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ON CONNECTIONS BETWEEN FUNCTION CLONES AND VARIETIES OF ALGEBRAS

Abstract

A lot of theorems are established which are inspired by the duality between congruences of clones and lattices of subvarieties.

This duality is given by theorem 2 below. It was explicitly stated about 15 years ago (independently by several authors), but has its roots in some works (1966-1973) of A.I. Mal'cev, E.Manes, W.Taylor and other authors. Using theorem 2 and some known results we obtain, for example:

(i) *the equational theory of congruence lattices of finite-valued clones is trivial;*

(ii) *if an algebra* $\mathbf{A}$ *generates the small variety* $\hat{\mathbf{A}}$ *and* $\hat{\mathbf{A}}$ *contains the global of* $\hat{\mathbf{A}}$, *then the clone of term functions of the global is isomorphic to the clone of term functions of* $\mathbf{A}$.

A majority of results of the article was published in [2-4] and in other our works. It is of review character (mainly) and its aim is to point out a lot of results inspired by the duality between congruences of clones and lattices of subvarieties. This duality in the exact form was formulated in the middle of 80-th years in [3, 17, 24] and is reminded below- see theorem 2, whose sources are perceived roughly (without precise formulation) in [9, 22, 29].

Theorems 1, 2 and a part of corollaries have been obtained in collaboration with I.A. Mal'cev - see [3, 4, 31] and bibliography in [2].

One can be acquainted with all undefined below notions in [10, 19, 28], while notations $\mathbf{F}_A$ (the clone of all finitary functions on a set A), var$(\mathbf{A})$ (the variety generated by an algebra $\mathbf{A}$), the finite-valued variant $\mathbf{F}_k$ of the clone $\mathbf{F}_A$ and others have been got from [2].

On establishment of desired duality the solution of the question "will a factor-algebra of a clone of functions be always isomorphic to a clone of functions" was found essential. If "yes", then the description of sets of truth values of functions from clones obtained by a factorization is of independent interest also. Theorem 1 answers the both questions. Let's give informations which are necessary for truing to understand it.

Trivial congruences of function clones are: the zero-congruence χ_0 (equality), the unit (universal) congruence χ_1 and the arity congruence χ_a. As is shown in [7, 12], in the lattice $\mathbf{Con}(\mathbf{F})$ of all congruences of any function clone $\mathbf{F}$ the set $C'on(\mathbf{F}) := \{\chi \in Con(\mathbf{F}) | \chi \neq \chi_1\}$ of its non-universal congruences coincides with the set $\{\chi \in Con(\mathbf{F}) | \chi \leq \chi_a\}$ of its subarity congruences and so forms the principal ideal in it. Further, any $\chi \in C'on(\mathbf{F})$ induces on $\mathbf{F}$ an out arity equivalence $\chi^* := (\chi \circ \nu) \cup (\nu \circ \chi)$, where ν is the thread equivalence on $\mathbf{F}$, i.e. the partition of F into threads (every thread begins with some function $f \in F$ and consists of $f, \nabla f, \nabla^2 f, \ldots, \nabla^n f, \ldots (n < \omega)$). In this inducing $\chi_a^* = \chi_1$ and $\chi^* \notin Con(\mathbf{F})$ for each $\chi < \chi_a$: here $\chi^* \neq \chi_1$ and χ^* is incomparable with χ_a. But $\chi^* \cap \chi_a = \chi$ is fulfilled for any $\chi \in C'on(\mathbf{F})$.

Theorem 1 ([2, 31]). *For any function clone* $\mathbf{F}$ *and any* $\chi \in C'on(\mathbf{F})$ *the factor-algebra* $\mathbf{F}/\chi$ *is isomorphic to some clone of functions on the set* F/χ^{*}.

In what follows $\mathbf{T}(\mathbf{A})$ denotes the clone of term functions of $\mathbf{A}$.

Theorem 2 ([2, 3, 31]). *For any algebra* $\mathbf{A}$ *the lattice* $\mathbf{C'on}(\mathbf{T}(\mathbf{A}))$ *is anti-isomorphic to the lattice* $\mathbf{L}_v(\hat{\mathbf{A}})$ *of all subvarieties of the variety* $\hat{\mathbf{A}} := \mathrm{var}(\mathbf{A})$.

Proof of theorem 2 which we gave in [31] using theorem 1 and the criterion of belongness of an algebra $\mathbf{B}$ to a variety $\hat{\mathbf{A}}$ looks more natural than its "ad hoc"- fachion proofs in [17, 24](the mentioned criterion generalizes the A.I. Mal'cev theorem [9, 10] on selector isomorphisms).

Theorem 2 permits to obtain results in the theory of clones (correspondingly in theory of varieties) for which the possibilities of the proof with "inner means" are by far not obvious.

At first let's give some corollaries immediately obtained by combination of theorem 2 with results known from references (see [23, 25-27] and other works) and then we'll formulate theorems 3-14 obtained by more complicated way.

Corollary 1. *For* $|A| \geq 2$ *the lattice* $\mathbf{C'on}(\mathbf{F})$ *is dually atomic for any clone* $\mathbf{F} \leq \mathbf{F}_A$.

Corollary 2. *For any clone* $\mathbf{F} \leq \mathbf{F}_k$ $(1 \leq k < \omega)$ *the set of dual atoms of the lattice* $\mathbf{C'on}(\mathbf{F})$ *is finite.*

In what follows we'll call subclones of clones $\mathbf{F}_k$ $(1 \leq k < \omega)$ as finite-valued clones.

Corollaries 1, 2 show essential difference of lattices of subarity congruences from lattices of subclones; by the same token they show essential difference of lattices of subclones from congruence lattices of clones also. It is interesting to compare the latter with the well-known fact on coincidence of abstract characterizations of congruence lattices and lattices of subalgebras of arbitrary algebras: both classes of lattices coincide up to isomorphism with the class of all algebraic (i.e. complete compactly generated) lattices.

Corollary 3. *For a nonsingleton semigroup* $\mathbf{S}$, *the lattice* $\mathbf{C'on}(\mathbf{T}(\mathbf{S}))$ *has a unique dual atom if and only if* $\mathbf{S}$ *is either equationally complete or is a group of exponent* p^n *(*p *is a prime,* $n \geq 1$*) or for some* $m \geq 2$ *satisfies the identities* $x^m = y^m, x^m = x^{m+1}$.

Corollary 4. *For a nonsingleton associative ring* $\mathbf{R}$ *the lattice* $\mathbf{C'on}(\mathbf{T}(\mathbf{R}))$ *has a unique dual atom if and only if* $\mathbf{R}$ *is either equationally complete or for some prime* p *and some* $n \geq 2$ *satisfies the identities* $x^{p^n} = x$ *and* $px = 0$ *or for some* $m \geq 1, n \geq 2$ *and a prime* p *satisfies the identities* $p^m x = 0 = x^n$.

The analogous corollaries are obtained similarly in all other cases when the description of varieties with unique atomic subvariety (for example, for varieties of semigroups with a signature involution- see [19]) is known. In particular, the following corollary is true in virtue of the uniqueness of the atomic variety of lattices. This is

Corollary 5. *The lattice* $\mathbf{C'on}(\mathbf{T}(\mathbf{L}))$ *has the unique dual atom for any nonsingleton lattice* $\mathbf{L}$.

Corollary 6. *The lattice* $\mathbf{C'on}(\mathbf{T}(\mathbf{R}))$ *is distributive for any associative ring* $\mathbf{R}$ *which satisfies for some prime* p *and some* $n \geq 1$ *the identities* $px = 0$ *and* $x^{p^n} = x$.

Corollary 7 (see also [17]). *There exists 1-generated subclone* $\mathbf{F} \leq \mathbf{F}_k$ *(for each* $k, 3 \leq k < \omega$*) with the continual congruence lattice.*

Corollary 8. *If for a clone* $\mathbf{F} \leq \mathbf{F}_A$ *all principal filters* $\mathbf{D} \leq \mathbf{C'on}(\mathbf{F})$ *are selfdual lattices, then* $\mathbf{C'on}(\mathbf{F})$ *(and therefore* $\mathbf{Con}(\mathbf{F})$*) is finite and distributive in the unit.*

Corollary 9. *The lattices* $\mathbf{C'on}(\mathbf{F})$, $\mathbf{Con}(\mathbf{F})$ *are distributive in the unit for any clone* $\mathbf{F}$ *generated by a binary associative function (i.e. for the clone of term functions of any semigroup).*

We'll call a clone $\mathbf{F} \leq \mathbf{F}_k$ as supconstant (respectively: supdiscriminatory, supmajoritary) if $\mathbf{F}$ contains all constant functions $c_a^n \in F_k$ (respectively: the ternary discriminator, a majority function).

Corollary 10. $\mathbf{C'on}(\mathbf{F})$ *and* $\mathbf{Con}(\mathbf{F})$ *are finite distributive lattices for any finite-valued* CD*-clone* $\mathbf{F}$, *in particular, for any finite-valued supmajoritary or supdiscriminatory clone* $\mathbf{F}$. *Consequently any plain-algebraic (in the sense of [1])* CD*-clone is simple, i.e. it hasn't congruences different from* χ_0, χ_1, χ_a.

From the latter statement of corollary 10 in the case of supconstant clones we obtain

Corollary 11. *Any supconstant finite-valued* CD*–clone* $\mathbf{F} \leq \mathbf{F}_k$, *which violets all non-trivial equivalences on* $\{0,\ldots,k-1\}$ *is simple.*

Corollary 12. *The lattices* $\mathbf{C'on}(\mathbf{F})$ *and* $\mathbf{Con}(\mathbf{F})$ *are completely distributive for any countable* CD*–clone.*

Now let's point out examples of corollaries of theorem 2 whose proofs require additional efforts.

Theorem 3 ([2, 3]). *The equational theory of congruence lattices of finite-valued clones is trivial. Further more, for any non-trivial lattice identity* ε *one can show a number* $k(\varepsilon) < \omega$ *such that for* $k \geq k(\varepsilon)$ *there exists a 1-generated subclone* $\mathbf{D}_k \leq \mathbf{F}_k$ *with finite* $\mathbf{Con}(\mathbf{D}_k)$ *on which* ε *is disturbed.*

Lately, the following question of A.V. Kusnetsov and I.S. Negru (see [15]) relatively to the lattice $\mathbf{S}(\mathbf{F}_k)$ of all subclones of the global clone $\mathbf{F}_k$ is of interest:

Does the lattice $\mathbf{S}(\mathbf{F}_k)$ satisfy any non-trivial identity or at least quasiidentity for $k \geq 3$? The case $k = 2$ isn't contained in the question, since some true on $\mathbf{S}(\mathbf{F}_2)$ non-trivial identities were ascertained in [15] with the help of straight (and bulky enough) calculations. To our mind, even more general questions (for example, does the lattice $\mathbf{S}(\mathbf{F}_k)$ satisfy some non-trivial $\forall\exists$- axiom, i.e. is its $\forall\exists$- theory non-trivial) is of interest.

This is more actual, since in [5, 6] lately the negative answer in the case of identities was obtained.

Theorem 4 ([2]). *If the* $\forall\exists$- *theory of the lattice* $\mathbf{S}(\mathbf{F}_k)\ (k \geq 3)$ *is trivial, then* $\mathbf{S}(\mathbf{F}_k)$ *is embeddable as a principal filter into the lattice of subarity congruences of none clone of functions.*

Note. The fact itself of existence of non-trivial identities on the lattice $\mathbf{S}(\mathbf{F}_2)$ may be very simply– without any calculation– proved (unlike the I.S. Negru [15] proof) with the help of belonging to Wille theorem 6 from §2 of V part of Grätzer's monograph [8].

Let D_ω be the class of all finite distributive lattices, AC and $ACJI$ -the classes of algebraic lattices with a compact unit and correspondingly with completely $\vee$-indecomposable unit (obviously $ACJI \subset AC$), $E(\mathbf{M})$- be the equational theory of an algebra $\mathbf{M}$. The filter $\mathbf{F}(\mathbf{M})$ of all equational theories extending $E(\mathbf{M})$ (with the same signature) belongs to AC and therefore, for the class E of all lattices which are isomorphic to appropriate $\mathbf{F}(\mathbf{M})$'s the inclusion $E \subseteq AC$ is fulfilled. The problem b) of part 2.1 of A.I. Mal'cev's survey [11] contains the question: is it strict, and if it is so then by what conditions E may be defined inside AC?

Rather long time only the results appeared that served as arguments in favor of hypothesis $E = AC$, in particular, the inclusions $D_\omega \subset E$ and $ACJI \subset E$ (implying $D_\omega \bigcup ACJI \subset E \subseteq AC$) first of which is a folklore fact and the second is quoted in [20] as the Pigozzi- Kogalovski theorem.

However in 1986-1988 Lampe [20], Erne [18] (independently, Tardos [32]) and Lampe [30] again disprove this hypothesis, besides each new refutation was stronger than previous; the total of these offers was the chain $E \subseteq V \subset ET \subset Z \subset AC$, where the class Z is picked out in AC the Zipper condition [20], the class V by the Velcro condition, and the class ET by the Erne-Tardos condition [see 18]. It is asked in [30], will the inclusion $E \subseteq V$ be strict and the assumption, that will be, is said.

The following result confirms this assumption.

Theorem 5 (see [2] also [21]). *The strict inclusion $E \subset V$ has place.*

The comparison of the Oates-Powell theorem on existence of a finite basis of identities of any finite group with Bryant's example [16] of finite pointed group which has no finite basis of its identifies, shows that the enrichment of signature of an algebra $\mathbf{A}$ by even one new 0-ary operation can sharply change the properties of the filter of supvarieties of the variety $\hat{\mathbf{A}}$ generated by it. The question about behaviour with respect to 0-ary (i.e. constant) enrichments for the lattice $\mathbf{L}_v(\hat{\mathbf{A}})$ of subvarieties of the variety $\hat{\mathbf{A}}$ is so interesting and natural. For $\mathbf{A} = \langle A; F \rangle$, we assume $\mathbf{A}^* := \langle A; F \bigcup C_A \rangle$, where C_A is the set of all constant functions on A.

Let $\mathbf{M}$ be the Murski groupoid (see [14, 23]).

Theorem 6 ([4]). $\mathrm{var}(\mathbf{M}^*)$ *is a minimal (atomic) variety, while* $\mathrm{var}(\mathbf{M})$ *is the union of an infinite increasing chain of proper subvarieties and the lattice* $\mathbf{L}_v(\hat{\mathbf{M}})$ *is continual.*

Some other cases of collapsing of continual lattice of subvarieties to the two-element chain under constant signature saturation of a generic algebra are ascertained.

Proposition 7 [2]. *The set of dual atoms of the lattice $\mathbf{L}_v(\mathbf{A})$ is finite for any finite algebra $\mathbf{A}$.*

This proposition complements the known D. Scott's theorem [25].

A clone $\mathbf{F} \le \mathbf{F}_A$ will be called a clone with separable selectors if $F^- := F \setminus Sel_A$ forms in $\mathbf{F}$ the iterative subalgebra $\mathbf{F}^-$. An algebra $\mathbf{A}$ is called an algebra with separable selectors, if $\mathbf{T}(\mathbf{A})$ is a clone with separable selectors. For such algebra $\mathbf{A}$ we'll denote by $i_{\mathbf{A}}$ the index (i.e. the power of the set of blocks) of the main out arity [12] congruence of the iterative algebra $(\mathbf{T}(\mathbf{A}))^-$.

A variety V is called constant, if it contains nonsingleton algebras and all object variables of all non-trivial V -terms are fictitious. We'll denote by N_Ω the least constant variety of signature Ω. As was shown in [13], for any algebra $\mathbf{A}$ with separable selectors the inclusion $\hat{\mathbf{A}} \supseteq N_\Omega$ (where Ω is signature of algebra $\mathbf{A}$) is fulfilled. Also it is obvious, that N_Ω is an atomic variety.

Theorem 8 [2]. *For any algebra* $\mathbf{A}$ *with separable selectors the interval* $\lfloor N_\Omega, \hat{\mathbf{A}} \rfloor$ *of the lattice* $\mathbf{L}_v(\hat{\mathbf{A}})$ *contains a principal ideal* $\lfloor N_\Omega, \hat{\mathbf{B}} \rfloor$, *which is dually isomorpjic to the lattice of partitions of* $i_\mathbf{A}$ *-element set.*

For an algebra $\mathbf{A}$ we'll denote by $I_\mathbf{A}^*$ the intersection of all ideals of the semigroup $\mathbf{T}^{(1)}(\mathbf{A})$ of its unary term functions and analogously, for the clone $\mathbf{F} \leq \mathbf{F}_A$ we'll denote by $I^*(\mathbf{F})$ the intersection of all ideals of the semigroup $\mathbf{F}^{(1)}$. Obviously, in case of its non-emptiness $I_\mathbf{A}^*$ (respectively $I^*(\mathbf{F})$) is the least ideal of the corresponding semigroup.

The tolerance $\pi(\mathbf{F}) \subseteq F \times F$ on a clone $\mathbf{F}$ with $I^*(\mathbf{F}) \neq \varnothing$ will be called pseudohomotopic iff $\langle f, g \rangle \in \pi(\mathbf{F})$ means $f, g \in F^{(n)}$ for some n and for any $\varphi \neq \psi$ from $I^*(\mathbf{F})$ there exists a function $h \in F^{(n+1)}$ such that $f(x_1,...,x_n) = h(x_1,...,x_n,\varphi(x_n))$ and $g(x_1,...,x_n) = h(x_1,...,x_n,\psi(x_n))$.

A set Φ of unary functions $\varphi : A \to A$ we'll call semitransitive, if for any $n \geq 1$ and $\langle a_1,...,a_n \rangle \in A^n$ there exist $\varphi_1,...,\varphi_n \in \Phi$ and element $b \in A$ such that $\varphi_1(b) = a_1,...,\varphi_n(b) = a_n$.

Theorem 9 [2]. *If the ideal* $I_\mathbf{A}^* \neq \varnothing$ *is semitransitive and the tolerance* $\pi(\mathbf{T}(A))$ *differs from diagonal of the square* $T(\mathbf{A}) \times T(\mathbf{A})$, *then* $\hat{\mathbf{A}}$ *has the greatest proper subvariety.*

In the case when $I_\mathbf{A}^* = \varnothing$ and therefore $\mathbf{T}^{(1)}(\mathbf{A})$ hasn't the least ideal, one can prove an analogue of this theorem in which the semitransitivity is required from the whole semigroup $\mathbf{T}^{(1)}(\mathbf{A})$ and the tolerance $\pi(\mathbf{T}(A))$ is replaced by suitable binary relation on $T(\mathbf{A})$ (or even on $T^{(1)}(\mathbf{A})$); generally, different ways on choosing such a relations are possible.

Namely: for a set $\Phi \subseteq T^{(1)}(\mathbf{A})$ we'll consider the relation $\pi_\Phi(\mathbf{A}) \subseteq T^1(\mathbf{A}) \times T^1(\mathbf{A})$, where $\langle f, g \rangle \in \pi_\Phi(\mathbf{A}) \overset{def}{\Leftrightarrow}$ for any two $\varphi \neq \psi$ from Φ there exists $h \in T^{(2)}(\mathbf{A})$ with the property $f(x) = h(x, \varphi(x)) \wedge g(x) = h(x, \psi(x))$; further, for $\Phi \leq \mathbf{T}^{(1)}(\mathbf{A})$, the entry $\Phi \overset{ess}{\leq} \mathbf{T}^{(1)}(\mathbf{A})$ will mean that $\mathbf{T}^{(1)}(\mathbf{A})$ is an essential extension of its subsemigroup Φ.

Theorem 10. *If* $T^{(1)}(\mathbf{A})$ *is semitransitive and there exists* $\Phi \overset{ess}{\leq} \mathbf{T}^{(1)}(\mathbf{A})$ *such that* $\pi_\Phi(\mathbf{A})$ *differs from the diagonal of the square* $T^{(1)}(\mathbf{A}) \times T^{(1)}(\mathbf{A})$, *then* $\hat{\mathbf{A}}$ *has the greatest proper subvariety.*

Let's point out some more results in this direction, had not reflected in [3, 4, 31].

For an algebra $\mathbf{A}$, we'll denote by $gl(\mathbf{A})$ and $gl^*(\mathbf{A})$ its global and, correspondingly, its extended global [2, 19], by $\overline{HS}(\mathbf{A})$ the set of its proper factors.

Let $K_1 \circ K_2$ be the Mal'cev product of classes K_1, K_2 of algebras with the same signature.

We'll say $\mathbf{A}$ has the Foster property, if $\hat{\mathbf{A}}$ contains the unique (up to $\cong$) subdirectly irreducible algebra.

An algebra $\mathbf{A} = \langle A; F \rangle$ we'll call weakly compressable if there exists an operation $f \in F$ such that the partition of A into blocks $f(A,...,A)$ and $A \setminus f(A,...,A) \neq \varnothing$ contains a non-diagonal congruence from $Con(\mathbf{A})$.

Theorem 11. *If* $\mathbf{A}$ *generates the small variety* $\hat{\mathbf{A}}$ *and* $gl(\mathbf{A}) \in \hat{\mathbf{A}}$, *then the clone* $\mathbf{T}(gl(\mathbf{A}))$ *is isomorphic to the clone* $\mathbf{T}(\mathbf{A})$.

Theorem 12. *For any* TC^* *- variety* V *with non-trivial spectrum,* $gl^*(V) \neq V$ *holds.*

Theorem 13. *If* $\mathbf{A}$ *is weakly compressiable and doesn't belong to* $\mathrm{var}(\overline{HS}(\mathbf{A}))$, *then for any variety* V *with the condition* $\mathrm{var}(\overline{HS}(\mathbf{A})) \subseteq V \subset \hat{\mathbf{A}}$ *at least one of equalities* $gl(V) = V$, $V \circ V = V$ *is disturbed.*

Theorem 14. *For* $k \geq 4$, *there exist* 2^{ω} *non-equivalent* k *-element functionally complete algebras without the Foster property.*

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ON GENERALIZED FRACTIONAL INTEGRALS

Abstract

The properties of generalized fractional integrals are studied.

It is known that the fractional integral

$$I_\alpha f(x)=\int_{R^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy,\qquad 0<\alpha<n,$$

is bounded from $L_p(R^n)$ to $L_q(R^n)$, when $1<p<n/\alpha$ and $1/q=1/p-\alpha/n$ as the Hardy-Littlewood-Sobolev theorem. We consider generalized fractional integrals and prove the Hardy-Littlewood-Sobolev theorem for them. Related questions were studied in works of Y.Mizuta, E.Nakai, A.Gadjiev and etc.

It is also known that the modified fractional integral

$$\widetilde{I}_\alpha f(x)=\int_{R^n}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|y|^{n-\alpha}}\right)\chi_{B_1^*}(y)f(y)dy$$

is bounded from $L_{n/\alpha}(R^n)$ to $BMO(R^n)$. Here B_1^* - exterior of unit ball $B_1=B(0,1)$, that is $B_1^*=R^n\setminus B_1$, χ_A - the characteristic function of set A. We also investigate the boundedness of modified generalized fractional integrals from $L_p(R^n)$ to $BMO(R^n)$.

The $L_p(R^n)$, $1\le p\le\infty$ spaces are defined as the set of all measurable functions $f(x)$, $x\in R^n$ on R^n with finite norm

$$\|f\|_{L_p}(R^n)=\left(\int_{R^n}|f(x)|^p dx\right)^{1/p},\quad 1\le p<\infty.$$

At $p=\infty$ the spaces $L_\infty(R^n)$ are defined by means of usual modification

$$\|f\|_{L_\infty}(R^n)=\operatorname*{esssup}_{x\in R^n}|f(x)|.$$

A measurable function $f(x)$, $x\in R^n$ belongs to $WL_p(R^n)$, $0<p<\infty$ (weak L_p space) by definition if the norm

$$\|f\|_{WL_p}(R^n)=\sup_{t>0}t\left|\left\{x\in R^n:|f(x)|>t\right\}\right|^{1/p},\ 0<p<\infty$$

is finite. At $p=\infty$ $WL_\infty(R^n)\equiv L_\infty(R^n)$ and

$$\|f\|_{WL_\infty}(R^n)=:\|f\|_{L_\infty}(R^n).$$

A locally integrable function f will be said to belong to BMO if the norm

$$\|f\|_{BMO}(R^n)=\sup_{x\in R^n,r>0}|B(x,r)|^{-1}\int_{B(x,r)}|f(y)-f_{B(x,r)}|dy$$

is finite; here