corollary 1. *Let* $1<p<\infty,\sigma=\frac{n}{p},f\in L_p(R^n)$ *and the kernel* K *satisfies* $(K_1),(K_2),(K_4)$ *and* (K_5). *If* $T_K f$ *absolutely converges a.e. in* R^n, *then* $T_K f\in BMO(R^n)$ *and*

$$\|T_K f\|_{BMO(R^n)}\le\|f\|_{L_p(R^n)}.$$

Remark 1. Let $0<\alpha<\frac{n}{p}, \sigma=\alpha, \gamma(p)=\frac{n}{p}-\alpha$ and $K(t)=t^{\alpha-n}$. Then the conditions (K_1), (K_2),(K_3),(K_4) and (K_5) are valid for the kernel K.

Remark 2. Let $0<\sigma<\alpha<n, \sigma=\alpha-1,\ 0<\gamma(p)=\frac{n}{p}+1-\alpha$ and $K(t)=t^{\alpha-n}\ln\left(1+\frac{1}{t}\right)$. Then the conditions (K_1), (K_2),(K_3),(K_4) and (K_5) are valid for the kernel K.

Proof. Let's us check (K_2):

$$\int_0^R K(t)t^{n-1}dt=\int_0^R t^{\alpha-1}\ln\left(1+\frac{1}{t}\right)dt\le\int_0^R t^{\alpha-2}dt=CR^{\alpha-1}=CR^{\sigma}.$$

Then

$$\left(\int_0^\infty K^{p'}(t)t^{n-1}dt\right)^{1/p'}=\left(\int_R^\infty t^{(\alpha-n)p'+n-1}\ln^{p'}\left(1+\frac{1}{t}\right)dt\right)^{1/p'}\le$$

$$\le\left(\int_R^\infty t^{(\alpha-n-1)p'+n-1}dt\right)^{1/p'}=CR^{\alpha-\frac{n}{p}-1}=CR^{-\gamma(p)}.$$

In order to prove (K_4) we apply mean-value theorem

$$|K(r)-K(s)|=|K'(\xi)|\,|r-s|=\left|\xi^{\alpha-n-1}\left((\alpha-n)\ln\left(1+\frac{1}{\xi}\right)-\frac{1}{1+\xi}\right)\right||r-s|.$$

Since

$$\frac{1}{1+\xi}\le\ln\left(1+\frac{1}{\xi}\right),$$

then we get

$$|K(r)-K(s)|\le C\frac{K(r)}{r}|r-s|\quad\text{for}\quad\frac{1}{2}\le\frac{s}{r}\le 2.$$

Finally, taking into account that $\alpha-1=\frac{n}{p}$ we have

$$\left(\int_0^\infty K^{p'}(t)t^{n-p'-1}dt\right)^{1/p'}=\left(\int_R^\infty t^{(\alpha-n-1)p'+n-1}\ln^{p'}\left(1+\frac{1}{t}\right)dt\right)^{1/p'}\le$$

$$\le\left(\int_R^\infty t^{(\alpha-n-1)p'-p'+n-1}dt\right)^{1/p'}=\left(\int_R^\infty t^{-p'-1}dt\right)^{1/p'}=CR^{-1}.$$

Corollary 2. *Let* $1<p<\infty, 0<\sigma<\alpha<\frac{n}{p}, 0<\gamma(p)<\frac{n}{p}-\alpha, \frac{1}{p}-\frac{1}{q}=$ $=\frac{\sigma}{p(\gamma(p+\sigma))}$ *and*

$$K(t)=\begin{cases} t^{\alpha-n}\ln\frac{1}{t} & for\ \ small\ \ t \\ t^{\alpha-n}(\ln t)^{-1} & for\ \ large\ \ t\ . \end{cases}$$

Then $T_k f\in L_q\left(R^n\right)$ *and*

$$\|T_K f\|_{L_q\left(R^n\right)}\le C\|f\|_{L_p\left(R^n\right)}.$$

Proof. For small R

$$\int_0^R K(t)t^{n-1}dt=\int_0^R t^{\alpha-1}\ln\frac{1}{t}dt=\frac{1}{\alpha}\int_0^R \ln\frac{1}{t}dt^{\alpha}=\frac{1}{\alpha}\ln\frac{1}{R}R^{\alpha}+\frac{1}{\alpha}\int_0^R t^{\alpha-1}dt=$$

$$=\frac{1}{\alpha}R^{\alpha}\left(\ln\frac{1}{R}+\frac{1}{\alpha}\right)\simeq R^{\alpha}\ln\frac{1}{R}\le CR^{\sigma}\quad if\ \sigma<\alpha.$$

For large R

$$\int_0^R K(t)t^{n-1}dt=\int_0^e t^{\alpha-1}\ln\frac{1}{t}dt+\int_e^R t^{\alpha-1}\frac{dt}{\ln t}\le C+\int_e^R t^{\alpha}d\ln\ln t=$$

$$=C+t^{\alpha}\ln\ln t\Big|_e^R-\frac{1}{\alpha}\int_e^R t^{\alpha-1}\ln\ln t dt\le CR^{\alpha}\ln\ln R\le CR^{\sigma}\quad \text{if}\quad \sigma<\alpha.$$

For large R

$$\int_R^{\infty}K^{p'}(t)t^{n-1}dt=\int_R^{\infty}t^{(\alpha-n)p'+n-1}\frac{dt}{\ln^{p'}t}=-\frac{1}{p'-1}\int_R^{\infty}t^{(\alpha-n)p'+n}d\frac{1}{\ln^{p'-1}t}=$$

$$=-\frac{1}{p'-1}t^{(\alpha-n)p'+n}\frac{1}{\ln^{p'-1}t}\Bigg|_R^{\infty}+\frac{1}{p'-1}\frac{1}{(\alpha-n)p'+n}\int_R^{\infty}t^{(\alpha-n)p'+n-1}\frac{dt}{\ln^{p'-1}t}\overset{\alpha<\frac{n}{p}}{\le}$$

$$\le CR^{(\alpha-n)p'+n}\frac{1}{\ln^{p'-1}t}.$$

Then for large R

$$\left(\int_0^{\infty}K^{p'}(t)t^{n-1}dt\right)^{1/p'}=CR^{\alpha-\frac{n}{p}}\frac{1}{(\ln R)^{\frac{1}{p}}}\le CR^{-\gamma(p)}\quad if\quad \gamma(p)<\frac{n}{p}-\alpha\ .$$

For small R

$$\int_R^{\infty}K^{p'}(t)t^{n-1}dt=\int_R^1 t^{(\alpha-n)p'+n-1}\left(\ln\frac{1}{t}\right)^{p'}dt+\int_1^{\infty}t^{(\alpha-n)p'+n-1}\frac{dt}{\ln^{p'}t}\le$$

$$\le\int_1^{\frac{1}{R}}t^{(n-\alpha)p'-n-1}(\ln t)^{p'}dt+C=\frac{1}{p'+1}\int_1^{\frac{1}{R}}t^{(n-\alpha)p'-n}d(\ln t)^{p'+1}+C=$$

$$=\frac{1}{p'+1}t^{(n-\alpha)p'-n}(\ln t)^{p'+1}\Big|_1^{\frac{1}{R}}-\frac{1}{p'+1}\int_1^{\frac{1}{R}}t^{(n-\alpha)p'-n-1}(\ln t)^{p'+1}dt+C\le$$

$$\le R^{(\alpha-n)p'+n}\left(\ln\frac{1}{R}\right)^{p'+1}+C\,.$$

Then

$$\left(\int_R^\infty K^{p'}(t)t^{n-1}dt\right)^{1/p'}\le CR^{\alpha-\frac{n}{p}}\left(\ln\frac{1}{R}\right)^{1+\frac{1}{p'}}\le CR^{-\gamma(p)}\ \text{ if }\ \gamma(p)<\frac{n}{p}-\alpha\,.$$

■

Corollary 3. *Let* $1<p<\infty,\ 0<\alpha<n,\ 0<\beta,\ 0<\frac{n}{p}-\alpha+\beta,\ \frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n+p\beta}$ *and* $K(t)=\frac{t^{\alpha-n}}{(1+t)^{\beta}}$. *Then* $T_K f\in L_q(R^n)$ *and*

$$\|T_K f\|_{Lq(R^n)}\le C\|f\|_{L_p(R^n)}\,.$$

Proof. It's obvious that the condition (K_1) is fulfilled. Let's check the conditions (K_2) and (K_3).

$$\int_0^R K(t)t^{n-1}dt=\int_0^R\frac{t^{\alpha-1}}{(1+t)^{\beta}}dt\le\int_0^R t^{\alpha-1}dt=CR^{\alpha}\le CR^{\sigma}\quad if\quad \sigma=\alpha\,.$$

$$\left(\int_R^\infty K^{p'}(t)t^{n-1}dt\right)^{1/p'}=\left(\int_R^\infty\frac{t^{(\alpha-n)p'+n-1}}{(1+t)^{\beta p'}}dt\right)^{1/p'}\le$$

$$\le\left(\int_R^\infty t^{(\alpha-n)p'+n-\beta p'-1}dt\right)^{1/p'}\overset{0<\frac{n}{p}-\alpha+\beta}{=}R^{-\left(\frac{n}{p}-\alpha+\beta\right)}\le CR^{-\gamma(p)}\,.$$

$$if\quad \gamma(p)=\frac{n}{p}-\alpha+\beta\,.$$

■

References

[1]. Gadjiev A.D. *On generalized potential-type integral operators.* Functiones Et Approximatio, Adam Mickiewicz Univercity Press, Poznan, 1997, v.25, p.37-44.
[2]. Guliev V.S. *Function spaces, integral operators and two-weighted inequalities on homogeneous groups.* Some applications. Baku, 1999, p.1-331.
[3]. Nakai E. *On generalized fractional integrals.* Proceedings of the Second ISAAC Congress, volume 1, Edited by H.Begehr, R.Gilbert and J.Kajiwara. Kluwer Academic Publishers, 2000, p.75-81.
[4]. Stein E.M. *Singular integrals and differentiability properties of functions.* Princeton Univ. Press, Princeton, 1970.

Vagif S. Guliyev, Rza Ch. Mustafayev
Institute of Mathematics & Mechanics of NAS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

Received March 17, 2000; Revised February 20, 2001.
Translated by authors.

IBIKLI E.

APPROXIMATION OF BOREL DERIVATIVES OF FUNCTIONS BY SINGULAR INTEGRALS

Abstract

The definition of right and left Borel derivatives are given and the theorems on approximation of functions, having Borel derivatives, by the sequences of linear integral operators with positive kernels are established.

Key words. Borel derivatives, singular integrals, approximation problem.

1. Approximation of functions and its derivatives by the sequences of integral operators with positive kernels (so called singular integrals) or, in general, by linear positive operators wave investigated by many authors. Many results in this direction may be found in books [1]-[4]. We also refer to papers [5]-[8].

This paper is devoted to a problem of approximation of functions, having a Borel derivative. The function f has a Borel derivative $B'f(x_0) \neq \pm\infty$ at x_0 if

$$\lim_{h\to 0}\frac{1}{h}\int_0^h \frac{f(x_0+t)-f(x_0)}{t}dt = B'f(x_0)$$

and has a symmetrical Borel derivative $B'_s f(x_0) \neq \pm\infty$ at x_0 if

$$\lim_{h\to 0}\frac{1}{h}\int_0^h \frac{f(x_0+t)-f(x_0-t)}{2t}dt = B'_s f(x_0)$$

where $\int_0^h = \lim_{\varepsilon\to 0}\int_\varepsilon^h$ (see, for example, [5]).

Obviously, if the ordinary first derivative $f'(x_0)$ exists, so does Borel derivatives and $B'f(x_0) = B'_s f(x_0) = f'(x_0)$. The converse is not hold.

As usual may be given a definition of right and left Borel derivatives $B'_+ f(x_0)$ and $B'_- f(x_0)$. Namely, we shall say that the function f has a right and left Borel derivatives at x_0 if

$$\lim_{h\to 0}\frac{1}{h}\int_0^h \frac{f(x_0+t)-f(x_0)}{t}dt = B'_+ f(x_0),$$

$$\lim_{h\to 0}\frac{1}{h}\int_0^h \frac{f(x_0)-f(x_0-t)}{t}dt = B'_- f(x_0).$$

Clearly, if $B'_+ f$ and $B'_- f$ there exists then there exists also

$$B'_s f(x_0) = \frac{1}{2}\left(B'_+ f(x_0) + B'_- f(x_0)\right).$$

2. Let $(-R,R)$, $R>0$ is finite or infinite interval and $n=1,2,\ldots$. Consider a sequence of integral operators

$$L_n(f;x) = \int_{-R}^{R} f(t)K_n(t-x)dt, \tag{1}$$