ASLANOV G.I.

PARTIAL OPERATOR-DIFFERENTIAL EQUATIONS IN SPACES OF TYPE S

Abstract

In the paper partial differential equations with unbounded operator coefficients in space of type S. Theorem on existence and asymptotic representation of solution of the given equation are proved. In conclusion examples of application of obtained abstract results to solution of Newmann problem for elliptic equations in unbounded cylindrical domains are give.

Differential equations with operator coefficients in Banach spaces were studied in many papers [1-3]. Results of these papers have a series of applications in theory of boundary-value problems. The main applications concern with the question of behaviour of solutions in infinitive cylinder [4] or in the neighborhood of conical point of he boundary [5].

In comparison with ordinary operator-differential equations few papers are devoted to investigation of solvability of partial operator-differential equations in Hilbert spaces. We'll note papers [6,7], where solvability of boundary-value problems for some classes of partial operators-differential equations in functional spaces are studied.

Partial differential equations with operator coefficients were investigated in a number of papers [8-16]. Theorems on single valued, normal and Fredholm solvability, on asymptotic behavior and smoothness of solution in Hilbert spaces such equations also have applications in theory of boundary-value problems.

1. Let H be a Hilbert space. Consider differential operator

$$L(D)u=\sum_{|\alpha|\le m}A_\alpha D^\alpha u,$$

where A_α are bounded operators: $H\to H$.

Denote by $R(\lambda)=\left[\sum_{|\alpha|\le m}(i\lambda)^\alpha A_\alpha\right]^{-1}$ the operator $H\to H$. At first we'll define the space $S(R^n;H)$. Let $u(x)\in H$ at any $x\in R^n$ and $D^\alpha u$ exists at all α. Suppose that $(1+|x|)^p\left\|D^\alpha u\right\|_H\le C_{\alpha,p}$ take place at any p,α.

Then we'll say that $u(x)\in S(R^n;H)$. If it's clear what H we are talking about then we'll use denotation $u(x)\in S(R^n)$. Let continuous mapping $T:u(x)\to H$ $u(x)\leftarrow S(R^n)$ be given. The set of all such T we'll denote by S^i.

Theorem 1. *If $R(\lambda)$ exists at all real λ, infinity differentiable function λ is such that*

$$\left\|D^\alpha R(\lambda)\right\|_H\in C_\alpha(1+|\lambda|)^s,$$

where s depends on α, then $L(D)$ maps λ on

$$\left\|D^\alpha R(\lambda)\right\|_H\le C_\alpha(1+|\lambda|)^s.$$

Proof. If $u(x)\in S(R^n)$, then it's clear that $Lu\in S(R^n)$. Let $f\in S(R^n)$. We have to prove that there exists such $u(x)\in S(R^n)$, that $Lu=f$. Consider $\tilde{u}(\lambda)=R(\lambda)\tilde{f}(\lambda)$, where $\widetilde{u}(\lambda)$ and $\widetilde{f}(\lambda)$ are Fourier transformation of the functions $u(x)$ and $f(x)$ respectively. It's clear that $\widetilde{u}(x)\in S(R^n)$. Inverse Fourier transformation $\widetilde{u}(\lambda)$ of $\widetilde{u}(\lambda)$ is such function $u(x)\in S(R^n)$ that $Lu=f$.

It's said that T is on the set $\Omega\subset R^n$, if from $u(x)\in S(R^n)$, $u(x)=0$ in the neighborhood Ω follows that $Tu=0$.

Lemma 1. *If $T\in S'$ is concentrated at one point $x=0$, then*

$$Tu=\sum_{|\alpha|\le k}T_\alpha D^\alpha u(x)\big|_{x=0}, \tag{1}$$

where T_α - are some bounded operators in H.

Proof. Since T continuously maps S into H, then we can find such k that if $D^\alpha u(0)=0$ at $|\alpha|\le k$, then $Tu=0$. Let $\widetilde{u}(x)\in S(R^n)$.

Consider the segment of its Taylor series

$$u(x)=\sum_{|\alpha|\le k}\frac{D^\alpha u(0)}{\alpha!}x^\alpha+u_1(x). \tag{2}$$

We'll rewrite formula (2) in the following form

$$\theta(x)u(x)=\sum_{|\alpha|\le k}\frac{D^\alpha u(0)}{\alpha!}x^\alpha\theta(x)+u_1(x)\theta(x), \tag{3}$$

where $\theta(x)\equiv 0$ at $|x|\ge 1$, $\theta(x)\equiv 1$ at $|x|<1$.

We'll apply the operator $T\in S'$ to the both sides of (3). As a result we have:

$$T[\theta(x)u(x)]=\sum_{|\alpha|\le k}T_\alpha\left[D^\alpha u(0)\right]. \tag{4}$$

Note that $T[\theta(x)u(x)]=Tu$, since $\theta(x)u(x)-u(x)=0$ in the neighborhood $x=0$. Therefore, necessary formula (1) follows from (4)

$$Tu=\sum_{|\alpha|\le k}T_\alpha\left[D^\alpha u(0)\right].$$

Also note that any element $T\in S'$ has the derivative $D^\alpha T\in S'$. Action of $D^\alpha T$ on $u(x)\in S(R^n)$ passes by the formula

$$\left(D^\alpha T,u\right)=(-1)^{|\alpha|}\left(T,D^\alpha u\right).$$

If $T\in S'$, A is a bounded operator $H\to H$, then $A\cdot T\in S'$ can be defined, putting

$$(AT,u(x))=A(Tu(x)).$$

The above said allows to define operator L on S'.

Theorem 2. *If $R(\lambda)=\left[\sum_{|\alpha|\le k}(i\lambda)^\alpha A_\alpha\right]^{-1}$ is defined at $\lambda\in R^n/0$ and there exists such homogeneous polynomial $P(\lambda)$ of the order k, that $P(\lambda)\ne 0$ at $\lambda\in R^n/0$ and $\left\|D^\alpha P(\lambda)R(\lambda)\right\|\le C_\alpha(1+|\lambda|)^s$, where s depends on α. Then any solution of the equation*

$$Lu=0,\ u\in S' \tag{5}$$

has the following form: $u=\sum_{|\alpha|\le l}T_\alpha x^\alpha$, where $T_\alpha: H\to H$ are bounded operators.

Proof. Fourier transformation in (5) gives

$$\left[\sum_{|\alpha|\le m} A_\alpha (i\lambda)^\alpha\right]\tilde{u}(\lambda)=0. \tag{6}$$

Hence it follows that $\tilde{u}(\lambda)=0$ at $\lambda\in R^n/0$.

Consider any element $v(\lambda)\in S(R^n)$ such that $D^\alpha v(0)=0$ at $|\alpha|\le k$, k is order of $P(\lambda)$. From (6) it follows that $(\tilde{u}(\lambda),\tilde{v}(\lambda))=0$. Thus, $\tilde{u}(\lambda)\in S$ satisfies the conditions of lemma 1. It means that

$$\tilde{u}(\lambda)=\sum_{|\alpha|\le l} T_\alpha\left(D^\alpha u(0)\right).$$

Passing here to $u(x)$ we obtain that

$$u(x)=\sum_{|\alpha|\le l} T_\alpha x^\alpha,$$

where T_α - are some operators

Theorem 3. *If $R(\lambda)$ satisfies conditions of theorem 2, $f(x)\in S(R^n)$, then there exists solution $u(x)$ of the equation*

$$Lu=f(x) \tag{7}$$

such that $u(x)\in S'$

$$\|u(x)\|\le c(1+|x|)^{k-n} \tag{8}$$

at odd n and N at $k<n$

$$\|u(x)\|\le C(1+\|x\|)^{k-n}(1+\ln|x|) \tag{8'}$$

at even $k\ge n$.

Proof. Let $V(x)$ be a solution of the equation

$$P\left[-i\frac{\partial}{\partial x}\right]v=g(x),$$

where $g(x)$ is such, that its Fourier transformation is $g(\lambda)=P(\lambda)R(\lambda)\tilde{f}(\lambda)$.

Note that $\left\|D^\alpha g(\lambda)\right\|(1+|\lambda|)^p\le c_{\alpha,p}$ at any α,p.

Such $v(x)$ can be obtained by the formula

$$v(x)=\Gamma(x)*g(x), \tag{9}$$

where $\Gamma(x)$ is a fundamental solution of the equation

$$P\left[-i\frac{\partial}{\partial x}\right]\Gamma(x)=\delta(x).$$

Function $\Gamma(x)$ is a scalar function defined in $R^n/0$. We'll show that $v(x)$ is a solution of equation (7).

Indeed,

$$\begin{aligned}\sum_{|\alpha|\le m} A_\alpha D^\alpha v&=\sum_{|\alpha|\le m} A_\alpha D^\alpha\left[\Gamma(x)*g(x)\right]=\sum_{|\alpha|\le m} A_\alpha\left[\Gamma(x)*D^\alpha g(x)\right]=\\&=\sum_{|\alpha|\le m}\left[\Gamma(x)*A_\alpha D^\alpha g(x)\right]=\Gamma(x)*\left[\sum_{|\alpha|\le m} A_\alpha D^\alpha g\right].\end{aligned} \tag{10}$$

We'll find Fourier transformation of the right-hand side of (10). It equals.

$$\widetilde{\Gamma}(\lambda)\left[\sum_{|\alpha|\le m} A_\alpha (i\lambda)^\alpha \widetilde{g}(x)\right] = \frac{1}{P(\lambda)} \sum_{|\alpha|\le m} A_\alpha (i\lambda)^\alpha \widetilde{g}(\lambda) =$$
$$= \frac{1}{P(\lambda)} \sum_{|\alpha|\le m} A_\alpha (i\lambda)^\alpha P(\lambda) R(\lambda) \widetilde{f}(\lambda) = \widetilde{f}(\lambda).$$

Consequently, the right-hand side of (10) coincides with $f(x)$.

Now we have to obtain estimation of function $v(x)$, defined by formulas (8), (8').

Let n be odd or $n-k$. Then $\Gamma(x,y) \le c|x-y|^{k-n}$. From (9) we have

$$|v(x)| \le c \int_{R^n} |x-y|^{k-n} g(y)dy \le c \int_{2|x|<|y|} |x-y|^{k-n} g(y)dy +$$
$$+ c \int_{\frac{|x|}{2}<|y|<2|x|} |x-y|^{k-n} g(y)dy + c \int_{|y|<\frac{|x|}{2}} |x-y|^{k-n} g(y)dy = J_1 + J_2 + J_3 .$$

We'll estimate J_1. We have:

$$J_1 \le c \int_{|y|>2|x|} |x-y|^{k-n} g(y)dy \le c \int_{|y|>2|x|} |y|^{k-n} (1+|y|)^{-N} dy \le c_1(N) \cdot |x|^{-N}$$

for any N. Estimate J_2:

$$J_2 \le c \int_{|y|>2|x|} |x-y|^{k-n} g(y)dy \le c_2 |x|^{-N} \int_{\frac{|x|}{2}<|y|<2|x|} |x-y|^{k-n} dy \le$$
$$\le c_2' |x|^{-N} |x|^{k+1} \le c_3 |x|^{-N'} ,$$

where N' is any number

$$J_3 \le \int_{|y|<\frac{|x|}{2}} |x-y|^{k-n} g(y)dy \le c|x|^{k-n} \int_{|y|<\frac{|x|}{2}} g(y)dy \le c^* |x|^{k-n} .$$

Required estimation (8) for $u(x)$ follows from these estimations.

In case $n \le k$ is even n

$$|v(x)| \le c \int_{R^n} |x-y|^{k-n} [1 + \ln|x-y|] g(y)dy .$$

If we divide the integral in the right-hand side by three analogous to J_1, J_2, J_3 addends I_1, I_2, I_3 then the similar estimations are obtained for I_1, I_2 and $I_3 \le c|x|^{k-1}(1+\ln|x|)$.

Hence estimation (8) follows. Theorem 3 is proved.

2. Equation

$$\sum_{|\alpha|\le m} A_\alpha D^\alpha u = f(x) \tag{11}$$

in a given form included only bounded operators A_α.

In order to consider the case of unbounded operators we consider a family of Hilbert spaces $H_0 \supset H_1 \supset \ldots \supset H_m$. Let $f(x) \in H_0, A_\alpha$, is bounded operator: $H_{m-|\alpha|} \to H_0$.

Such a function $u(x) \in H_m$ is considered a solution, that $D^\alpha u \in H_{m-|\alpha|}$ at $|\alpha| \le m$ and equality (11) is satisfied at almost all $x \in R^n$. Let $S_j(R^n, H_j)$ be a space defined with the help of H_j. It's clear that $S_j \supset S_{j+1}$, correspondingly $S''_j \supset S'_{j+1}$. These definitions allow to extend the notion of solution of equation $S_j \supset S_{j+1}$. As a solution we can consider elements S'_m. Consider asymptotic properties of solutions of equation (11) in case $f(x) \in S_0$ and when $f(x)$ has a compact support.

Theorem 4. *Let* $R(\lambda) = \left[\sum_{|\alpha| \le m} (i\lambda)^m A_\alpha\right]^{-1}$ *be bounded operator:* $H_0 \to H_m$ *at any* $\lambda \in R^n / 0$. *Suppose that there exists a homogeneous polynomial* $P(\lambda)$ *of order* k *such that* $P(\lambda) \ne 0$ *and*

$$P(\lambda) R(\lambda) = a(\lambda) + P(\lambda) R_1(\lambda)$$

where $R_1(\lambda)$ *is an analytical function in the layer* $|Jm\lambda_j| \le c$ *such that* $\|R_1(\lambda)\|_{H_0 \to H_m} < c(1+|\lambda|)^s$, *at some* C^n, *is bounded in whole* $u(x)$ *operator function.*

Let $u(x)$ *be a solution of equation (11) such that*

$$\int_{R^n} \|u(x)\|_{H_m}^2 dx < \infty \ .$$

Then $u(x) = u_1(x) + u_2(x)$, *where*

$$\|u_2(x)\|_{H_m} \le c_1 e^{-c_2|x|}, \tag{12}$$

$$P\left[-i\frac{\partial}{\partial x}\right] u_1 = 0 \ \ at \ |x| \ge N \ \ and \ \|u(x)\|_{H_m} \le c. \tag{13}$$

Proof. Applying Fourier transformation in (11) we obtain

$$\sum_{|\alpha| \le m} (i\lambda)^\alpha A_\alpha \tilde{u}(\lambda) = \tilde{f}(\lambda).$$

From here

$$P(\lambda) \sum_{|\alpha|} (i\lambda)^\alpha A_\alpha \tilde{u}(\lambda) = P(\lambda) \tilde{f}(\lambda)$$

consequently

$$P(\lambda)\tilde{u}(\lambda) = R(\lambda)P(\lambda)\tilde{f}(\lambda) = A(\lambda)\tilde{f}(\lambda) + R_1(\lambda)P(\lambda)\tilde{f}(\lambda). \tag{14}$$

Assume $\tilde{u}_2(1) = R_1(\lambda)\tilde{f}(\lambda)$.

Then $u_2(x)$ we'll satisfy (12). From (14) it follows that $P(\lambda)\tilde{u}_1(\lambda) = A(\lambda)\tilde{f}(\lambda)$. Consequently,

$$-P\left[-i\frac{\partial}{\partial x}\right] u_1(x) = F^{-1}\left[A(\lambda)\tilde{f}(\lambda)\right],$$

where F^{-1} is inverse Fourier transformation of entire function. It's known that such function has a compact support. It means that $u_1(x)$ satisfies conditions (13). The theorem is proved.

3. Operator $\Delta\Delta u - \Delta u$ satisfies the conditions of theorem 3.

In this case $P(\lambda) = |\lambda^2|$.

4. Operator of Neuman problem on infinite layer satisfies the conditions of theorem.

Let $\Pi = \{x : 0 < x_n < 1, \ \ (x_1, x_2, \dots, x_{n-1}) \in R^{n-1}\}$.

Consider equation

$$\sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} + a(x_n)\frac{\partial^2 u}{\partial x_n^2} = f(x), \quad x \in \Pi$$

with boundary conditions $\left.\frac{\partial u}{\partial x_n}\right|_{\partial\Pi} = 0$. Suppose $H_0 = W_2^2[0,1]$, $H_1 = W_2^1[0,1]$, $H_2 = L_2(0,1)$.

In this case $P(\lambda) = |\lambda|^2$. The function $R_1(\lambda)$ will be regular in the layer $|Jm\lambda_k| \leq \pi$.

The author expresses his deep gratitude to academician of AS of Azerbaijan Gasymov M.G. and professor Bayramogli M. for their content interest to his paper and also to professor Mirzoyev S.S. for his helpful comments.

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Gamidulla I. Aslanov
Institute of Mathematics & Mechanics of NAS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.:39-47-20.

Received February 21, 2001; Revised August 20, 2001.
Translated by Agayeva R.A.

BAIRAMOV R.A.

ON CONNECTIONS BETWEEN FUNCTION CLONES AND VARIETIES OF ALGEBRAS

Abstract

A lot of theorems are established which are inspired by the duality between congruences of clones and lattices of subvarieties.

This duality is given by theorem 2 below. It was explicitly stated about 15 years ago (independently by several authors), but has its roots in some works (1966-1973) of A.I. Mal'cev, E.Manes, W.Taylor and other authors. Using theorem 2 and some known results we obtain, for example:

(i) the equational theory of congruence lattices of finite-valued clones is trivial;

(ii) if an algebra $\mathbf{A}$ generates the small variety $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}$ contains the global of $\hat{\mathbf{A}}$, then the clone of term functions of the global is isomorphic to the clone of term functions of $\mathbf{A}$.

A majority of results of the article was published in [2-4] and in other our works. It is of review character (mainly) and its aim is to point out a lot of results inspired by the duality between congruences of clones and lattices of subvarieties. This duality in the exact form was formulated in the middle of 80-th years in [3, 17, 24] and is reminded below- see theorem 2, whose sources are perceived roughly (without precise formulation) in [9, 22, 29].

Theorems 1, 2 and a part of corollaries have been obtained in collaboration with I.A. Mal'cev - see [3, 4, 31] and bibliography in [2].

One can be acquainted with all undefined below notions in [10, 19, 28], while notations $\mathbf{F}_A$ (the clone of all finitary functions on a set A), var$(\mathbf{A})$ (the variety generated by an algebra $\mathbf{A}$), the finite-valued variant $\mathbf{F}_k$ of the clone $\mathbf{F}_A$ and others have been got from [2].

On establishment of desired duality the solution of the question "will a factor-algebra of a clone of functions be always isomorphic to a clone of functions" was found essential. If "yes", then the description of sets of truth values of functions from clones obtained by a factorization is of independent interest also. Theorem 1 answers the both questions. Let's give informations which are necessary for truing to understand it.

Trivial congruences of function clones are: the zero-congruence χ_0 (equality), the unit (universal) congruence χ_1 and the arity congruence χ_a. As is shown in [7, 12], in the lattice $\mathbf{Con}(\mathbf{F})$ of all congruences of any function clone $\mathbf{F}$ the set $C'on(\mathbf{F}) := \{\chi \in Con(\mathbf{F}) | \chi \neq \chi_1\}$ of its non-universal congruences coincides with the set $\{\chi \in Con(\mathbf{F}) | \chi \leq \chi_a\}$ of its subarity congruences and so forms the principal ideal in it. Further, any $\chi \in C'on(\mathbf{F})$ induces on $\mathbf{F}$ an out arity equivalence $\chi^* := (\chi \circ \nu) \cup (\nu \circ \chi)$, where ν is the thread equivalence on $\mathbf{F}$, i.e. the partition of F into threads (every thread begins with some function $f \in F$ and consists of $f, \nabla f, \nabla^2 f, \ldots, \nabla^n f, \ldots (n < \omega)$). In this inducing $\chi_a^* = \chi_1$ and $\chi^* \notin Con(\mathbf{F})$ for each $\chi < \chi_a$: here $\chi^* \neq \chi_1$ and χ^* is incomparable with χ_a. But $\chi^* \cap \chi_a = \chi$ is fulfilled for any $\chi \in C'on(\mathbf{F})$.